ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

OPEN MORPHISMS OF REAL CLOSED SPACES

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Introduction. In [**8**] and [**9**] it is shown that the theory of semi-algebraic spaces developed by H. Delfs and M. Knebusch (see [**3**, **4**]) can be extended by the theory of real closed spaces much as Grothendieck's theory of schemes extends the classical theory of varieties. In this paper the systematic study of real closed spaces is continued by looking at open and generalizing morphisms. The reason for the interest in these morphisms is that in the theory of schemes openness implies that there is some regularity in the behavior of the fibres of a morphism (see [**6**]).

In §1, some basic properties of open, generalizing, universally open, universally generalizing morphisms are collected. In the theory of schemes, there are connections between algebraic properties of a morphism and openness. For example, a locally finitely presented flat morphism of schemes is universally open [**5**, I.7.3.10]. Inspired by this result, algebraic properties of generalizing morphisms are investigated in §2. A valuative characterization of universally generalizing morphisms is in §3. Finally, in §4, the fibres of affine finitely presented morphisms of real closed spaces are studied. The fibres of these morphisms are affine semi-algebraic spaces. So, from semi-algebraic geometry there are numerous numerical invariants of these spaces (for example: dimension, number of connected components, Betti numbers). It is easy to construct examples of morphisms of semi-algebraic spaces in which these invariants of the fibres change very drastically. Openness (in connection with other hypotheses) brings some measure of regularity into the behavior of the fibres as far as dimension and number of connected components are concerned. For example, if $f : X \to Y$ is affine, finitely presented and universally open, then $\dim f^{-1}(y') \geq \dim f^{-1}(y)$ if y is a specialization of y'.

1. Open morphisms. The basic notions of open and generalizing morphisms of real closed spaces are adapted from the theory of schemes:

DEFINITION 1. Let $f : X \to Y$ be a morphism of real closed spaces.

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(a) f is *generalizing in* $x \in X$ if, for all $y' \subset y = f(x)$, there is some $x' \subset x$ such that $f(x') = y'$ (cf. [**5**, I.3.9.2]).

(b) f is *open in* $x \in X$ if, for all neighborhoods $U \subset X$ of x, $f(U) \subset Y$ is a neighborhood of $y = f(x)$ (cf. [**5**, $\mathrm{O_I}$, 2.10.1]).

(c) f is *generalizing* (*open*) if f is generalizing (open) in all points $x \in X$.

(d) f is *universally generalizing* (*universally open*) if, for all base extensions

$$\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y_1 \end{array}$$

f' is generalizing (open) (cf. [**5**, I.3.9.2], [**5**, I.3.8.1]).

The most elementary properties of these notions are similar to properties of generalizing and open morphisms of schemes.

PROPOSITION 2. (cf. [**5**, I 3.9.1]) *Let $f : X \to Y$ be a morphism of real closed spaces, let $x \in X$, $y = f(x)$. The following statements are equivalent*:

(a) *f is generalizing in x.*

(b) *The local morphism $f_x : X_x \to Y_y$* [**9**; V 2.10, V 2.20] *is surjective.*

(c) *For any irreducible subspace $\overline{\{z\}} = Z \subset Y$ with $y \in Z$, there is an irreducible component $T = \overline{\{t\}}$ of $f^{-1}(Z)$ with $f(t) = z$, $x \in T$.*

PROOF. (a) $\Leftrightarrow$ (b) is immediate from Definition 1.

(a) $\Rightarrow$ (c). With $y \in \overline{\{z\}} = Z \subset Y$ we find some $t \in X$, $x \in \overline{\{t\}}$ such that $f(t) = z$. Let $\overline{\{u\}} \subset f^{-1}(Z)$ be an irreducible component with $t \in \overline{\{u\}}$. Then $f(u) = z$.

(c) $\Rightarrow$ (a). Given $y' \subset y$ find some irreducible component $T = \overline{\{t\}}$ of $f^{-1}(\overline{\{y'\}})$ such that $f(t) = y'$ and $x \in T$. □

For generalizing morphisms the following characterization is obtained:

PROPOSITION 3. *For $f : X \to Y$ the following statements are equivalent:*

(a) *f is generalizing.*

(b) *For all irreducible subspaces $Z = \overline{\{y\}} \subset Y$ and for all irreducible components $T = \overline{\{t\}}$ of $f^{-1}(Z)$, $f(T) = Z$.*

(c) *For all subspaces $i : Z \to Y$, the restriction $f' : f^{-1}(Z) \to Z$ of f is generalizing.*

(d) *For all subspaces $i : Z \to Y$ with $|Z| \leq 2$, the restriction $f' : f^{-1}(Z) \to Z$ of f is generalizing.*

PROOF. (a) $\Rightarrow$ (b). Let Z, T be as in the statement of (b). If $y \subsetneq f(t)$ then there is some $t' \subset t$ such that $f(t') = y$. But then $t' \in f^{-1}(Z)$, and $\overline{\{t'\}} \supsetneq T$, a contradiction.

(b) $\Rightarrow$ (c). Let $x \in f^{-1}(Z)$, $y = f(x)$, $y' \subset y$ with $y' \in Z$. Then set $Y' = \overline{\{y'\}} \subset Y$. Let $T \subset f^{-1}(Y')$ be an irreducible component containing x, say $T = \overline{\{t\}}$. Then $t \subset x$ and $f(t) = y'$.

(c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (a). Let $x \in X$, $y = f(x)$, $y' \subset y$. Then let Z be the subspace of Y consisting of the two points $\{y', y\}$ [**9**, V 2.11]. By (d), there is some $x' \in f^{-1}(Z)$ with $x' \subset x$, $f(x') = y'$. □

Between open and generalizing morphisms we have the same connections as in the theory of schemes.

