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ON THE STRUCTURE OF EVEN UNIMODULAR EXTREMAL LATTICES OF RANK 40

MICHIO OZEKI

In memory of the late Professor Hel Braun

1. Introduction. Let $\Gamma_{8k}(k \geq 1)$ be the genus consisting of all equivalence classes of positive definite even unimodular quadratic lattices of rank $8k$. In an element L of Γ_{8k}, a vector x in L is called a $2m$-vector if x satisfies $(x,x) = 2m$, where $(\,,\,)$ is the inner product of L and $2m$ is an even integer. In obtaining a complete picture of the configurations of all equivalence classes in $\Gamma_{8k}(k \geq 4)$, the classes of lattices without 2-vectors would be a main obstacle.

In this paper, we study the subfamily $\Gamma_{40,0}$ of Γ_{40} consisting of all equivalence classes of lattices without 2-vector. As in [**7**], we use $\mathcal{L}_{2m}(L)$ (respectively, $\mathcal{L}_{2m_1+2m_2}(L)$) to denote the sublattice of L generated by all $2m$-vectors (respectively $2m_1$-vectors and $2m_2$-vectors) in L. In §2, we prove

THEOREM 1. *Let L be a lattice in $\Gamma_{40,0}$. Then we have*

$$L = \mathcal{L}_{4+6}(L).$$

In §3,we shall introduce the notion of the c-sublattice of a lattice in $\Gamma_{40,0}$. We expect this notion would play a role in the study of the structures of lattices in $\Gamma_{40,0}$, and also in $\Gamma_{32,0}$. However our present study of the c sublattice is merely a beginning of exploration.

We collect some standard notations used throughout the paper: $\mathbf{Q}$ is the field of rational numbers, $\mathbf{Z}$ is the ring of rational integers, $\mathbf{M}(1,k)$ (respectively $\mathbf{S}(1,k)$) is the linear space of modular (respectively cusp) forms of degree 1 and weight k, $\mathbf{E}_k(\mathbf{z})$ is Eisenstein series of degree 1 and weight k, $\Delta_{12}(\mathbf{z})$ is the normalized cusp form of degree 1 and weight 12. Special notations are explained in the appropriate places if necessary.

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2. Some preliminaries and the proof of Theorem 1. Let L be an element of $\Gamma_{40,0}$. Then theta-series of degree 1 attached to L is defined by

$$\Theta(\mathbf{z}, L) = \sum_{\mathbf{x}\in L} \mathbf{e}((\mathbf{x},\mathbf{x})\mathbf{z}),$$

where $\mathbf{z}$ is the variable on the upper-half plane H and $e(\cdot) = \exp(\pi i)$. The theta-series with spherical function P_ν of degree ν attached to L is defined by

$$\Theta(\mathbf{z}, P_\nu, L) = \sum_{\mathbf{x}\in L} P_\nu(\mathbf{x};\alpha)\mathbf{e}((\mathbf{x},\mathbf{x})\mathbf{z}),$$

where α is a vector in $L \otimes_{\mathbf{Z}} \mathbf{Q}$. For the spherical function $P_\nu(\mathbf{x};\alpha)$, one may refer to [**1**], [**11**] or [**7**].

If we use the set $\Lambda_{2t}(L)$ defined by

$$\Lambda_{2t}(L) = \{\mathbf{x} \in L | (x,x) = 2t\},$$

and its cardinality

$$a(t,L) = \text{cardinality of } \Lambda_{2t}(L),$$

then $\Theta(z,L)$ can be rewritten as

$$\begin{aligned}\Theta(\mathbf{z}, L) &= \sum_{t=0}^{\infty} \sum_{\mathbf{x}\in\Lambda_{2t}(L)} \mathbf{e}((\mathbf{x},\mathbf{x})\mathbf{z})\\ &= \sum_{t=0}^{\infty} a(t,L)\mathbf{e}(2tz).\end{aligned}$$

The number $a(t,L)$ is the number of the solutions $\mathbf{x}$ in L of $(\mathbf{x},\mathbf{x}) = 2t$. Similarly we have

$$\Theta(\mathbf{z}, P_\nu, L) = \sum_{t=1}^{\infty} \sum_{\mathbf{x}\in\Lambda_{2t}(L)} P_\nu(\mathbf{x};\alpha)\mathbf{e}((\mathbf{x},\mathbf{x})\mathbf{z}).$$

