BETWEEN GROUPS AND RINGS

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Introduction. The semilinear sets-those subsets of $\mathbf{R}^m$ defined with real linear equalities and inequalities–form only a part of the semialgebraic sets-those subsets of $\mathbf{R}^m$ defined with arbitrary real polynomial equalities and inequalities. Yet in one dimension, the semilinear and semialgebraic sets are the same: the finite unions of points and open intervals. Since subtle properties of definable sets in higher dimensions may be established uniformly when the one-dimensional definable sets are so simple [4], the number of examples covered by these explanations becomes a matter of interest. Van den Dries has asked whether we can add new operations or relations to the ordered group $(\mathbf{R}, +, 0 <)$ so that the system of definable sets lies strictly between the semilinear and the semialgebraic sets [3]. Since the one-dimensional semialgebraic sets are semilinear, such an expansion would provide a new ordered structure whose one-dimensional definable sets are finite unions of points and open intervals: that is, a new o-minimal structure [6].

For $a \in \mathbf{R}$ let $f_a : \mathbf{R} \to \mathbf{R}$ be multiplication by a. Using this notation, we see that the sets definable over $(\mathbf{R}, +, 0, <, f_a)_{a \in \mathbf{R}}$ are the semilinear sets. We conjectured that the expansion of $(\mathbf{R}, +, O, <, f_a)_{a \in \mathbf{R}}$ by the restriction of multiplication to a bounded interval - to $[-1, 1]$, say–would have all the properties desired. Although sets which are not semilinear become definable, every definable set is semialgebraic; and though multiplication on arbitrarily large intervals $[-n, n]$ becomes definable, the definitions seem to depend on n. The results of §2 show that this dependence is inevitable: multiplication on all of $\mathbf{R}$ is not definable over the new structure.

To simplify the notation, we will work with $(\mathbf{R}, +, 0, <)$ instead of $(\mathbf{R}, +, 0, <, f_a)_{a \in \mathbf{R}}$, although all the proofs in §1-3 actually handle the more complicated structure. Corollary 2.2 says that multiplication on

R is not definable over the expansion of $(\mathbf{R}, +, 0, <)$ by multiplication restricted to $[-1, 1]$. This result follows from Theorem 2.1, which says that no expansion of $(\mathbf{R}, +, 0, <)$ by a relation on a bounded subset of $\mathbf{R}^m$ will permit the definition of multiplication everywhere. So even if we give up o-minimality, we cannot define multiplication unless we add an unbounded relation to $(\mathbf{R}, +, 0, <)$.

Behind the proof of Theorem 2.1 lies Theorem 1.1, which concerns pairs $\mathcal{M} \prec \mathcal{N}$ of o-minimal or stable structures which have a two-cardinal formula: that is, a formula with the same extension A in both models. Theorem 1.1 allows us to expand the models by any relation on A^n without destroying the elementary embedding; in the proof of Theorem 2.1, A is a bounded interval.

Returning to the example which inspired all these efforts, we show in §3 that the theory of the expansion of $(\mathbf{R}, +, 0, <)$ by multiplication on $[-1, 1]$ admits elimination of quantifiers. We thus obtain both a more down-to-earth proof of Corollary 2.2 and a closer analogy between $(\mathbf{R}, +, 0, <)$, $(\mathbf{R}, +, \cdot, 0, 1, <)$, and our structure.

[**6**] contains all the facts about o-minimality which figure in what follows. We will exploit especially the monotonicity theorem and the existence and uniqueness of prime models over sets. The reader may assume, for the sake of simplicity, that all o-minimal structures are dense linear orders without end points. [**6**] contains all the results we take from stability theory.

In what follows $\mathcal{M}, \mathcal{N}, \ldots$ are structures, $A, B, \ldots$ are subsets of structures, $a, b, c, \ldots$ are elements of structures, $\bar{a}, \bar{b}, \bar{c}, \ldots$ are n-tuples of elements, $x, y, z, \ldots$ are variables, and $\bar{x}, \bar{y}, \bar{z}, \ldots$ are n-tuples of variables. If $\bar{a}$ is an n-tuple from some large saturated $\mathcal{N} \prec \mathcal{M}$ and $A \subseteq \mathcal{M}$, then the type of $\bar{a}$ over A is

$$t(\bar{a}, A) = \{\varphi(\bar{x}, \bar{b}) : b \text{ is a tuple from } A \text{ and } \mathcal{N} \models \varphi(\bar{a}, \bar{b})\}.$$

If $\varphi(x)$ is a formula over the L-structure $\mathcal{M}$, we let

$$\varphi(\mathcal{M}) = \{a \in M : \mathcal{M} \models \varphi(a)\}.$$

If $R \subseteq \varphi(\mathcal{M})^n$ is a new relation, $L(R)$ is the expansion of L by a new n-ary relation symbol, which is interpreted by R when we add it to $\mathcal{M}$ to obtain the $L(R)$-structure $\mathcal{M}(R)$.