PROPOSITION 4. (cf. [**5**, I.7.3.10]) *Let $f : X \to Y$ be a morphism of real closed spaces, let $x \in X$, $y = f(x)$. Consider the following statements:*

(a) *f is open in x.*

(b) *f is generalizing in x.*

Then (a) $\Rightarrow$ (b) *holds. If f is finitely presented in x, then the converse is also true.*

PROOF. Suppose f is open in x. Pick $y' \subset y = f(x)$. Fix some open affine neighborhood U of x. For any open constructible neighborhood

$U' \subset U$ of x, $y' \in f(U')$. $f^{-1}(y') \cap U$ is a pro-constructible subset and is compact in the constructible topology. Since $U' \cap f^{-1}(y') \neq \emptyset$ for all U' as above, there is some $x' \in f^{-1}(y') \cap \bigcap_{U'} U'$. Since

$$X_x = \bigcap_{U'} U',$$

we see that $x' \subset x$ and $f(x') = y'$. Assume that f is finitely presented in x and that (b) holds. Let $U \subset X$, $V \subset Y$ be open affine neighborhoods of x and y such that $f(U) \subset V$ and that there is a finite presentation

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\cong"'] & V \\
Z \arrow[r, hook] & V \times_{R_0} \tilde{R}_0^n . \arrow[u]
\end{tikzcd}$$

Let $U' \subset U$ be an open constructible neighborhood of x. Then $f(U') \subset V$ is constructible [**9**, V 6.5]. Moreover, since f is generalizing in x, $V_y = Y_y \subset f(U')$. But then $f(U')$ contains some neighborhood of y. □

Without proof we record

PROPOSITION 5. (cf. [**5**, I.3.8.2]).

(a) *If $U \subset X$ is an open subspace and $i : U \to X$ is the inclusion, then i is universally open.*

(b) *For $x \in X$, the inclusion $i : X_x \to X$ is universally generalizing.*

(c) *If $f : X \to Y$, $g : Y \to Z$ are generalizing (open, universally generalizing, universally open) then so is gf.*

(d) *If $f : X \to Y$ is universally generalizing (universally open), then so is $f' : X' \to Y'$ for any base extension*

$$\begin{tikzcd}
X' = X \times_y Y' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

(e) *If* $f : X \to Y$, $f' : X' \to Y'$ *are universally generalizing (universally open) morphisms over a real closed space* Z, *then* $f \times f' : X \times_Z X' \to Y \times_Z Y'$ *is universally generalizing (universally open).*

(f) *Let* $f : X \to Y$, $g : Y \to Z$ *be such that* gf *is generalizing (open, universally generalizing, universally open). If* f *is surjective, then* g *is generalizing (open, ...). If* g *is a monomorphism, then* f *is generalizing (open, ...).*

Both the properties of being generalizing and of being universally generalizing can be recognized locally. Without proof we record

PROPOSITION 6. *For* $f : X \to Y$ *the following statements are equivalent:*

(a) f *is generalizing.*

(b) *For all* $x \in X$, $f_X : X_x \to Y_{f(x)}$ *is generalizing.*

(c) *For all open affine subspaces* $U \subset X$, $V \subset Y$ *such that* $f(U) \subset V$, $f : U \to V$ *is generalizing.*

PROPOSITION 7. *For* $f : X \to Y$ *the following conditions are equivalent:*

(a) f *is universally generalizing.*

(b) *For all* $x \in X$, $f_x : X_x \to Y_{f(x)}$ *is universally generalizing.*

(c) *For all open affine* $U \subset X$, $V \subset Y$ *with* $f(U) \subset V$, $f : U \to V$ *is universally generalizing.*

PROOF. (a) $\Rightarrow$ (b). $X_x \xrightarrow{i_x} X \xrightarrow{f} Y$ is universally generalizing. fi_x factors in the following way:

$$\begin{tikzcd}
X_x \arrow[rr, "fi_x"] \arrow[dr, "f_x"'] & & Y \\
& Y_{f(x)} \arrow[ur, "i_{f(x)}"'] &
\end{tikzcd}$$

Let $g : Z \to Y_{f(x)}$ be any morphism. In the diagram

$$\begin{tikzcd}
Z \times_y X_x \arrow[rr, "(fi_x)'"'] \arrow[dr, "f_x{}'"] & & Z \\
& Z \times_y Y_{f(x)} \arrow[ur, "i_{f(x)}{}'"'] \arrow[d] & \\
& Y_{f(x)} \arrow[dr, "i_{f(x)}"] & \\
X_x \arrow[uuu] \arrow[ur, "f_x"] \arrow[rr, "fi_x"'] & & Y \arrow[uuu, "gi_{f(x)}"']
\end{tikzcd}$$

$i_{f(x)}{}'$ is a monomorphism and $(fi_x)'$ is universally generalizing. So is $f_x{}'$.

(a) $\Rightarrow$ (c). This is proved in the same way.

(b) $\Rightarrow$ (a). Consider a pull-back diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

Pick $x' \in X'$ and set $y' = f'(x')$, pick $y_0' \subset y'$. Setting $g'(x') = x$, $g(y') = y$, $g(y_0') = y_0$, we consider the following diagram:

$$\begin{tikzcd}
X'_{x'} \arrow[drr, bend left, "f'_{x'}"] \arrow[dr, "h"] \arrow[ddr, "g'_{x'}"'] & & \\
& X_x \times_{y_y} Y'_{y'} \arrow[r, "f_{x'}"] \arrow[d] & Y'_{y'} \arrow[d, "g_{y'}"] \\
& X_x \arrow[r, "f_x"] & Y_y
\end{tikzcd}$$

$X_x \times_{Y_y} X'_{y'}$ and $X'_{x'}$ are both subspaces of X' which are closed under generalization. $h : X'_{x'} \to X_x \times_{Y_y} X'_{y'}$ is the inclusion. Thus, h is generalizing. So is f'_x, by hypothesis. There is some $x'_0 \subset x'$ such that $f(x'_0) = f'_{x'}(x'_0) = y'_0$.

(c) $\Rightarrow$ (a). For $x \in X$, $y = f(x) \in Y$ let $U \subset X$, $V \subset Y$ be open affine neighborhoods of x and y such that $f(U) \subset V$. If $y' \subset y$, (c) shows that there is some $x' \subset x$ with $f(x') = y'$. □

2. Algebraic properties of generalizing morphisms. In the theory of schemes it is known that there are close connections between flatness of a morphism and the property of being universally generalizing (e.g., [**5**, I.3.9.4]). Because of these connections we will take a look at algebraic properties of generalizing morphisms.

THEOREM 8. *For a morphism $f : X \to Y$ of affine real closed spaces the following conditions are equivalent:*

(a) *f is generalizing.*

(b) *For all closed subspaces $C \subset Y$ and affine generalizing subspaces $C' \subset C$, if $f' : f^{-1}(C') \to C'$ is the restriction of f and $g : B \to A$ is the homomorphism of the rings of global sections belonging to f', then g maps non-zero divisors to non-zero divisors, or $f^{-1}(C') = \emptyset$.*

(c) *For all $x \in X$ and all closed irreducible subspaces $Z \subset Y_{f(x)'}$ if $f' : f_x^{-1}(Z) \to Z$ is the restriction of f and $g : B \to A$ is the homomorphism of the rings of global sections corresponding to f', then*

g maps non-zero divisors to non-zero divisors.