It is known that

(1) $$\Theta(\mathbf{z}, L) \in \mathbf{M}(1,20), \qquad \Theta(\mathbf{z}, P_\nu, L) \in \mathbf{S}(1, 20+\nu)$$

and

(2) $$\dim \mathbf{M}(1,20) = 2, \quad \dim \mathbf{S}(1,20+\nu) = 1 \quad \text{for } \nu = 2,6$$
$$\text{and } \dim \mathbf{S}(1, 20+\nu) = 2 \text{ for } \nu = 4, 8.$$

We take $\mathbf{E}_4^5(\mathbf{z})$ and $E_4^2(\mathbf{z})\Delta_{12}(z)$ as the basis of $\mathbf{M}(1,20)$, and $\Theta(z, L)$ is expressed as

(3) $$\Theta(z, L) = E_4^5(\mathbf{z}) - 1200E_4^2(\mathbf{z})\Delta_{12}(\mathbf{z}).$$

The equation (3) is obtained by comparing the Fourier coefficients of three series in (3). We give some values of $a(t, L)$'s:
(4)
$$a(0,L) = 1, \quad a(1,L) = 0, \quad a(2,L) = 39600 \quad \text{and} \quad a(3,L) = 87859200.$$

Since $\Lambda_2(L)$ is the empty set, we must have

(5) $$\Theta(\mathbf{z}, P_2, L) = 0,$$
(6) $$\Theta(\mathbf{z}, P_4, L) = c_1\Delta_{12}^2(\mathbf{z}) \quad \text{with a constant } c_1,$$
(7) $$\Theta(\mathbf{z}, P_6, L) = 0$$
and
(8) $$\Theta(\mathbf{z}, P_8, L) = c_2E_4(\mathbf{z})\Delta_{12}^2(\mathbf{z}) \quad \text{with a constant } c_2.$$

Now we assume that $L \neq \mathcal{L}_{4+6}(L)$. Then the quotient module $L/\mathcal{L}_{4+6}(L)$ is not equal to $\{0\}$. We take a minimal non-zero representative w of $L/\mathcal{L}_{4+6}(L)$. Namely, w satisfies the conditions

(9) $$(\mathbf{w}, \mathbf{w}) \geq 8$$
and
(10) $$(\mathbf{w}, \mathbf{w}) \leq (\mathbf{x}, \mathbf{x}) \text{ for any } \mathbf{x} \equiv \mathbf{w} \bmod \mathcal{L}_{4+6}(L).$$

If we can show a contradiction, then we can conclude that $L = \mathcal{L}_{4+6}(L)$. This will imply Theorem 1.

To get more precise informations on w, we use the quantities

$$M_k(\mathbf{w}) = \text{the cardinality of } \{\mathbf{x} \in \Lambda_4(L) | (\mathbf{x}, \mathbf{w}) = k\}$$

and

$$N_k(\mathbf{w}) = \text{the cardinality of } \{y \in \Lambda_6(L) | (\mathbf{y}, \mathbf{w}) = k\}.$$

Clearly we have $M_k(\mathbf{w}) = M_{-k}(\mathbf{w})$ and $N_k(\mathbf{w}) = N_{-k}(\mathbf{w})$. The following lemma is easy to prove, and we give it without proof .

LEMMA 2-1. *Let w be a minimal non-zero representative in a residue class of $L/\mathcal{L}_{4+6}(L)$. Then we have*

$$M_k(\mathbf{w}) \neq 0 \text{ only when } k = 0, \pm 1, \pm 2 \tag{11}$$

and

$$N_k(\mathbf{w}) \neq 0 \text{ only when } k = 0, \pm 1, \pm 2, \pm 3. \tag{12}$$

Putting $\alpha = \mathbf{w}$ in the equations (5) and (7), it follows that

$$\sum_{\mathbf{x} \in \Lambda_4(L)} (\mathbf{x}, \mathbf{w})^2 = \frac{(\mathbf{w}, \mathbf{w})}{40} \sum_{\mathbf{x} \in \Lambda_4(L)} (\mathbf{x}, \mathbf{x}) = 3960(\mathbf{w}, \mathbf{w}), \tag{13}$$

and

$$\begin{aligned} \sum_{y \in \Lambda_6(L)} (\mathbf{y}, \mathbf{w})^2 &= \frac{(\mathbf{w}, \mathbf{w})}{40} \sum_{\mathbf{u} \in \Lambda_6(L)} (\mathbf{y}, \mathbf{y}) \\ &= 13178880(\mathbf{w}, \mathbf{w}). \end{aligned} \tag{14}$$

