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SIMULTANEOUS SIMILARITIES OF PAIRS OF 2×2 INTEGRAL SYMMETRIC MATRICES

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This is a continuation of a previous paper [**12**]. The issue of the present paper is stated in a different way as the title indicates. The title links it with S. Friedland's important paper [**2**] called 'Simultaneous similarity of matrices'. However, he deals with matrix pairs with complex entries, even pairs of symmetric ones. My paper is a small inroad in the case of integral matrices.

I will first report briefly on my previous work, including [**12**], and then come to new material. Both parts deal with integral matrices $\mathbf{A}$ with characteristic polynomial $x^2 - m$, $m \equiv 2$, $3(4)$ and square free. The matrix $\mathbf{A} = (a_{ik})$ is 2×2 and belongs to a matrix class in the sense of the theorem of Latimer and MacDuffee.

By a theorem of Frobenius, $\mathbf{A}$ can be expressed as $\mathbf{S}_1\mathbf{S}_2$, with $\mathbf{S}_i$ symmetric and rational. I had studied the problem to characterize the $\mathbf{A}$'s with both factors integral [**10**]. The factorization can be linked to a similarity, say $\mathbf{S}$, between $\mathbf{A}$ and its transpose $\mathbf{A}'$:

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \quad \text{or} \quad \mathbf{A} = \mathbf{S}\mathbf{A}'\mathbf{S}^{-1}$$

It is known, see, e.g., [**14**] that $\mathbf{S}$ can be chosen symmetric and rational, even integral. In this case also $\mathbf{A}'\mathbf{S}^{-1}$ turns out symmetric and rational. In 1973 I showed that both factors can be chosen integral if and only if the ideal class corresponding to the matrix class of $\mathbf{A}$ is of order 1, 2, 4, apart from a set of m's which will be discussed again in Part II.

Part I. I made an attempt to unify all the m's by expressing $\mathbf{A}$ as the product of two rational matrices $\mathbf{T}_1, \mathbf{T}_2$ with $\mathbf{T}_i = \mathbf{S}^{-1}\mathbf{S}_i\mathbf{S}$ so that $\mathbf{A} = \mathbf{S}^{-1}\mathbf{S}_1\mathbf{S} \cdot \mathbf{S}^{-1}\mathbf{S}_2\mathbf{S}$.

This was done in [**12**] paper in the following way: Instead of studying a single matrix class, all matrix classes corresponding to m are considered, and, in particular, the classes of order 1 or 2 or 4. While there may not be any of order 2 or 4, there is certainly one of order 1, namely the class of the companion matrix

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$$

whose symmetric factors can be taken as $\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

At this stage it is appropriate to use the following facts from [**7**]:

THEOREM 1. *A matrix* $\mathbf{A}$ *is obtained from a matrix in the principal class via an ideal matrix corresponding to the ideal which corresponds to the class of* $\mathbf{A}$.

THEOREM 2. *Let* $\mathbf{A}, \mathbf{B}$ *define two matrix classes. Then* $\mathbf{B} = \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}$ *where* $\mathbf{Y}$ *is of the form* $\mathbf{X}_1\mathbf{X}_2^{-1}$ *and* $\mathbf{X}_1, \mathbf{X}_2$ *ideal matrices, one corresponding to an ideal corresponding to the class of* $\mathbf{A}$, *the other one to the class of* $\mathbf{B}$.

For these theorems the concept of ideal matrix has to be introduced (see [**7**, **12**]). The definition is as follows: Let $\omega_1, \dots, \omega_n$ be a basis for the maximal order $\mathcal{O}$ of the field in question, in our case $\mathbf{Q}(\sqrt{m})$. Let $\alpha_1, \dots, \alpha_n$ be a basis for the ideal $\mathfrak{u}$. Then the ideal matrix $X_\mathfrak{u}$ of $\mathfrak{u}$ is given by

$$X_\mathfrak{u} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

With changes of bases for $\mathcal{O}$ and for $\mathfrak{u}$, $X_\mathfrak{u}$ is determined up to left or right unimodular factors.

Theorems 1, 2 are applied here to use $X_\mathfrak{u}$ to transform the companion matrix $\mathbf{C}$ into our given matrix A, while any matrix in an ideal class of order 2, or 4 (if not in the excluded class) leads to a factorization via a quotient of ideal matrices.

This is the similarity corresponding to the title when it is applied to the symmetric factors of $\mathbf{C}$ or the factors of the other cases.

Examples were computed to show that, under not yet fully understood circumstances, the two rational factors of $\mathbf{A}$ obtained can be integral, and to find other information on the nature of the factors.