(c′) *With the same notation as in* (c), *let* $(x) \subset B$ *be a principal ideal and consider* A *as a* B*-module via* g. *Then the canonical homomorphism*

$$(x) \otimes_B A \to A$$

is injective.

PROOF. (a) $\Rightarrow$ (b). Let $b \in B$ be a non-zero divisor and assume that $g(b)$ is a zero-divisor. Since A is reduced, the set of zero divisors of A is the union of all minimal prime ideals of A. Thus, $g(b) \in x$ for some minimal element $x \in f^{-1}(C')$. Since b is not a zero-divisor, $f'(x)$ is not a minimal element of C'. Let $y' \subset f'(x)$ be a minimal element of C' (this exists by definition of C'). Since f is generalizing, there is some $x' \in X$, $x' \subset x$ such that $f(x') = y'$. But then $x' \in f^{-1}(C')$, $x' \subsetneq x$. However, x was chosen to be minimal in $f^{-1}(C')$.

(b) $\Rightarrow$ (c). Let $Z \subset Y_{f(x)}$ be closed irreducible. Then $Z = \overline{\{z\}} \cap Y_{f(x)}$ with $z \in Y_{f(x)}$. Setting $C = \overline{\{z\}}$, $C' = Z$ we are in the situation of (b). Thus, if A' is the ring of global sections of $f^{-1}(C')$ and $g' : B \to A'$ is the homomorphism corresponding to $f^{-1}(C') \to Z$, then g' maps non-zero-divisors to non-zero-divisors. Since the inclusion $f^{-1}(C') \cap X_x \subset f^{-1}(C')$ is generalizing, the corresponding homomorphism $h : A' \to A$ maps non-zero-divisors to non-zero-divisors. Since $g = hg'$, the claim follows.

(c) $\Rightarrow$ (c'). Let $\varphi : (b) \otimes_B A \to A$ be the canonical homomorphism. If $b = 0$, then the claim is clear. If $b \neq 0$, then b is a non-zero-divisor, since B is an integral domain. Now let

$$g(b)a = \varphi(b \otimes a) = 0.$$

Since $g(b)$ is not a zero-divisor (by (c)), it follows that $a = 0$. Hence, $b \otimes a = 0$.

(c′) $\Rightarrow$ (a). Pick $x \in X$, set $y = f(x)$ and pick $y' \subset y$. Set $Z = \overline{\{y'\}} \subset Y_y$. Let $f' : f_x^{-1}(Z) \to Z$ be the restriction of f. Let $g : B \to A$ be the homomorphism of the rings of global sections. It suffices to show that there is some minimal prime ideal $x' \subset A$ with $f(x') = y'$. Assume that this is not true. Then there is some $0 \neq b \in B$

with $g(b) \in x'$ for some minimal prime ideal $x' \subset A$. Thus, $g(b)$ is a zero-divisor, i.e., there is some $a \in A$ with $0 \neq a$, $g(b)a = 0$. Since $\varphi : (b) \otimes_B A \to A$ is injective, $b \otimes a = 0$ in $(b) \otimes_B A$. But since B is an integral domain, $B \to (b) : c \to cb$ is an isomorphism. Then

$$A \cong B \otimes_B A \to (b) \otimes_B A$$

is an isomorphism as well, and we see that $a = 0$ which is a contradiction. □

If both X and Y are affine real closed schemes, then the result of Theorem 8 can be improved.

PROPOSITION 9. *Let $f : X \to Y$ be a morphism of affine real closed schemes. Then the following conditions are equivalent:*

(a) *f is generalizing.*

(b) *For all closed subspaces $C \subset Y$, if $f' : f^{-1}(C) \to C$ is the restriction of f and $g : B \to A$ is the corresponding homomorphism of the rings of global sections, then either $f^{-1}(C) = \emptyset$ or g maps non-zero-divisors to non-zero-divisors.*

(c) *With the notation of* (b), *if $(b) \subset B$ is a principal ideal, then the canonical homomorphism*

$$\varphi : (b) \otimes_B A \to A$$

is injective.

PROOF. (a) $\Rightarrow$ (b) is a special case of Theorem 8.

(b) $\Rightarrow$ (a). Let $x \in X$, $y = f(x)$, $y' \subset y$. If it is possible to find $x'' \subset x$ with $f(x'') = y'' \subset y'$ minimal, then set $C = \overline{\{y''\}}$ and have a monomorphism $B/y'' \to A/x''$ of totally ordered integral domains. The convex hull of y'/y'' in A/x'' is a prime ideal x'/x'', where $x' \subset A$ is a prime ideal with $x'' \subset x' \subset X$. Clearly, $f(x') = y'$. Assume that y' is minimal in Y and $y' \notin f(X_x)$. Set $C = \overline{\{y'\}}$. For every $x' \in X_x \cap f^{-1}(C)$ there is some $0 \neq b(x') \in B$ with $g(b(x')) \in x'$. Set

$$N(x') = \{\alpha \in f^{-1}(C) \,|\, g(b(x'))(\alpha) = 0\}.$$

The $N(x')$ cover the pro-constructible set $X_x \cap f^{-1}(C)$. So there is a finite subcover

$$X_x \cap f^{-1}(C) \subset \cup_{i=1}^{r} N(x_i').$$

Set $b = b(x_1') \cdot \ldots \cdot b(x_r')$. Clearly, $0 \neq b \in B$ and $g(b)(\alpha) = 0$ for all $\alpha \in X_x \cap f^{-1}(C)$. In particular, $g(b)(\alpha) = 0$ for some minimal $\alpha \in f^{-1}(C)$. But then there is an open constructible neighborhood $U \subset f^{-1}(C)$ of α with $g(b) \mid U = 0$. If $U = \{\alpha \in f^{-1}(C) \mid a(\alpha) > 0\}$ for some $0 \leq a \in A$, then $g(b)a = 0$. However, B is an integral domain, hence b is a non-zero-divisor, but $g(b)$ is a zero-divisor which is a contradiction.