1. We devote this section to the proof of

THEOREM 1.1.

(a) *Suppose that $\mathcal{M}$ is an ω-saturated, o-minimal L-structure, $\varphi(x)$ is a formula over $\mathcal{M}$, and $\mathcal{N} \succ \mathcal{M}$ is an ω-saturated L-structure for which $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$. If $R \subseteq \varphi(\mathcal{M})^n$ is a new relation, then $\mathcal{M}(R) \prec \mathcal{N}(R)$.*

(b) *Suppose that T is a stable L-theory, $\mathcal{M} \models T$ is ω_1 saturated, $\varphi(x)$ is a formula over $\mathcal{M}$, and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ for some ω_1-saturated $\mathcal{N} \succ \mathcal{M}$. If $R \subseteq \varphi(\mathcal{M})^n$ is a new relation, then $\mathcal{M}(R) \prec \mathcal{N}(R)$.*

To prepare the way for Theorem 1.1(a), we need

LEMMA 1.2. *Let $\mathcal{M}$ be o-minimal, $\varphi(x)$ be a formula over $\emptyset$, and $\overline{a} \in \mathcal{M}^n$ be algebraically independent over $\varphi(\mathcal{M})$. If $\overline{b} \in \mathcal{M}^n$ and $t(\overline{b}, \emptyset) = t(\overline{a}, \emptyset)$, then $t(\overline{b}, \varphi(\mathcal{M})) = t(\overline{a}, \varphi(\mathcal{M}))$.*

PROOF. The proof will go by induction on the length $n = \mathrm{lh}(\overline{a})$ of $\overline{a}$.

If there us just one element a, we start from the assumption that a does not belong to the definable closure dcl $(\varphi(\mathcal{M}))$ of $\mathcal{M}$. By induction on m we will show that, for all formulas $\psi(x, \overline{y})$ over $\emptyset$ with $\mathrm{lh}(\overline{y}) = m$,

$$(*) \qquad \mathcal{M} \models \psi(a, \overline{c}) \Leftrightarrow \mathcal{M} \models \psi(b, \overline{c})$$

whenever $t(a, \emptyset) = t(b, \emptyset)$ and $\overline{c} \in \varphi(\mathcal{M})^m$.

When $m = 1$, o-minimality implies that $A = \{c \in \varphi(\mathcal{M}) : \psi(a, c)\}$ is a finite union of points and open intervals, all definable over $\{a\}$. With A we may associate the boundary points of these intervals and the isolated points of A. These are either boundary points of $\varphi(\mathcal{M})$, isolated points of $\varphi(\mathcal{M})$, or interior points of $\varphi(\mathcal{M})$, and all except possibly the last

are $\emptyset$-definable. If all the points associated with A are $\emptyset$-definable, then $(*)$ clearly follows: so suppose that one of the points d associated with A is not $\emptyset$-definable. Then d is an interior point of $\varphi(\mathcal{M})$ that is definable over $\{a\}$ but not over $\emptyset$: by the exchange principle for o-minimal theories [6, p. 577], a is definable over $\{d\} \subseteq \varphi(\mathcal{M})$, contrary to hypothesis. Thus $(*)$ holds when $m = 1$.

Assume now that $(*)$ holds when $\mathrm{lh}\,(\overline{c}) = m$. If $\psi(x, \overline{y}, z)$ is a formula over $\emptyset$ with $\mathrm{lh}\,(\overline{y}) = m$ and $\overline{c}_o c \in \varphi(\mathcal{M})^{m+1}$, then $A = \{c' \in \varphi(\mathcal{M}) : \psi(a, \overline{c}_o, c')\}$ is again a finite union of points and open intervals. If, as before, we associate with A the end points of these intervals and the isolated points of A, these points are $\{a, \overline{c}_o\}$-definable, and the induction hypothesis yields $(*)$ if all the points associated with A are $\{\overline{c}_o\}$-definable. We may therefore suppose that some point d associated with A is not $\{\overline{c}_o\}$-definable. Just as before, d must belong to the interior of $\varphi(\mathcal{M})$, and the exchange principle makes a definable over $\{d, \overline{c}_o^{\frown}\} \subseteq \varphi(\mathcal{M})$, contrary to hypothesis. So $(*)$ holds for all $\overline{c}_o^{\frown} c \in \varphi(\mathcal{M})^{m+1}$ if $(*)$ holds for all $\overline{c}_o \in \varphi(\mathcal{M})^m$, and induction yields the lemma for $n = 1$.