Part II. I will now report on the additions which are at issue in this paper:

1. *The fact that the* 1, 2, 4 *cases remain somehow 'aloof'.* Although the main factorization applies to them, too, an example is given which shows that their special virtue, of leading to symmetric factors is not obtained by similarity via an ideal matrix.

I tried a simple example of a field of class number 2. I then knew that the matrix class belonging to the non-principal ideal would factorize into 2 symmetric integral factors. The example is the field $\mathbf{Q}(\sqrt{15})$, with the non-principal ideal with bases $(2, 3+\sqrt{15})$ and bases $(1, \sqrt{15})$ as basis of $\mathcal{O}$. The matrix class belonging to the ideal can be represented by $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 3 & 3 \end{pmatrix}$ and an ideal matrix $\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$. It turns out that

$$\mathbf{A} = \mathbf{S}_1\mathbf{S}_2 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 8 \end{pmatrix}.$$

This would be a stronger result than is needed for the problem of the title. However, it does not fit into this problem and I cannot obtain it by my method as I will explain.

The symmetric factor $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ is not similar to one of the factors of the companion matrix. First of all, it is not similar to $\begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix}$, but also not to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ because the traces do not coincide. A more detailed examination of $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 15 & 0 \end{pmatrix}$ shows: $\mathbf{C}$ cannot be split into two symmetric factors of determinants 3, 5 nor is there an essentially different factorization possible of determinants 1, 15 than the one mentioned. These statements follow from the fact that the transformation $\mathbf{S} = (s_{ik})$ of $\mathbf{C}$ into $\mathbf{C}'$ leads only to the equation $s_{22} = 15s_{11}$. Hence, for $\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & 15s_{11} \end{pmatrix}$, we have determinants either $15s_{11}^2 - s_{12}^2 = \pm 1$, or ± 3, or ± 5 from which, by simple diophantine arguments, the result announced is achieved. This example indicates that if there is a factorization into two integral symmetric factors, they cannot be obtained by my method of mapping via ideal matrices.

2. *The similarity between* $\mathbf{A}$ *and its transpose* $\mathbf{A}'$. At this point I want to make another remark concerning the factorization of an integral

matrix **A** arising from an algebraic number field with the maximal order $\mathcal{O}$ generated by a single element α with characteristic polynomial, say $f(x)$. As said before, $\mathbf{A} = \mathbf{S}_1\mathbf{S}_2$ with rational entries, where the entries of $\mathbf{S}_1$ can be assumed integral. Further, $\mathbf{S}_1$ can be obtained from the similarity of **S** between **A** and $\mathbf{A}'$. I have a new remark here: The matrix class of **A** corresponds to an ideal class in $\mathcal{O}$, but $\mathbf{A}'$ is another matrix for $f(x) = 0$. The ideal class to which it corresponds is discussed in [**6**] for the case that $\mathcal{O} = \mathbf{Z}[\alpha]$. Let me denote it as the class of the ideal $\mathfrak{u}'$. By Theorem 2 of [**12**] it follows that, for $\mathfrak{u}$ and $\mathfrak{u}'$ suitably chosen in their classes, we have: $\mathbf{S}_1$ *is given by the ideal matrix of* $\mathfrak{u}$ *divided by the inverse of the ideal matrix of* $\mathfrak{u}'$.

If the matrix **A** is in a class of order 2, then the class of **A** coincides with the class of $\mathbf{A}'$. As pointed out in [**12**], the similarity between **A** and $\mathbf{A}'$ can be obtained by a unimodular matrix. In this paper a case of cyclic class group was discussed. Since it contained an element of order 2 it could again be concluded that **A** and $\mathbf{A}'$ were in the same class without recourse to [**6**]. A similar conclusion could not be made for class of order 4, for the class of $\mathbf{A}^3$ is again of order 4.

3. *The exceptional values of* m. For this purpose, a quadratic form, usually denoted by $a(\lambda, \mu)$, is introduced. The matrix class of **A** and its corresponding ideal class are not the only concepts considered here. For Gauss the quadratic form $f(x,y) \equiv a_{21}x^2 + (a_{22} - a_{11})\, xy - a_{12}y^2$ would have been the starting point. It has the discriminant $4m$. This is also the discriminant of $x^2 - m$. The form $a(\lambda, \mu)$ is the negative square of $f(x,y)$, see [**8**]. Its class is of order 1 or 2. In the case under consideration it represents factors of the discriminant. If these are factors of m the factorization of A can be achieved, but it can happen that all discriminantal divisors represented by the form contain a factor 2. Then the factorization cannot take place. These are the exceptional numbers, an example of them was constructed by Estes and Kisilevsky for $m = 1139 = 17 \cdot 67$. Another example, namely $m = 1299 = 3 \cdot 433$, will be discussed below.