(b) $\Rightarrow$ (c). If $b = 0$ everything is trivial. Assume that $b \neq 0$ and that there is some $0 \neq a \in A$ with $g(b)a = \varphi(b \otimes a) = 0$. Set

$$N = \{\alpha \in C \mid b(\alpha) = 0\}$$
$$U = \{\alpha \in f^{-1}(C) \mid a(\alpha) \neq 0\}.$$

Then $f(U) \subset N$. By (a), $f(U)$ is closed under generalization. By quasi-compactness of $f(U)$ (U is constructible) and by constructibility of N, there is some open constructible neighborhood $V \subset N$ of $f(U)$. By [**9**, II 4 .14],

$$V = \{\alpha \in C \mid v(\alpha) > 0\}$$

for some $0 \leq v \in B$. But then $b \cdot v = 0$ and $a \in \sqrt{g(v)A}$. Thus, there is some $c \in B$, some $d \in A$, some $m \in \mathbf{N}$ such that $v = c^m$, $a = g(c)d$. Then, we also have $bc = 0$. It follows that

$$b \otimes a = b \otimes g(c)d = bc \otimes d = 0.$$

(c) $\Rightarrow$ (b). Let $b \in B$ be a non-zero-divisor, $g(b) \in A$ a zero-divisor and $g(b)a = 0$ for $0 \neq a \in A$. From injectivity of

$$(b) \otimes_B A \to A$$

we see that $b \otimes a = 0$. Since

$$B \to (b) : c \to bc$$

is an isomorphism,

$$A \cong B \otimes_B A \to (b) \otimes_B A$$

is an isomorphism as well, and $a = 0$, another contradiction. □

As a consequence of Theorem 8 we can prove the following connection with flatness:

COROLLARY 10. *Let $f : X \to Y$ be a flat morphism of real closed spaces. Suppose that X is a scheme. Then f is generalizing.*

PROOF. Let $x \in X$, $y = f(x)$ and consider the local morphism $f_x : X_x \to Y_y$. Let $y' \in Y_y$. Set $Z = \overline{\{y'\}} \subset Y_y$, $T = f_x^{-1}(Z)$. Let A, B be the rings of global sections of X_x, Y_y and $\overline{A}, \overline{B}$ the rings of global sections of T, Z. Then we have the local homomorphisms

$$g : B \to A, \quad \overline{g} : \overline{B} \to \overline{A}$$

corresponding to f_x and the restriction $\overline{f_x} : T \to Z$ of f_x. By hypothesis, g is flat. $\overline{g}$ is obtained from g by base extension, and is flat as well. But then $\overline{g}$ maps non-zero-divisors to non-zero-divisors, and the claim is proved by Theorem 8. □

This result is no longer true if we drop the hypothesis that X is a scheme. For example, let X be an affine real closed space which is *not* a scheme. Let $i : X \to \mathrm{Aff}(X)$ be the natural morphism into the associated affine real closed scheme [**9**, V 2.24]. i is clearly flat. But i is not generalizing since the convex hull of $i(X)$ in $\mathrm{Aff}(X)$ is all of $\mathrm{Aff}(X)$ [**9**, II 4.7], but $i(X) \neq \mathrm{Aff}(X)$.

3. A valuative criterion. In the theory of real closed spaces there is valuative criteria for separatedness and universal closedness as in the theory of schemes ([**9**, V 3.19], [**5**, I.5.5.4] and [**9**, V.5.4], [**5**, I.5.5.8]). We will see that there is a valuative criterion for a morphism to be universally generalizing. This is reminiscent of the valuative criterion for flatness of a morphism of schemes in [**6**, IV 11.8.1].

THEOREM 11. *For a morphism $f : X \to Y$ of affine real closed spaces the following conditions are equivalent:*

(a) *f is universally generalizing.*

(b) *For all valuative real closed spaces* Y' (*cf.* [**9**, V 2.11]) *and all morphisms* $g : Y' \to Y$, *the projection* $f' : X' \times_Y Y' \to Y'$ *is generalizing.*

(c) *For all morphisms* $g : Y' \to Y$ *with* Y' *valuative such that, if* $y'_0 \in Y'$ *is the generic point,* $\rho(g(y'_0)) \to \rho(y_0)$ *is an isomorphism, the projection* $f' : X' = X \times_y Y' \to Y'$ *is generalizing.*

PROOF. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold trivially. We will prove (c) $\Rightarrow$ (a). Consider any pullback diagram

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of real closed spaces. If f' is not generalizing then there is some $x'_1 \in X'$ and some $y'_0 \subset y'_1 = f'(x'_1)$ such that $y'_0 \notin f'_{x'_1}(X'_{x'_1})$. Replacing Y' by an open affine neighborhood of y'_1 we may assume that Y' is affine as well. Let $C \subset \rho(y'_0)$ be the largest convex subring such that the image of the natural homomorphism $0_{Y',y'_1} \to \rho(y'_0)$ is dominated by C. Let Y'' be the valuative real closed space associated with C. Let $h : Y'' \to Y'$ be induced by $0_{Y',y'_1} \to C$ ([**9**, V 2.21]). Consider

$$\begin{tikzcd} X'' = X \times_Y Y' \arrow[r, "f''"] \arrow[d, "h'g'"] & Y'' \arrow[d, "hg"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $y''_0 \subset y''_1$ be the points of Y''. Let $x''_1 \in X''$ be such that $f''(x''_1) = y''_1$, $h'(x''_1) = x'_1$. If there is some $x''_0 \in X''_{x''_1}$ such that $f''(x''_0) = y''_0$ then we find $x'_0 = h'(x''_0) \in X'_{x'_1}$ and $f'(x'_0) = y'_0$, a contradiction. Thus we may assume that Y' itself is valuative. g induces $\rho(g(y'_0)) \to \rho(y'_0)$. Let $C \subset \rho(y'_0)$ be the ring of global sections of Y', let $B = C \cap \rho(g(y'_0))$ and let Y'' be the valuative real closed space associated with B. Then g factors as $Y' \xrightarrow{g_1} Y'' \xrightarrow{g_2} Y$. In the diagram