Assume that the lemma holds when $\mathrm{lh}\,(\overline{a}) = n$, and let $\overline{a}_o^{\frown} a, \overline{b}_o^{\frown} b$ be $(n+1)$-tuples from $\mathcal{M}$ with $t(\overline{a}_0 a, \emptyset) = t(\overline{b}_0^{\frown} b, \emptyset)$. By the induction hypothesis, $t(\overline{a}_0, \varphi(M)) = t(\overline{b}_0, \varphi(M))$, and so an automorphism of some large saturated $\mathcal{N} \prec \mathcal{M}$ fixes $\varphi(\mathcal{M})$ and moves $\overline{a}_0$ to $\overline{b}_0$ We may therefore assume that $\overline{a}_0 = \overline{b}_0$. By adding constants for the elements of $\overline{a}_0$ to L, we reduce the present case of the lemma to the $n = 1$ case, which we have already established. So the lemma holds for all $\overline{a}_o^{\frown} a \in \mathcal{M}^{n+1}$ if it holds for all $\overline{a} \in \mathcal{M}^n$, and by induction we are finished. $\square$

An easy corollary of Lemma 1.2 is

LEMMA 1.3. *Let $\mathcal{M}$ be o-minimal, $\varphi(x)$ be a formula over $\emptyset$, and $\overline{a} \in \mathcal{M}^n$. There is a finite $A \subseteq \mathcal{M}$ for which*

$$t(\overline{a}, \varphi(\mathcal{M})) = t(\overline{b}, \varphi(\mathcal{M}))$$

whenever $\overline{b} \in \mathcal{M}^n$ and $t(\overline{a}, A) = t(\overline{b}, A)$.

LEMMA 1.4. *Let T be a stable theory, $\mathcal{M} \models T$ be ω_1-saturated, and $\varphi(x)$ be a formula over $\emptyset$. For every $\overline{a} \in \mathcal{M}^n$ there is a countable $A \subseteq \varphi(\mathcal{M})$ for which*

$$t(\overline{a}, \varphi(\mathcal{M})) = t(\overline{b}, \varphi(\mathcal{M}))$$

whenever $\overline{b} \in \mathcal{M}^n$ and $t(\overline{a}, A) = t(\overline{b}, A)$.

PROOF. Let M_0 be a countable elementary substructure of $\mathcal{M}$ with $\overline{a} \in (\mathcal{M}_0)^n$. If $\overline{c}, \overline{c}' \in (\varphi(\mathcal{M}))^p$ and

$$t(\overline{c}, \varphi(\mathcal{M}_0)) = t(\overline{c}', \varphi(\mathcal{M}_0)),$$

then, by Corollary 7.3 of [**5**],

$$t(\overline{c}, \mathcal{M}_0) = t(\overline{c}', \mathcal{M}_0).$$

Let $A = \varphi(\mathcal{M}_0)$. For any $\overline{b} \in \mathcal{M}^n$ with $t(\overline{a}, A) = t(\overline{b}, A)$, we must show that

(∗) $$t(\overline{a}, \overline{c}) = t(\overline{b}, \overline{c})$$

for any $\overline{c} \in \varphi(\mathcal{M})^p$. Because $\mathcal{M}$ is ω_1-saturated, there is a countable $\mathcal{M}_0' \prec \mathcal{M}$, isomorphic to $\mathcal{M}_0$ over $\varphi(\mathcal{M}_0)$, such that $\overline{b} \in (\mathcal{M}_0')^n$ and

$$t(\overline{a} \,\widehat{}\, \overline{c}, \emptyset) = t(\overline{b} \,\widehat{}\, \overline{c}', \emptyset),$$

where $\overline{c}' \in \varphi(\mathcal{M})^p$ is a tuple for which

$$t(\overline{c}, \varphi(\mathcal{M}_0)) = t(\overline{c}', \varphi(\mathcal{M}_0)).$$

Since $\varphi(\mathcal{M}_0) = \varphi(\mathcal{M}_0')$, we may invoke the result at the start of the proof to conclude that

$$t(\overline{c}, \mathcal{M}_0') = t(\overline{c}', \mathcal{M}_0').$$

Thus $t(\overline{c}, \overline{b}) = t(\overline{c}', \overline{b})$ and (∗)follows. □

We are now ready for the proof of Theorem 1.1. Since we focus here on o-minimal structures, we will prove Theorem 1.1(a) from Lemma

1.3, and leave to the reader the analogous proof of Theorem 1.1(b) from Lemma 1.4.