By means of (6) and the values of the Fourier coefficients of $\Delta_{12}^2(\mathbf{z})$, we get

$$\sum_{y \in \Lambda_6(L)} P_4(\mathbf{y}; \mathbf{w}) = -48 \sum_{\mathbf{x} \in \Lambda_4(L)} P_4(\mathbf{x}; \mathbf{w}). \tag{15}$$

Using the explicit expression of P_4, the equation (15) becomes
(16)

$$\sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^4 - \frac{9}{11}(\mathbf{w},\mathbf{w}) \sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^2 + \frac{(\mathbf{w},\mathbf{w})^2}{616} \sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{y})^2$$
$$= -48 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^4 + \frac{288(\mathbf{w},\mathbf{w})}{11} \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^2 - \frac{6(\mathbf{w},\mathbf{w})^2}{77} \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{x})^2.$$

Substituting (13) and (14) into (16), we get

$$\sum_{y\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^4 = -48 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^4 + 5702400(\mathbf{w},\mathbf{w})^2 \tag{17}$$

From (7), we obtain
(18)

$$\sum_{\mathbf{x}\in\Lambda_4(L)} P_6(\mathbf{x},\mathbf{w})$$
$$= \sum_{\mathbf{x}\in\Lambda_4(L)} \Big((\mathbf{x},\mathbf{w})^6 - \frac{15}{48}(\mathbf{x},\mathbf{w})^4(\mathbf{x},\mathbf{x})(\mathbf{w},\mathbf{w}) + \frac{45}{48\cdot 46}(\mathbf{x},\mathbf{w})^2(\mathbf{x},\mathbf{x})^2(\mathbf{w},\mathbf{w})^2 - \frac{15(\mathbf{x},\mathbf{x})^3(\mathbf{w},\mathbf{w})^3}{48\cdot 46\cdot 44}\Big) = 0,$$

and
(19)

$$\sum_{\mathbf{y}\in\Lambda_6(L)} P_6(\mathbf{y};\mathbf{w})$$
$$= \sum_{\mathbf{y}\in\Lambda_6(L)} \Big((\mathbf{y},\mathbf{w})^6 - \frac{15}{48}(\mathbf{w},\mathbf{w})(\mathbf{y},\mathbf{y})(\mathbf{y},\mathbf{w})^4 + \frac{45}{48\cdot 46}(\mathbf{y},\mathbf{y})^2(\mathbf{w},\mathbf{w})^2(\mathbf{y},\mathbf{w})^2 - \frac{15(\mathbf{y},\mathbf{y})^3(\mathbf{w},\mathbf{w})^3}{48\cdot 46\cdot 44}\Big) = 0.$$

Using (13) and (14), the equations (18) and (19), respectively, become

$$\sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^6 = \frac{5}{4}(\mathbf{w},\mathbf{w}) \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^4 - 900(\mathbf{w},\mathbf{w})^3 \tag{20}$$

and

$$\sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^6 = \frac{15}{8}(\mathbf{w},\mathbf{w}) \sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^4 - 6739200(\mathbf{w},\mathbf{w})^3, \tag{21}$$

respectively. By means of (8) and the values of the Fourier coefficients of $\mathbf{E}_4(\mathbf{z})\Delta_{12}^2(\mathbf{z})$ we get

$$\sum_{\mathbf{y}\in\Lambda_6(L)} P_8(\mathbf{y};\mathbf{w}) = 192 \sum_{\mathbf{x}\in\Lambda_4(L)} P_8(\mathbf{x};\mathbf{w}). \tag{22}$$

Using explicit expression of P_8 given in [**7**], the equation (22) becomes

(23)

$$\begin{aligned}
&\sum_{\mathbf{y}\in\Lambda_6(L)} \Big((\mathbf{y},\mathbf{w})^8 - \frac{7}{13}(\mathbf{y},\mathbf{w})^6(\mathbf{y},\mathbf{y})(\mathbf{w},\mathbf{w}) + \frac{21}{260}(\mathbf{y},\mathbf{w})^4(\mathbf{y},\mathbf{y})^2(\mathbf{w},\mathbf{w})^2 \\
&\quad - \frac{7}{2080}(\mathbf{y},\mathbf{w})^2(\mathbf{y},\mathbf{y})^3(\mathbf{w},\mathbf{w})^3 + \frac{7}{382720}(\mathbf{y},\mathbf{y})^4(\mathbf{w},\mathbf{w})^4\Big) \\
&= 192 \sum_{\mathbf{x}\in\Lambda_4(L)} \Big((\mathbf{x},\mathbf{w})^8 - \frac{7}{13}(\mathbf{x},\mathbf{w})^6(\mathbf{x},\mathbf{x})(\mathbf{w},\mathbf{w}) + \frac{21}{260}(\mathbf{x},\mathbf{w})^4(\mathbf{x},\mathbf{x})^2(\mathbf{w},\mathbf{w})^2 \\
&\quad - \frac{7}{2080}(\mathbf{x},\mathbf{w})^2(\mathbf{x},\mathbf{x})^3(\mathbf{w},\mathbf{w})^3 + \frac{7(\mathbf{x},\mathbf{x})^4(\mathbf{w},\mathbf{w})^4}{382720}\Big).
\end{aligned}$$