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "f'"] \arrow[d, "g_1'"] & Y' \arrow[d, "g_1"] \\ X'' = X \times_Y Y'' \arrow[r, "f''"] \arrow[d, "g_2'"] & Y'' \arrow[d, "g_2"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

every square is cartesian. Set $x_1'' = g_1'(x_1')$, $y_0'' = g_1(y_0')$ and $y_1'' = g_1(y_1')$. By (c), f'' is generalizing, i.e., we find some $x_0'' \in X''_{x_1''}$ with $f''(x_0'') = y_0''$. We have a commutative diagram

$$\begin{tikzcd} \rho(x_1'') & \rho(y_1'') \arrow[l] \\ \rho(x_1) \arrow[u] & \rho(y_1) \arrow[l] \arrow[u] \end{tikzcd}$$

of real closed residue fields. Let $D \subset \rho(x_0'')$ be the largest convex subring such that $0_{x'', x_1''} \to \rho(x_0'')$ factors through D via a local homomorphism. Let R be the residue field of D. By choosing a large enough η_α-field E (cf. [**7**, Chapter IV]) we obtain a diagram

$$\begin{tikzcd} E & \rho(x_1'') \arrow[l] \\ R \arrow[u] & \rho(x_1) \arrow[l] \arrow[u] \end{tikzcd}$$

Let $\Gamma_B, \Gamma_C, \Gamma_D$ be the value groups of B, C, D. There is some totally ordered group Γ such that we have a commutative diagram

$$\begin{tikzcd} \Gamma_B \arrow[r] \arrow[d] & \Gamma_D \arrow[d] \\ \Gamma_C \arrow[r] & \Gamma. \end{tikzcd}$$

Then we have a diagram

$$\begin{tikzcd} \rho(y_1'')((\Gamma_B)) \arrow[r] \arrow[d] & R((\Gamma_0)) \arrow[d] \\ \rho(y_1')((\Gamma_C)) \arrow[r] & E((\Gamma)) \end{tikzcd}$$

of formal power fields. Valuation preserving embeddings

$$\begin{aligned} &\rho(y_0'') \to \rho(x_1'')((\Gamma_B)) \\ &\rho(y_0') \to \rho(x_1')((\Gamma_C)) \\ &\rho(x_0') \to R((\Gamma_D)) \end{aligned}$$

can be chosen such that

$$\begin{tikzcd} \rho(y_0'') \arrow[rr] \arrow[dr] \arrow[dd] & & \rho(x_0') \arrow[d] \\ & \rho(y_1'')((\Gamma_B)) \arrow[r] \arrow[d] & R((\Gamma_D)) \arrow[d] \\ \rho(y_0') \arrow[r] & \rho(y_1')((\Gamma_C)) \arrow[r] & E((\Gamma)) \end{tikzcd}$$

commutes [**7**, Chapter II §5]. If A is the natural valuation ring of $E((\Gamma))$ and Z is the corresponding valuative real closed space then we get

$$\begin{tikzcd} Z \arrow[dr, "i"] \arrow[drr, "h"] \arrow[ddr] & & \\ & X' \arrow[r, "f'"] \arrow[d, "g_1'"] & Y' \arrow[d, "g_1"] \\ & X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$

such that $h(z_0) = y_0'$, $h(z_1) = y_1'$ (z_0 is the generic point of Z, z_1 the closed point). But then $i(z_0) \subset i(z_1) = x_1'$ and $f'i(z_0) = y_0'$ which is a contradiction. □

COROLLARY 12. *If Y is a valuative real closed space or a real closed space with one point then any generalizing morphism $f : X \to Y$ is universally generalizing.*

Up to this point it is not clear if there is any difference between the notions of generalizing and universally generalizing morphisms. However, by comparing Proposition 3 and Theorem 11 a very clear picture of the difference between these notions emerges. $f : X \to Y$ is generalizing if the projection $f' : X \times_Y Y' \to Y'$ is generalizing for all base extensions by subspaces $Y' \subset Y$ with $|Y'| \leq 2$. f is universally generalizing if the projection f is generalizing for all base extensions by valuative real closed spaces.

We will now see that a morphism may be generalizing without being universally generalizing.

EXAMPLE 13. Let R be a real closed field with rank 2 natural valuation. Thus, the valuation ring C has the prime ideals $(0) \subset P \subset M$. Let R_1 be the residue field. Identify R_1 with a field of representatives. Then set $B = R_1 + P$. Let R_2 be the residue field of the valuation ring $B' = C_P$. Identify R_2 with a field of representatives such that $R_1 \subset R_2$. Let $0 < r \in R_2$ be infinitesimal with respect to R_1. Let $Y = \mathrm{Sper} B$ and Y' the valuative real closed space associated with B'. Let $g : Y' \to Y$ be the natural morphism. Now define X to be the following constructible subspace of $Y \times_{R_0} \widetilde{R_0}$: If $y_0 \subset y_1$ are the points of Y then X is the union of the constructible subsets of $p^{-1}(y_0) = \widetilde{R}$, $p^{-1}(y_1) = \widetilde{R_1}$ ($p : Y \times_{R_0} \widetilde{R_0} \to Y$ projection) corresponding to the semi-algebraic spaces

$$\{r\} \subset R, \quad \{0\} \subset R_1.$$

Then $f : X \to Y \times_{R_0} \widetilde{R_0} \xrightarrow{p} Y$ is generalizing. However $f' : X' = X \times_Y Y' \to Y'$ is not generalizing. Because X' is the union of the constructible subsets of $p'^{-1}(y_0') = \widetilde{R}$, $p'^{-1}(y_1') = \widetilde{R_2}$ corresponding to the semi-algebraic spaces

$$\{r\} \subset R, \quad \{0\} \subset R_2.$$

The diagram

$$\begin{array}{ccc} B'[X] & \longrightarrow & B' \subset R \\ \Big\downarrow & \begin{array}{c} X \longrightarrow r \\ X \longrightarrow 0 \end{array} & \Big\downarrow \\ R_2[X] & \longrightarrow & R_2 \end{array}$$

does not commute.

It is shown in [**10**] that for morphisms of semi-algebraic spaces the properties of being generalizing and of being universally generalizing agree. Since the morphism $f : X \to Y$ is finitely presented, we see that this equivalence is a property which does not carry over to finitely presented morphisms of real closed spaces.

4. Fibres of open morphisms.

If $f : X \to Y$ is a finitely presented morphism of real closed spaces then the fibres of f are semi-algebraic spaces. Two important numerical invariants of such spaces are their dimensions and their numbers of connected components. In this section the question to be studied is to what extent the fibres of an open morphism behave regularly. More specifically, we investigate the functions

$$c : Y \to \mathbf{Z} : y \to \sharp\{\text{ connected components of } f^{-1}(y)\},$$
$$d : Y \to \mathbf{Z} : y \to \dim f^{-1}(y)$$

for f finitely presented and open. (We set $\dim \emptyset = -1$.)