Without loss of generality we may assume that L contains only relation and constant symbols and that $\varphi(x)$ is a formula over $\emptyset$. By the back-and-forth criterion for elementary inclusion [1, p. 16], we need show merely that

$$\mathcal{F} = \{f : f \text{ is a function and dom } f = \varphi(\mathcal{M}) \cup \{\overline{a}\} \subseteq \mathcal{M}$$
$$\text{and ran } f = \varphi(\mathcal{M}) \cup \{\overline{b}\} \subseteq \mathcal{N} \text{ and } f|\varphi(\mathcal{M}) = \text{identity}$$
$$\text{and } f(\overline{a}) = \overline{b} \text{ and } t_L(\overline{a}, \varphi(\mathcal{M})) = t_L(\overline{b}, \varphi(\mathcal{M}))\}$$

has the back-and-forth property: for every $f \in \mathcal{F}$, f fixes $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ and so preserves the new relation $R \subseteq \varphi(\mathcal{M})^n$. We will establish the backward direction, and leave the other to the reader. Given $f : \varphi(\mathcal{M}) \cup \{\overline{a}\} \rightarrow \varphi(\mathcal{M}) \cup \{\overline{b}\}$ in $\mathcal{F}$, and $b' \in \mathcal{N}$, we must find $a' \in \mathcal{M}$ for which

$$t_L(\overline{a} \,\frown a', \varphi(\mathcal{M})) = t_L(\overline{b} \,\frown b', \varphi(\mathcal{M})).$$

Lemma 1.3 provides a finite $A \subseteq \varphi(\mathcal{N}) = \varphi(\mathcal{M})$ for which

$$t_L(\overline{d} \,\frown d', \varphi(\mathcal{M})) = t_L(\overline{b} \,\frown b', \varphi(\mathcal{M}))$$

whenever $\overline{d} \,\frown d'$ is a tuple from $\mathcal{N}$ and $t_L(\overline{d} \,\frown d', A) = t_L(\overline{b} \,\frown b', A)$. Since $\mathcal{M}$ is ω-saturated, $t_L(\overline{a} \,\frown a', A) = t_L(\overline{b} \,\frown b', A)$ for some $a' \in \mathcal{M}$, and so $t_L(\overline{a} \,\frown a', \varphi(\mathcal{M})) = t_L(\overline{b} \,\frown b'(\mathcal{M}))$ as desired.

Lemma 1.3 fails for ordered structures which are not o-minimal. Though the theory of $(\mathbf{R}, <, \mathbf{Q})$ lacks the independence property, sufficiently saturated elementary extensions of the model violate Lemma 1.3 for any A with $|A| < |\varphi(\mathcal{M})|$.

2. We now have assembled the machinery needed to prove.

THEOREM 2.1. *If $S \subseteq \mathbf{R}^n$ is bounded, then multiplication is not definable over $(\mathbf{R}, +, 0, <, S)$.*

PROOF. Suppose that $S \subseteq [-m, m]^n$ and let

$$\mathcal{M} = (M, +, 0, <, S')$$

be an ω-saturated elementary extension of $(\mathbf{R}, +, 0, < S)$. $\mathcal{M}_L$, the restriction of $\mathcal{M}$ to the language $L = \{+, 0, <\}$, is an ω-saturated divisible ordered Abelian group. If we order $\mathcal{M}_L \oplus \mathcal{M}_L$ lexicographically and identity $\mathcal{M}_L$ with $\{0\} \times \mathcal{M}_L$, we obtain another divisible ordered Abelian group which is an end extension of $\mathcal{M}_L$. Being definable inside $\mathcal{M}_L$, this new group is also ω-saturated. Since

$$-m \leq x \leq m$$

has the same extension in both $\mathcal{M}_L$ and $\mathcal{M}_L \oplus \mathcal{M}_L$, we may invoke Theorem 1.1(a) to conclude that

$$\mathcal{M} \prec (\mathcal{M}_L \oplus \mathcal{M}_L)(S') = \mathcal{N}.$$

Because $\mathcal{N}$ contains no elements infinitesimal with respect to $\mathcal{M}$, but does contain elements infinite with respect to $\mathcal{M}$, multiplication cannot be definable over $(\mathbf{R}, +, 0, <, S), \mathcal{M}$, or $\mathcal{N}$. $\square$

An immediate consequence is

COROLLARY 2.2. *Let $B \subseteq \mathbf{R}^3$ be the graph of the restriction of multiplication to $[-1, 1]$: then multiplication is not definable over the proper o-minimal expansion $(\mathbf{R}, +, 0, <, B)$ of $(\mathbf{R}, +, 0, <)$.*

PROOF. Since every set definable over $(\mathbf{R}, +, 0, <, B)$ is semialgebraic, the new structure is o-minimal; and since B is not semilinear, $(\mathbf{R}, +, 0, <, B)$ is a proper expansion of $(\mathbf{R}, +, 0, <)$. The undefinability of multiplication follows directly from Theorem 2.1. $\square$

A simple direct argument produces an ω-saturated proper elementary end extension of the divisible ordered Abelian group $\mathcal{M}_L$ in Theorem 2.1. But we may get a proper elementary end extensions of κ-saturated o-minimal structures in more general situations as well.