Substituting (13), (14), (20), (21) into (23) and rearranging the equation, we get

$$\begin{aligned}
&\sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^8 - \frac{63}{20}(\mathbf{w},\mathbf{w})^2 \sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^4 + 14031360(\mathbf{w},\mathbf{w})^4 \\
&\qquad = 192 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^8 - \frac{1344}{5}(\mathbf{w},\mathbf{w})^2 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^4.
\end{aligned} \tag{24}$$

The equation (24) is transformed, by using (17), into

(25)

$$\begin{aligned}
\sum_{\mathbf{y}\in\Lambda_6(L)} (\mathbf{y},\mathbf{w})^8 = 192 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^8 - 420(\mathbf{w},\mathbf{w})^2 \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^4 \\
+ 3931200(\mathbf{w},\mathbf{w})^4.
\end{aligned}$$

Taking Lemma 2-1 into account, we see that

$$(26)\qquad \sum_{\mathbf{x}\in\Lambda_4(L)} (\mathbf{x},\mathbf{w})^r = \Big(M_1(\mathbf{w}) + 2^r M_2(\mathbf{w})\Big)$$

and

$$(27)\qquad \sum_{\mathbf{y}\in\Lambda_6(L)} (y,w)^r = 2\Big(N_1(\mathbf{w}) + 2^r N_2(\mathbf{w}) + 3^r N_3(\mathbf{w})\Big),$$

where r is a positive even integer ≥ 2. In terms of (26) and (27), the equations (13), (14),(17), (20) and (25) respectively can be transformed, by putting $(\mathbf{w},\mathbf{w}) = m$, into

$$(28)\, M_1(\mathbf{w}) + 4M_2(\mathbf{w}) = 1980m,$$

$$(29)\, N_1(\mathbf{w}) + 4N_2(\mathbf{w}) + 9N_3(\mathbf{w}) = 6589440m,$$

$$(30)\, N_1(\mathbf{w}) + 16N_2(\mathbf{w}) + 81N_3(\mathbf{w}) = -48\Big(M_1(\mathbf{w}) + 16M_2(\mathbf{w})\Big) + 2851200m^2,$$

$$(31)\, M_1(\mathbf{w}) + 64M_2(\mathbf{w}) = \frac{5m}{4}\Big(M_1(\mathbf{w}) + 16M_2(\mathbf{w})\Big) - 450m^3,$$

$$(32)\qquad N_1(\mathbf{w}) + 64N_2(\mathbf{w}) + 729N_3(\mathbf{w}) = \frac{15m}{8}\Big(N_1(\mathbf{w}) + 16N_2(\mathbf{w}) + 81N_3(\mathbf{w})\Big) - 3369600m^3$$

and

$$(33)\, N_1(\mathbf{w}) + 256N_2(\mathbf{w}) + 6561N_3(\mathbf{w}) = 192\Big(M_1(\mathbf{w}) + 256M_2(\mathbf{w})\Big) - 420m^2\Big(M_1(\mathbf{w}) + 16M_2(\mathbf{w})\Big) + 1965600m^4$$

respectively. We can easily solve the equations (28), (29), (30), (31) and (32), and the solutions are given by

$$(34)\, M_1(w) = -24m(5m^2 - 110m + 352)/(m-4),$$

$$(35)\, M_2(w) = m(30m^2 - 165m + 132)/(m-4),$$

$$(36)\, N_1(w) = m(81000m^3 - 1864440m^2 + 16085520m - 39701376)/(m-4),$$

$$(37)\, N_2(w) = m(-32400m^3 + 604080m^2 - 2898720m + 4004352)/(m-4)$$

and

(38) $N_3(w) = m(5400m^3 - 61320m^2 + 233200m - 297088)/(m-4)$.