4. *Numerical examples*: ($m = 79$, class number 3; $m = 235$, class number 6). Details of integral factorizations are given in [**12**].

($m = 226$, class number 8). Cyclic class group, generating ideal

$(3, 14+\sqrt{226})$, corresponding (For clarity we repeat here, once for all, that the correspondence (in the sense of Latimer and MacDuffee) is given by the fact that $\begin{bmatrix} -14 & 3 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} 3 \\ 14+\sqrt{226} \end{bmatrix} = \sqrt{226}\begin{bmatrix} 3 \\ 14+\sqrt{226} \end{bmatrix}$.) matrix $\begin{pmatrix} -14 & 3 \\ 10 & 14 \end{pmatrix}$. Square of generator $(9, 8+\sqrt{226})$, corresponding matrix $\begin{pmatrix} -8 & 9 \\ 18 & 8 \end{pmatrix}$ factors into

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -8 & 9 \\ 9 & 4 \end{pmatrix}.$$

Fourth power of generator $(2, 1+\sqrt{226})$, corresponding matrix $\begin{pmatrix} -14 & 2 \\ 15 & 14 \end{pmatrix}$ factors into

$$\begin{pmatrix} 0 & 1 \\ 1 & 14 \end{pmatrix} \begin{pmatrix} 211 & -14 \\ -14 & 2 \end{pmatrix}.$$

Corresponding ideal matrices:

$$\begin{pmatrix} 3 & 0 \\ 14 & 1 \end{pmatrix}, \quad \begin{pmatrix} 9 & 0 \\ 8 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 14 & 1 \end{pmatrix}.$$

Similarity to factors of the companion matrix

$$\begin{pmatrix} 1 & 0 \\ -14\cdot 75 & 226 \end{pmatrix}, \quad \begin{pmatrix} -14 & 3 \\ -65 & 14 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -200 & 226 \end{pmatrix}, \quad \begin{pmatrix} -8 & 9 \\ -7 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -7\cdot 225 & 226 \end{pmatrix}, \quad \begin{pmatrix} -14 & 2 \\ -195/2 & 14 \end{pmatrix}.$$

(The fractional nature of only one of the factors here will occur in another example under similar circumstances.)

I will now use the factorization of the square of the generator, an element of order 4, and study the factorization of the matrix corresponding to the generator, by using Theorem 2 of [**7**]. (This case was not yet studied in [**12**].) Hence, instead of using the factors of the companion matrix, I use the pair of integral symmetric matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -8 & 9 \\ 9 & 4 \end{pmatrix}$$

and transform them by the matrix $\mathbf{Y} = \mathbf{X}_1\mathbf{X}_2^{-1}$, where $\mathbf{X}_1$ is the ideal matrix of the square of the generator and $\mathbf{X}_2$ that of the generator, i.e.,

$$\mathbf{Y} = \begin{pmatrix} 9 & 0 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 14 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}.$$

Now apply the similarity via $\mathbf{Y}$ to the matrix which corresponds to the square of the generator, obtaining the matrix corresponding to the generator. Next apply $\mathbf{Y}$ to the two factors of the latter obtaining 2 integral matrices:

$$\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} -14 & 3 \\ -9 & 10 \end{pmatrix}.$$

$(m = 1090 = 10 \cdot 109)$. Class group of order 12, noncyclic, one cycle of order 6, generator $(3, 32 + \sqrt{1090})$, the other one with generator $(2, 32 + \sqrt{1090})$ of order 2.

The corresponding matrices of these two ideals are

$$\begin{pmatrix} -32 & 3 \\ 22 & 32 \end{pmatrix}, \quad \begin{pmatrix} -32 & 2 \\ 33 & 32 \end{pmatrix}.$$

Their ideal matrices are

$$\begin{pmatrix} 3 & 0 \\ 32 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 32 & 1 \end{pmatrix}.$$

Transformations for the first one, via the companion matrix, are

$$\begin{pmatrix} 1 & 0 \\ -32 \cdot 363 & 1090 \end{pmatrix}, \quad \begin{pmatrix} -32 & 3 \\ -341 & 32 \end{pmatrix}.$$

Transformations for the second one, again via the companion matrix, are

$$\begin{pmatrix} 1 & 0 \\ 16(-1089) & 1090 \end{pmatrix}, \quad \begin{pmatrix} -32 & 2 \\ -1023/2 & 32 \end{pmatrix}.$$