THEOREM 2.2. *Let $\kappa > \omega, \mathcal{M}_1$ be a κ-saturated o-minimal structure, and $\mathcal{M}_2$ be a proper elementary end extension of $\mathcal{M}_1$. Then there is a κ-saturated proper elementary end extension $\mathcal{M}_3$ of $\mathcal{M}_1$.*

PROOF. We show that if $A \subseteq \mathcal{M}_2, |A| < \kappa$, and $p \in S_1(A)$ is not realized in $\mathcal{M}_2$, then $\mathcal{M}_2(p)$, the model prime over a realization of p, is an elementary end extension of $\mathcal{M}_1$ (all models here are to be elementary submodels of some huge saturated model). From this result the theorem follows by a union-of-chains argument.

Suppose, then, that a realizes p, and identify $\mathcal{M}_2(p)$ with $\mathcal{M}_2(a)$. Aiming towards a contradiction, we suppose that some $b \in \mathcal{M}_2(a) - \mathcal{M}_1$ belongs to an interval (c, d) with $c, d \in M_1$. b satisfies an isolated type over $M_2 \cup \{a\}$ [**6**, p. 583], and so is either definable from $M_2 \cup \{a\}$ or lies in an open interval, isolating b's type over $M_2 \cup \{a\}$, whose endpoints are definable from $M_2 \cup \{a\}$. In the latter case, at least one of the end points does not belong to M_2 because $b \notin M_2$: without loss of generality, we may suppose that b is definable from $M_2 \cup \{a\}$. Thus $b = f(a)$ for some $\mathcal{M}_2$-definable function $f : M_2 \to M_2$, and the monotonicity theorem provides an $\mathcal{M}_2$-definable interval I, with $a \in I$, such that $f|I$ is a monotone bijection onto some $\mathcal{M}_2$-definable interval $J \subseteq (c, d)$. Assume that f is increasing on I, and let

$$A' = \text{dcl}\,(A) \cap I.$$

Since $A' \subseteq M_2, f(A') \subseteq (c, d) \cap M_2$, and so $f(A') \subseteq M_1$ since $\mathcal{M}_2$ is an end extension of $\mathcal{M}_1$. Because $|f(A')| < \kappa_{\mathrm{i}}$ the cut in $\mathcal{M}_1$ given by

$$\{x > f(a') : a' \in A' \text{ and } a > a'\} \cup \{x < f(a') : a' \in A' \text{ and } a < a'\}$$

must be realized by some $b' \in M_1$. But then $f^{-1}(b') \in M_2$ must realize p, although $\mathcal{M}_2$ omits p, and this contradiction finishes the proof. $\square$

3. Another approach to Corollary 2.2 relies on the elimination of quantifiers. Let $g : \mathbf{R} \to \mathbf{R}$ be given for $x \in \mathbf{R}$ by

$$g(x) = \begin{cases} x, & \text{if } |x| \leq 1, \\ \text{the sign of } x, & \text{if } |x| > 1, \end{cases}$$

and let $* : \mathbf{R}^2 \to \mathbf{R}$ be given for $x, y \in \mathbf{R}$ by

$$x * y = g(x)g(y).$$

Clearly $(\mathbf{R}, +, *, 0, 1, <)$ and $(\mathbf{R}, +, 0, < B)$ produce the same definable sets. An easy inductive argument shows that any subset of $\mathbf{R}^2$ which

is quantifier-free-definable over $(\mathbf{R}, +, *, 0, 1, <)$ is semilinear outside of a finite union of strips, where a strip is a set

$$\{\overline{a} \in \mathbf{R}^2 : |p(\overline{a})| \le h\}$$

with p a linear polynomial. Since the graph of $y = x^2$ is not semilinear outside a finite union of strips, that graph is not quantifier-free-definable over $(\mathbf{R}, +, *, 0, 1, <)$. So if the theory of this structure admits elimination of quantifiers, the graph of $y = x^2$ is not definable over $(\mathbf{R}, +, *, 0, 1)$, and Corollary 2.2 follows.

Though we finish the proof of §1-2 before we eliminated quantifiers in $T = Th(\mathbf{R}, +, *, 0, 1, <)$, they maybe shown redundant without too much trouble. Before proving four preliminary lemmas, we introduce some notation. If $\mathcal{M} = (M, +, *, 0, 1, <) \models T$, let $\sim_{\mathcal{M}}$ be the equivalence relation on M given by

$$x \sim_{\mathcal{M}} y \Leftrightarrow \exists n < \omega(|x - y| \le n),$$

and let $[a]_{\mathcal{M}}$ be the $\sim_{\mathcal{M}}$-equivalence class of $a \in M$. $M^R = [0]_{\mathcal{M}}$ is the domain of a divisible convex subgroup of $(M, +, 0, 1, <)$; we call the elements of M^R the finite elements of M. If $M^O = M/\sim_{\mathcal{M}}$ and $\varphi_{\mathcal{M}} : M \to \mathcal{M}^O$ is the usual quotient map, we may make M^O the domain of a divisible ordered Abelian group $\mathcal{M}^O$-perhaps trivial-so that $\varphi_{\mathcal{M}}$ is a surjective group homomorphism preserving $\le$. When no confusion can arise, we will drop the subscript '$\mathcal{M}$'.