First we record a few results about constructibility of certain sets. The basic tool for this is the real version of the Theorem of Chevalley ([**9**, V 6.5], [**2**, Proposition 2.3]) along with the description of constructible sets by logical formulae ([**2**]). For the sake of simplicity we consider only affine spaces.

PROPOSITION 14. (cf. [**6**, IV 9.6.1]) *Let $f : X \to Z$, $g : Y \to Z$ be finitely presented morphisms of affine real closed spaces, and $h : X \to Y$*

be a morphism over Z. Let E be the set of those $z \in Z$ for which $h_z : f^{-1}(z) \to g^{-1}(z)$ has one of the following properties:

(a) *h_z is dominant,*

(b) *$h_z(f^{-1}(z)) \subset g^{-1}(z)$ has nonempty interior,*

(c) *h_z is open,*

(d) *For $T \subset Y$ constructible, $h_z(f^{-1}(z)) \subset T \cap g^{-1}(z)$,*

(e) *h_z is injective ($=$ a monomorphism - see* [**3**]*),*

(f) *$h_z = i_z$ if $i : X \to Y$ is another morphism over Z,*

(g) *h_z is an isomorphism,*

(h) *h_z is surjective,*

(i) *h_z has finite fibres,*

(j) *$|h_z^{-1}(y)| \leq n$ for all $y \in g^{-1}(z)$.*

Then E is constructible.

PROOF.

(a). Let

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow \simeq & & \uparrow \\ T & \longrightarrow & Z \times_{R_0} \widetilde{R_0^n} \end{array}$$

be a finite presentation of g. We identify $Y = T$. $h(X) \subset Y$ is constructible. Let Φ, Ψ be logical formulae defining $h(X), Y$ as subsets of $Z \times_{R_0} \widetilde{R_0^n}$. Let Ω be the formula

$$\forall x_1, \ldots, x_n \; \exists \epsilon > 0 (\Psi(x_1, \ldots, x_n) \to \exists y_1, \ldots, y_n$$
$$(\Psi(y_1, \ldots, y_n) \& \sum_{i=1}^{n} |x_i - y_i|^2 < \epsilon)).$$

If Ω_z is the specialization of Ω over $\rho(z)$ then $\rho(z) \models \Omega_z$ is equivalent to

$$h_z(f^{-1}(z)) \subset g^{-1}(z)$$

is dense. Thus,

$$E = \{z \in Z \mid \rho(z) \models \Omega_z$$

is constructible ([**2**]).

(b). This clearly follows from (a).

(c). Finite presentations of g and h yield

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
Y \times_{R_0} \widetilde{R_0^n} \arrow[r] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[d, "g"] \\
Z \times_{R_0} \widetilde{R_0^{m+n}} \arrow[r] & Z \times_{R_0} \widetilde{R_0^n} \arrow[r] & Z
\end{tikzcd}$$

Let Φ be a logical formula defining X as a constructible subset of $Z \times_{R_0} \widetilde{R_0^{m+n}}$.

Let Ψ be a logical formula defining Y as a constructible subset of $Z \times_{R_0} \widetilde{R_0^m}$. Then consider the formula Ω:

$$\begin{aligned}
&\forall x_1, \dots, x_m \, \forall y_1, \dots, y_n \\
&(\Phi(x, y) \to \forall \epsilon > 0 \exists \delta > 0 \forall z_1, \dots, z_n \\
&(\Psi(z) \& \sum_{i=1}^{n} (y_i - z_i)^2 < \delta \to \exists t_1, \dots, t_m \\
&(\Phi(t, z) \& \sum_{i=1}^{m} (x_i - t_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 < \epsilon))).
\end{aligned}$$

Then

$$E = \{z \in Z \mid \rho(z) \models \Omega_z\}.$$

(d). $E = \{z \in Z \mid h(X) \cap g^{-1}(z) \subset T \cap g^{-1}(z)\} = Z \backslash g(h(X) \backslash T)$ is constructible by the Theorem of Chevalley.

(e). Use the same notation as in the proof of (c) up to the definition of the formula Ω. Now, let Ω be the formula

$$\exists x_1, \dots, x_m \, \exists y_1, \dots, y_n \, \exists z_1, \dots, z_n :$$

$$(y_1 \neq z_1 \vee \cdots \vee y_n \neq z_n) \;\&\; \Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$$
$$\&\; \Phi(x_1, \ldots, x_m, z_1, \ldots, z_m).$$

Then Ω_z holds precisely if h_z is not injective. Thus,

$$E = Z \backslash \{z \in Z | \rho(z) \models \Omega_z\}$$

is constructible.

(f). From finite presentations of g, h, i we obtain the diagram

$$\begin{tikzcd}
X \arrow[rrr, "="] \arrow[dd, "="'] \arrow[dr, "j"] & & & X \arrow[dr, hook, "i"] \arrow[dd] & \\
& Z \times_{R_0} \tilde{R}_0^{\,l+m+2n} \arrow[rrr] \arrow[dd] & & & Z \times_{R_0} \tilde{R}_0^{\,m+n} \arrow[d] \\
X \arrow[rrr, "h"'] \arrow[dr, hook] & & & Y \arrow[r, hook] \arrow[d, hook] & Z \times_{R_0} \tilde{R}_0^{\,n} \arrow[d] \\
& Z \times_{R_0} \tilde{R}_0^{\,l+m} \arrow[rr] & & Z \times_{R_0} \tilde{R}_0^{\,n} \arrow[r] & Z.
\end{tikzcd}$$

Let Ψ be a formula defining X as a subset of $Z \times_{R_0} \tilde{R}_0^{\,l+m+2n}$, let Ω be the formula

$$\exists x_1, \ldots, x_1 \; \exists y_1, \ldots, y_m \; \exists z_1, \ldots, z_n \; \exists t_1, \ldots, t_n :$$
$$(z_1 \neq t_1 \vee \cdots \vee z_n \neq t_n) \;\&\; \Psi(x, y, z, t).$$

Then Ω_z holds over $\rho(z)$ exactly if $h_z \neq i_z$. Thus,

$$E = Z \backslash \{z \in Z \,|\, \rho(z) \models \Omega_z\}$$

is constructible.