Two exceptional cases: $(m = 1139 = 17 \cdot 67)$. This case was discovered by D. Estes and H. Kisilevsky, answering an inquiry by myself, see [**8**]. $\mathbf{Q}(\sqrt{m})$ has class group of order 4, generated by the ideal $(5, 33+\sqrt{1139})$

corresponding to the class of the matrix $\begin{pmatrix} -33 & 5 \\ 10 & 33 \end{pmatrix}$ and ideal matrix $\begin{pmatrix} 5 & 0 \\ 33 & 1 \end{pmatrix}$. Hence the matrix splits into the factors

$$\begin{pmatrix} 5 & 0 \\ 33(-1138) & 5 \cdot 1139 \end{pmatrix} \frac{1}{5}, \quad \begin{pmatrix} -5 \cdot 33 & 25 \\ -1085 & 5 \cdot 33 \end{pmatrix} \frac{1}{5}$$

both fractional. The quadratic form associated with this case is

$$a(\lambda, \mu) = \lambda^2 + 66\lambda\mu - 25\mu^2.$$

It represents $2, 34, -2 \cdot 67, 2 \cdot 17 \cdot 67$, but no odd divisor of the discriminant, positive or negative. It follows from Gauss' work on ambiguous forms (Disquisitiones Arithmeticae, Article 258) that only four divisors can be represented.

($m = 1299 = 3 \cdot 433$). This exceptional case is somehow different from the first one since it has a class group of order 8 with a cyclic subgroup of order 4. The generator of the whole class group is given by $(5, 33 + \sqrt{1299})$ with corresponding matrix $\begin{pmatrix} -33 & 5 \\ 42 & 33 \end{pmatrix}$ and ideal matrix equal to $\begin{pmatrix} 5 & 0 \\ 33 & 1 \end{pmatrix}$. It can then be shown that the matrix can be factorized into the fractional matrices

$$\begin{pmatrix} 5 & 0 \\ -33 \cdot 1298 & 5 \cdot 1299 \end{pmatrix} \frac{1}{5}, \quad \begin{pmatrix} -5 \cdot 33 & 25 \\ -1088 & 5 \cdot 33 \end{pmatrix} \frac{1}{5}.$$

I now proceed to the ideal which generates the subgroup of order $4 : (25, 32 + \sqrt{1299})$ and ideal matrix $\begin{pmatrix} 25 & 0 \\ 18 & 1 \end{pmatrix}$ with corresponding matrix $\begin{pmatrix} -18 & 25 \\ 39 & 18 \end{pmatrix}$.

The factorization is

$$\begin{pmatrix} 25 & 0 \\ 18(-1298) & 25 \cdot 1299 \end{pmatrix} \frac{1}{25}, \quad \begin{pmatrix} -18 \cdot 25 & 25^2 \\ -323 & 18 \cdot 25 \end{pmatrix} \frac{1}{25}.$$

The quadratic form $a(\lambda, \mu)$ attached to the matrix is

$$2\lambda^2 + 62\lambda\mu - 169\mu^2.$$

It represents -6 for $\lambda = 5$, $\mu = 2$.

Further,

$$-3 = s^2 - 1299t^2, \quad \text{for } s = 36, \quad t = 1$$

$$433 = u^2 - 1299v^2, \quad \text{for } u = 433, \quad v = 12$$

and $2 \cdot 433$ is represented by the form in consequence of the fact that -6, but neither 3, -3, nor $433, -433$ are represented by the form since they are norms of integers from $\mathbf{Q}(\sqrt{1299})$, but not of ideals.

This case is similar to the example found previously by Estes and Kisilevsky and another example by Estes, namely $579 = 3 \cdot 193$.

The following remark by D. Estes helps in the discussion of the exceptional cases.

Consider $f(x,y) = ax^2 + bxy + cy^2$ with discriminant $\Delta = (b^2 - 4ac)$. Then always $f(1,0) = a$. However, if a is a discriminantal divisor, since $b^2 = \Delta + 4ac$, it follows that $a|b$. Then $f(-b/a, 2) = -\Delta/a$ so that f also represents the (negative) residual factor.

5. *Remark.* It was pointed out to me by R. Guralnick and D. Estes that the companion matrix for polynomials of higher degrees can still be factorized into 2 symmetric integral factors. Since factorization into integral symmetric factors for higher degrees is studied in [**9**], the problems started in this paper could be extended to higher degrees.

D. Estes furthermore aided by explaining Gauss' work on quadratic forms.

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