LEMMA 3.1. *If $A \subseteq M$ is a divisible subgroup on which $\varphi_{\mathcal{M}}$ is injective, there is a divisible subgroup B, with $A \subseteq B \subseteq M$, for which $\varphi_{\mathcal{M}}|B : B \twoheadrightarrow M^O$ is an isomorphism.*

PROOF. If φ is injective on a subgroup G of M, φ must also be injective on the divisible hull $\overline{G}$ of G in M, since φ is $\mathbf{Q}$-linear and $\overline{G}$ consists of the rational multiples of elements of G. So if we invoke Zorn's lemma to obtain a $\subseteq$-maximal subgroup $B \supseteq A$ on which φ is injective, B must be divisible. If $k \in M - B, \varphi$ is not injective on $B + (k)$, and so

$$b + ak \ne 0 \text{ and } [b + ak] = [0]$$

for some $a \in \mathbf{Z}$ and $b \in B$. Thus

$$[k] = \left[\frac{-b}{a}\right]$$

and, since B is divisible, $[k] \in \varphi(B)$. $\varphi|B$ is therefore an isomorphism of B onto $\varphi(B) = M^O$. $\square$

A real-closed ring [2] is an ordered integral domain which obeys the intermediate value theorem for polynomials.

LEMMA 3.2. *We may supply M^R with a multiplication $\cdot : (M^R)^2 \to M^R$ which extends $*|[-1,1]^2$ and makes $\mathcal{M}^R = (M^R, +, \cdot, 0, 1, <)$ a real-closed ring.*

PROOF. If $a, b \in M^R$, both $|a|$ and $|b|$ are at most m for some positive $m < \omega$, and so we should let

$$a \cdot b = m^2 \left(\frac{a}{m} * \frac{b}{m}\right).$$

Using universal sentences about $*$ true in $\mathcal{M} \models T$, we may easily show that $\cdot$ is well-defined and makes $\mathcal{M}^R$ an ordered integral domain.

To obtain the intermediate value theorem for polynomials over M^R, we need

LEMMA 3.3. *For any $m, n \geq 1$ and any $q(\overline{x}) \in \mathbf{Z}[x_1, \ldots, x_m]$, there is an L_T-formula $\psi(\overline{x}, y)$ such that*

$$T \vdash \text{``ψ defines a continuous function } \overline{x} \mapsto y\text{''}$$

and

$$q|[-n, n]^m = t|[-n, n]^m$$

if $t : M^m \to M$ is the function defined by ψ over $\mathcal{M}$.

PROOF. The lemma obviously holds for linear polynomials, and if it holds for $q_1(\overline{x})$ and $q_2(\overline{x})$, it obviously holds for $q_1(\overline{x}) + q_2(\overline{x})$. If we

have formulas ψ_1, ψ_2 corresponding to polynomials $q_1(\overline{x}), q_2(\overline{x})$ as in the lemma, then since

$$T \vdash \text{``}\psi_i \text{ defines a continuous function''}$$

for $i = 1, 2$, there is a positive $k < \omega$ for which

$$T \vdash \forall \overline{x} \, (|\overline{x}| \leq n \text{ and } \psi_i(\overline{x} \, y) \to |y| \leq k).$$

The formula corresponding to $q_1(\overline{x}) q_2(\overline{x})$ should therefore be

$$\exists \, y_1, y_2 (y = k^2 (y_1 * y_2) \text{ and } \psi(\overline{x}, ky_1) \text{ and } \psi_2(\overline{x}, ky_2)),$$

that is

$$t(\overline{x}) = k^2 \left(\frac{t_1(\overline{x})}{k} * \frac{t_2(\overline{x})}{k} \right). \quad \square$$

Suppose now that $p(\overline{x}, y) \in \mathbf{Z}[x_1, \ldots, x_k y], a_1, \ldots, a_k, b$, and $c > b$ belongs to M^R, and

$$p(\overline{a}, b) < 0 < p(\overline{a}, c).$$

Since the a's, b, and c are finite, there is a positive $n < \omega$ bounding the $|a|$'s, $|b|$, and $|c|$. Applying Lemma 3.3, we obtain a continuous, definable $f : M^{k+1} \to M$ for which

$$p|[-n, n]^{k+1} = f|[-n, n]^{k+1}.$$

So

$$f(\overline{a}, b) < 0 < f(\overline{a}, c),$$

and the intermediate value theorem for continuous, definable functions provides a $d \in (b, c) \subseteq [-n, n]$ for which

$$f(\overline{a}, d) = 0.$$

So d is a finite root of $p(\overline{a}, x)$ between b and c, and $\mathcal{M}^R$ is a real-closed ring. $\square$