(g). h_z is an isomorphism if and only if h_z is open, injective and surjective. Therefore, it suffices to prove (h) (in view of (c), (e)).

(h) and (i) follow from:

THEOREM 15. *If $f : X \to Y$ is a finitely presented morphism of affine spaces then, for all $n \in \mathbf{Z}$,*

$$D_n = \{y \in Y \mid \dim f^{-1}(y) = n\}$$

is constructible.

PROOF OF (h). Set $T = Y \backslash h(X)$. Then $T \subset Y$ is constructible, i.e., $i : T \to Y \xrightarrow{g} Z$ is finitely presented. By Theorem 15,

$$E = g(Y) \cap \{z \in Z \mid \dim\, i^{-1}(z) = -1\}$$

is constructible. □

PROOF OF (i). In a first step,

$$D = \{y \in Y \mid \dim\, h^{-1}(y) = -1 \text{ or } 0\}$$

is a constructible subset of Y. Thus $Y \backslash D$ is constructible as well and

$$E = Z \backslash g(Y) \cup g(Y) \backslash g(Y \backslash D)$$

is constructible. □

PROOF OF THEOREM 15. Let B be the ring of global sections of Y. Then consider a finite presentation of f:

$$\begin{tikzcd}
 & Y \arrow[r] & \operatorname{Sper} B \\
X \arrow[ur, "f"] \arrow[dr, hook] & & \\
 & Y x_{R_0} \widetilde{R_0^n} \arrow[uu, "p"'] \arrow[r] & \operatorname{Sper} B[X_1, \ldots, X_n]. \arrow[uu, "\pi"']
\end{tikzcd}$$

Pick any $y \in Y$. Then $f^{-1}(y) \subset \widetilde{\rho(y)}^n$ is a semi-algebraic subspace. $\rho(y)$ is the quotient field of B/y. Thus, there are polynomials $P_1, \ldots, P_s \in B[X_1, \ldots, X_n]$ such that their images $\overline{P}_1, \ldots, \overline{P}_s$

in $\rho(y)[X_1,\dots,X_n]$ generate the ideal of all polynomials vanishing on $f^{-1}(y)$. Set

$$V = Y \times_{R_0} \overset{\sim n}{R_0} \cap V(P_1,\dots,P_s)$$

with $V(P_1,\dots,P_s) = \{z \in \operatorname{Sper} B[X_1,\dots,X_n] \mid P_1(z) = \cdots = P_s(z) = 0\}$. Let

$$A = \{z \in Y[f^{-1}(z) \subset V \cap p^{-1}(z)\}.$$

This is constructible by Proposition 14 (d). Suppose that $\dim f^{-1}(y) = m$ and set

$$B = \{z \in A \mid \rho(z) \models \Phi_z\}$$

where Φ is the following formula:

$$\exists x_1,\dots,x_n \, \exists \epsilon > 0 \, \forall y_1,\dots,y_n :$$
$$(P_1(y) = 0 \,\&\dots\&\, P_s(y) = 0 \,\&\sum_{i=1}^{n}(x_i - y_i)^2 < \epsilon$$
$$\& Rg\Big(\frac{\partial P_i}{\partial x_j}(x_1,\dots,x_n)\Big) = n - m) \to \Psi(y_1,\dots y_n)$$

(Ψ is a formula describing the constructible set $X \subset Y \times_{R_0} \overset{\sim n}{R_0}$). If the dimension of the algebraic variety $V_z = V \cap p^{-1}(z)$ is m then this means that $f^{-1}(z)$ contains some open subset of the set of regular points of the variety V_z. Then $f^{-1}(z)$ has dimension m as a semi-algebraic space. So, finally set

$$C = \{z \in B \mid \dim V_z = m\}.$$

By ([**6**, IV 9.9.1]) this is constructible.

Starting from $y \in Y$ we have constructed the constructible set C. We change notation now and denote C by C_y. By construction, $y \in C_y$. The sets C_y, $y \in Y$ form a constructible cover of Y. Hence, there is a finite subcover

$$Y = \cup_{i=1}^{r} C_{y_i}.$$

For each i, $\dim f^{-1}(z) = \dim f^{-1}(y_i)$ for all $z \in C_{y_i}$. Thus,

$$D_n = \cup\{C_{y_i} \mid \dim f^{-1}(y_i) = n\}$$

is constructible. □

To complete the proof of Proposition 14 we must still prove (j). This follows from

THEOREM 16. *Let $f : X \to Y$ be a finitely presented morphism of affine real closed spaces. For all $n \in \mathbf{Z}$ set*

$$C_n = \{y \in Y \mid f^{-1}(y) \text{ has } n \text{ connected components}\}.$$

Then C_n is constructible.

PROOF. Let B be the ring of global sections of Y. We have a finite presentation

$$\begin{tikzcd}
& Y \arrow[r, hook] & \operatorname{Sper} B \\
X \arrow[ur, "f"] \arrow[dr] & & \\
& Y \times_{R_0} \widetilde{R}_0^n \arrow[uu, "p"'] \arrow[r, hook] & \operatorname{Sper} B[X_1, \dots, X_n]. \arrow[uu, "\pi"']
\end{tikzcd}$$

of f. Let Φ be a formula with constants from B defining X as a constructible subset of $Y \times_{R_0} \widetilde{R}_0^n$. Pick $y \in Y$ and let

$$f^{-1}(y) = C(y)_1 \vee \cdots \vee C(y)_{r(y)}$$

be the decomposition into connected components. Let Ψ_y be a formula with constants from B/y expressing that the $C(y)_i$ are precisely the connected components of $f^{-1}(y)$ (cf. [**2**], Lemma 5.6). Let $\Psi(y)$ be a formula with constants from B lifting Ψ_y. Then

$$C(y) = \{z \in Y \mid \rho(z) \models \Psi(y)_z\}$$

is constructible and $y \in C(y)$. Thus,

$$Y = \cup_{y \in Y} C(y)$$

is a constructible cover, and there is a finite subcover $Y = C(y_1) \cup \cdots \cup C(y_r)$. Since the number of connected components of $f^{-1}(z)$ is constant as z varies in $C(y_i)$, the claim is proved. □

As an immediate consequence of Proposition 14 we have

COROLLARY 17. *Let $f : X \to Y$ be a finitely presented morphism of affine real closed spaces. Suppose that f is open. If*

$$\begin{tikzcd} & & Y \\ X \arrow[urr, "f"] \arrow[drr, hook] & & \\ & & Y \times_{R_0} \tilde{R}_0^{\,n} \arrow[uu, "p"'] \end{tikzcd}$$

is a finite presentation and $Z \subset Y \times_{R_0} \tilde{R}_0^{\,n}$ is closed constructible, then

$$E = \{y \in Y \mid X \cap p^{-1}(y) \subset Z \cap p^{-1}(y)\}$$

is closed constructible.