Finally, if B is any subgroup of M as in Lemma 3.1, $\mathcal{M}$ is, as an Abelian group, the internal direct sum $B \oplus M^R$ of B and M^R. If we

order $B \oplus M^R$ lexicographically, we obtain the original order on M. Since, for $b_1, b_2 \in B$ and $m_1, m_2 \in M^R$, we have

$$(b_1 + m_1) * (b_2 + m_2) = h(m_1)h(m_2),$$

where

$$h(b_1 + m_1) = \begin{cases} 1, & \text{if } b_1 > 0 \vee (b_1 = 0 \text{ and } m_1 \geq 1), \\ -1, & \text{if } b_1 < 0 \vee (b_1 = 0 \text{ and } m_1 \leq -1), \\ m_1 & \text{otherwise,} \end{cases}$$

we may also recover $*$ on M from B and $\mathcal{M}^R$. To sum up all these remarks, we have

LEMMA 3.4. $\mathcal{M}$ is the internal direct sum of B and $\mathcal{M}^R$, if B is as in Lemma 3.1.

We may now move on to

THEOREM 3.5. $T = Th(\mathbf{R}, +, *, 0, 1, <)$ admits elimination of quantifiers.

PROOF. We will apply the model-theoretic criterion, for elimination of quantifiers, given in Theorem 13.1(2) of [**7**, p. 63]. We thus begin with models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T having a common substructure $\mathcal{N}_1$, and we must embed $\mathcal{M}_2$ in an elementary extension $\mathcal{N}_2$ of $\mathcal{M}_1$ so that

$$\begin{array}{ccc} N_2 & \leftarrow & \mathcal{M}_2 \\ \vee & & \cup \\ M_1 & \supseteq & \mathcal{N}_1 \end{array}$$

commutes.

Let $\mathcal{M}_2$ be a $\lambda = \sup(|\mathcal{M}_1|, |\mathcal{M}_2|)^+$-saturated elementary extension of $\mathcal{M}_1$. Since L_T generates terms for all the integers, we have the inclusions

$$\begin{array}{ccc} \mathcal{N}_2^R & & \mathcal{M}_2^R, \\ \vee & & \cup \\ \mathcal{M}_1^R & \supseteq & \mathcal{N}_1^R \end{array}$$

where $\mathcal{N}_1^R$ is the substructure of $\mathcal{N}_1$ consisting of the finite elements of N_1. If $\overline{\mathcal{M}_1^R}$ $(\overline{\mathcal{M}_2^R}, \overline{\mathcal{N}_2^R})$ is the field of fractions of $\mathcal{M}_1^R(\mathcal{M}_2^R, \mathcal{N}_2^R)$, then, since $\mathcal{M}_1^R(\mathcal{M}_2^R, \mathcal{N}_2^R)$ is a real-closed ring, $\overline{\mathcal{M}_1^R}$ $(\overline{\mathcal{M}_2^R}, \overline{\mathcal{N}_2^R})$ is a real-closed field in which $\mathcal{M}_1^R, (\mathcal{M}_2^R, \mathcal{N}_2^R)$ is a convex subring [**2**, p. 214]. Because $\mathcal{N}_2$ is λ-saturated, $(N_2^R, <)$ realizes any type, with fewer than λ parameters from N_2^R, which includes '$|x| \leq n$' for some $n < \omega$. Using reciprocals, we may therefore show that $(\overline{N_2^R}, <)$ is λ-saturated; and since types over a real-closed field amount to cuts, $\overline{N_2^R}$ is a λ-saturated real-closed field. Since $|N_1^R|, |\overline{M_2^R}| < \lambda$, we may embed $\overline{\mathcal{M}_2^R}$ in $\overline{\mathcal{N}_2^R}$ by a map f which fixes every element of $\mathcal{N}_1^R$; that is,

$$
\begin{array}{ccc}
\overline{\mathcal{N}_2^R} & \xleftarrow{\;f\;} & \overline{\mathcal{M}_2^R} \\[4pt]
\vee & & \sqcup\!\sqcap \\[4pt]
\overline{\mathcal{M}_1^R} & \supseteq & \overline{\mathcal{N}_1^R}
\end{array}
$$

commutes. Because $\mathcal{M}_1^R$, $\mathcal{M}_2^R$, and $\mathcal{N}_2^R$ are convex subrings of $\overline{\mathcal{M}_1^R}, \overline{\mathcal{M}_2^R}$, and $\overline{\mathcal{N}_2^R}$,

$$
\begin{array}{ccc}
\mathcal{N}_2^R & \xleftarrow{\;f\;} & \mathcal{M}_2^R \\[4pt]
\sqcup\!\sqcap & & \sqcup\!\sqcap \\[4pt]
\mathcal{M}_1^R & \supseteq & \mathcal{N}_1^R
\end{array}
$$

also commutes.

Again because L_T generates terms for all the integers, we have the inclusions

$$
\begin{array}{ccc}
\mathcal{N}_2^0 & & \mathcal{M}_2^0 \\[4pt]
\sqcup\!\sqcap & & \sqcup\!\sqcap \\[4pt]
\mathcal{M}_1^0 & \supseteq & \mathcal{N}_1^0
\end{array}
$$

where $\mathcal{N}_1^0$ is defined from $\mathcal{N}_1$ as $\mathcal{M}_1^0, \mathcal{M}_2^0, \mathcal{N}_2^0$ are defined from $\mathcal{M}_1, \mathcal{M}_2$, $\mathcal{N}_2(\mathcal{N}_1^0$ is an ordered group, but may not be divisible). Since types over the divisible ordered Abelian group $\mathcal{N}_2^0$ amount to cuts, and $\mathcal{N}_2$ is λ-saturated, $\mathcal{N}_2^0$ is also λ-saturated. The divisible hull of $\mathcal{N}_1^0$ in $\mathcal{M}_1^0$ or $\mathcal{M}_2^0$ consists of the rational multiples of elements of $\mathcal{N}_1^0$, and these divisible hulls are isomorphic by an isomorphism which fixes $\mathcal{N}_1^0$. Identifying these divisible hulls, we obtain the inclusions

$$\mathcal{N}_2^0 \qquad\qquad \mathcal{M}_2^0,$$
$$\cup\!\! \mid \qquad\qquad\quad \cup\!\! \mid$$
$$\mathcal{M}_1^0 \quad \supseteq \quad \mathcal{N}_1^0$$

where $\overline{\mathcal{N}_1^0}$ is the divisible hull of $\mathcal{N}_1^0$. If $\overline{\mathcal{N}_1}$ is the divisible hull of the additive group determined by $\mathcal{N}_1$, Lemma 3.1 provides divisible subgroups N_1' of $\overline{\mathcal{N}_1}, M_1'$ of $\mathcal{M}_1, M_2'$ of $\mathcal{M}_2$, and N_2' of $\mathcal{N}_2$ such that $N_1' \subseteq M_1' \cap M_2', M_1' \subseteq N_2'$ and $\phi_{\mathcal{M}_1} \!\restriction\! M_1' : M_1' \twoheadrightarrow \mathcal{M}_1^0, \phi_{\mathcal{M}_2} \!\restriction\! M_2' : M_2' \twoheadrightarrow \mathcal{M}_2^0$ and $\phi_{\mathcal{N}_2} \!\restriction\! N_2' : N_2' \twoheadrightarrow \mathcal{N}_2^0$ are isomorphisms of ordered groups. Since $N_2' \simeq \mathcal{N}_2^0$ is λ-saturated, there is an embedding $h : M_2' \to N_2'$ making

$$\mathcal{N}_2' \quad \overset{h}{\leftarrow} \quad \mathcal{M}_2',$$
$$\cup\!\! \mid \qquad\qquad\quad \cup\!\! \mid$$
$$\mathcal{M}_1' \quad \supseteq \quad \mathcal{N}_1'$$

commute.

By Lemma 3.4, we may define a map $k : M_2 \to N_2$ by

$$k(a + b) = h(a) + f(b)$$

when $a \in M_2'$ and $b \in M_2^R$. k is obviously a homomorphism of additive groups, and since ran $h \subseteq N_2'$ and ran $f \subseteq N_2^R$, Lemma 3.4 makes k injective. Because both h and f are order-preserving, Lemma 3.4 also makes k order-preserving. Since $k(1) = 1, k$ will preserve $*$ if

$$k(b * b') = k(b) * k(b')$$

when $|b|, |b'| \leq 1$; but, in this case,

$$k(b * b') = k(bb') = f(bb') = f(b)f(b') = k(b) * k(b').$$

Finally, if $a \in N_1$, then

$$[a]_{\mathcal{M}_2} = \left[\frac{a'}{l}\right]_{\mathcal{M}_2},$$

where $a' \in N_1, l \geq 1$, and $a'/l \in N_1' \subseteq M_2'$. So $a - a'/l \in M_2^R$ and

$$k(a) = k\left(\frac{a'}{l} + \left(a - \frac{a'}{l}\right)\right) = h\left(\frac{a'}{l}\right) + f\left(\frac{la - a'}{l}\right).$$

Since h fixes N_1', f fixes N_1^R, and $la - a' \in N_1^R$,

$$k(a) = \frac{a'}{l} + \frac{la - a'}{l} = a.$$

So

$$
\begin{array}{ccc}
\mathcal{N}_2 & \xleftarrow{k} & \mathcal{M}_2, \\
\vee & & \cup \\
\mathcal{M}_1 & \supseteq & \mathcal{N}_1
\end{array}
$$

commutes, and T admits elimination of quantifiers.

L. Harrington has suggested that by proving the converse of Lemma 3.4, i.e., models of T amount to products of divisible ordered Abelian groups with real-closed rings, we could give another proof that some semialgebraic sets are not definable over $(\mathbf{R}, +, *, 0, 1, <)$.

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