PROOF. Constructibility follows from Proposition 14(d). For closedness we must show that E is closed under specialization ([**1**, Proposition 1]). Pick $\alpha \in E$, $\beta \supset \alpha$ a specialization in Y. Assume that $X \cap p^{-1}(\beta) \not\subset Z \cap p^{-1}(\beta)$, i.e., there is some $\gamma \in X \cap p^{-1}(\beta) \backslash Z$. Since f is generalizing there is some $\delta \in X \cap p^{-1}(\alpha)$ such that $\delta \subset \gamma$. Thus, $\delta \in Z$. Since Z is constructible, Z is closed under specialization, i.e., $\gamma \in Z$, a contradiction. □

Next we consider the function c defined at the beginning of this section:

THEOREM 18. *Let $f : X \to Y$ be a finitely presented morphism of affine real closed spaces. Suppose that f is open and closed. Then*

$$C_{\leq n} = \{y \in Y \mid c(y) \leq n\}$$

is closed constructible for all $n \in \mathbf{Z}$.

PROOF. $C_{\leq n}$ is constructible by Theorem 16. It remains to prove that $C_{\leq n}$ is closed under specialization. Pick α, $\beta \in Y$, $\alpha \subset \beta$ such that $\alpha \in C_{\leq n}$. The properties of f are preserved under base extension by $Y_\beta \to Y$. Therefore, assume that β is the unique closed point of Y. Then $f^{-1}(\beta) \subset X$ is closed and the connected components $C(\beta)_1, \ldots, C(\beta)_t$ of $f^{-1}(\beta)$ are closed in X. By ([**9**, II 4.16]), there are $f_1, \ldots, f_t \in 0_X(X)$ such that $f_i \mid C(\beta)_i > 0$, $f_i \mid C(\beta)_j < 0$ for $j \neq i$. Assume that $f_i(\gamma) = 0$ for $\gamma \in f^{-1}(\alpha)$. By closedness of f, there is some $\delta \in f^{-1}(\beta)$, $\gamma \subset \delta$. But then $f_i(\delta) = 0$ also. However, f_i was chosen such that there is no zero of f_i on $f^{-1}(\beta)$. Thus, f_i does not have a zero on $f^{-1}(\alpha)$. By openness of f, there are $\gamma_1, \ldots, \gamma_t \in f^{-1}(\alpha)$ such that $\overline{\{\gamma_i\}} \cap C(\beta)_i \neq \emptyset$. It follows that

$$f_i(\gamma_j) \begin{cases} > 0 & \text{for } i = j \\ < 0 & \text{for } i \neq j. \end{cases}$$

This shows

$$f^{-1}(\alpha) = C_1 \cup \cdots \cup C_t$$

with

$$C_i = \{\gamma \in f^{-1}(\alpha) \mid f_i(\gamma) > 0\}$$

is a partition into nonempty open constructible subsets. Thus,

$$c(\beta) = t \leq c(\alpha) \leq n. \ \square$$

Finally, we take a look at the dimensions of the fibres:

THEOREM 19. *Let $f : X \to Y$ be a finitely presented morphism of affine real closed spaces. Suppose that f is universally open. Then*

$$D_{\leq n} = \{y \in Y \mid d(y) \leq n\}$$

is closed and constructible for all $n \in \mathbf{Z}$.

PROOF. From Theorem 15 $D_{\leq n}$ is constructible. By ([**1**, Proposition 1]) it remains to show that $D_{\leq n}$ is closed under specialization. It suffices to show this for the case that Y is a valuative real closed space. Let y_0, y_1 be the generic and the closed points of Y. By ([**2** Proposition

8.11]), $\dim f^{-1}(y_1)$ is the length of a longest chain of specializations in $f^{-1}(y_1)$. Let

$$x_0 \subset x_1 \subset \cdots \subset x_m$$

be such a chain. Then $\operatorname{trdgr}_{\rho(y_1)}\rho(x_0) = m$. For, consider $f^{-1}(y_1)$ as a semi-algebraic subspace of $\rho(y_1)^r$ for some $r \in \mathbf{N}$. If V is the Zariski closure of $f^{-1}(y_1) \subset \rho(y_1)^r$ we decompose V into its irreducible components $V_1, \ldots, V_s$. Then $x_0 \in V_1$. By [**2**, Proposition 8.3], the function field of V has transcendence degree m over $\rho(y_1)$. $\rho(x_0)$ is the real closure of this function field with respect to the total order belonging to x_0.

Since f is open (i.e., generalizing), there is some $z \in f^{-1}(y_0)$ with $z \subset x_0$. Let B be the ring of global sections of Y and $A = 0_{x,x_0}/z \subset \rho(z)$. Let $C \subset \rho(z)$ be a convex subring dominating A. Then we have local homomorphisms

$$B \to A \to C$$

and homomorphisms

$$\rho(y_1) \to \rho(x_0) \to C/m_C$$

of residue fields. Thus, $\operatorname{trdgr}_{\rho(y_1)} C/m_C \geq m$. The place $B \to \rho(y_1)$ is a restriction of the place $C \to C/m_C$. Hence,

$$\operatorname{trdgr}_{\rho(y_0)}\rho(z) \geq \operatorname{trdgr}_{\rho(y_1)} C/m_C \geq m.$$

Now the dimension of $f^{-1}(y_0)$ is the length of a longest chain of specializations in $f^{-1}(y_0)$ ([**2**, Proposition 8.11]). By [**2**, Proposition 8.3], this implies

$$\dim f^{-1}(y_0) \geq \operatorname{trdgr}_{\rho(y_0)}\rho(z) \geq m. \ \square$$

The last two results show that the hypothesis of openness brings some measure of regularity to the fibres of a finitely presented morphism of affine real closed spaces. Examples in [**10**] show that the results presented here are the best one may expect without stronger hypotheses.

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