Here we show that if there is a minimal representative $\mathbf{w}$ of $L/\mathcal{L}_{4+6}(L)$ with $(\mathbf{w}, \mathbf{w}) \geq 8$, for a lattice L in $\Gamma_{40,0}$, then we reach a contradiction. We remark that $N_k(\mathbf{w})$ and $M_k(\mathbf{w})$ are non-negative integers. If $(\mathbf{w}, \mathbf{w}) = m \geq 20$, then we get $M_1(\mathbf{w}) < 0$, which contradicts the nature of $M_1(\mathbf{w})$. Therefore, we may assume $m \leq 18$.

For even m with $8 \leq m \leq 18$, we give the reasons for the impossibility of $(w, w) = m$. For $m = 14$ (respectively $m = 18$), we have

$$M_2(w) = 14 \times 3702/10 \notin \mathbf{Z}$$

$$(\text{respectively } M_2(w) = 18 \times 6882/14 \notin \mathbf{Z}).$$

For $m = 16, N_2(\mathbf{w}) = -4 \times 20441088/3 < 0$. For $m = 12$, 10 or 8, we get the positive integers $M_1(\mathbf{w}), M_2(\mathbf{w}), N_1(\mathbf{w}), N_2(\mathbf{w})$ and $N_3(\mathbf{w})$, but these values do not satisfy the condition (33). We have thus established the Theorem 1. □

REMARK 1. One may suspect that the equality $\mathcal{L}_4(L) = L$ would hold for $L \in \Gamma_{40,0}$. However, this is not always true. The examples in [**9**] show that $\mathcal{L}_{4+6}(L) = L$ and $\mathcal{L}_4(L) \neq L$ for some $L \in \Gamma_{40,0}$. This question is closely connected with the existence of the c-sublattice, which is introduced in the next section, of type D_{40} in $L \in \Gamma_{40,0}$.

3. Introduction of the notion of the c-sublattice. Throughout this section, L is a lattice in $\Gamma_{40,0}$. A *c-sublattice* M of L is defined as follows.

DEFINITION. Let T be a system of 4-vectors satisfying

(i) for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in T, $(\mathbf{x}, \mathbf{y}) \equiv 0 (\text{mod}\, 2)$;

(ii) there is no larger system of 4-vectors T_1, which contains T, satisfying (i).

M is the sublattice of L generated by T over $\mathbf{Z}$.

We say an integral lattice is a root lattice if it has a basis consisting of 2-vectors.

PROPOSITION 3-1. *Let the notations be as above. The c-sublattice M of L is similar to a root lattice.*

PROOF. Let $\mathbf{x}_1, \ldots, \mathbf{x}_t$ be all elements of T, which defines M. Putting

$$\mathbf{y}_i = 1/\sqrt{2}\mathbf{x}_i, \quad 1 \leq i \leq t,$$

then $\mathbf{y}_1, \ldots, \mathbf{y}_t$ *are 2-vectors satisfying*

$$(\mathbf{y}_i, \mathbf{y}_j) \in \mathbf{Z} \tag{39}$$

The last condition (39) is guaranteed by the condition (i) for T. Therefore the lattice U generated by 2-vectors $\mathbf{y}_1, \ldots, \mathbf{y}_t$ is an integral lattice. By Proposition 2-2 in [**5**], the lattice U has a basis consisting of 2-vectors, and U is a root lattice. □

By the above proposition, the c-sublattice M is similar to an orthogonal sum of irreducible root lattices $A_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7 and E_8 (see, e.g., [**5**]), and the scaling factor is always $1/\sqrt{2}$.

In what follows, we assume that

$$(*) \qquad \text{the } c-\text{sublattice } M \text{ of } L \text{ is similar to } D_{40}.$$

We can give examples of lattices L in $\Gamma_{40,0}$ satisfying the above assumption. In fact, let C be any one of doubly even [**40, 20, 8**] binary codes given in [**8**], then the construction A in [**2**] gives the lattice L_1 in Γ_{40} of type $40 \times A_1$. An adjacent lattice L of L_1 in the sense of M. Kneser, which has no 2-vectors, is shown to have the c-sublattice satisfying the assumption (*).

REMARK 2. It is obvious that the notion of the c-sublattice can be defined also for $\Gamma_{24,0}$ and $\Gamma_{32,0}$. and for any $L \in \Gamma_{24,0}$, we can show that the c-sublattice of L is similar to D_{24}. In this case, the modular form theory works quite well, and this fact leads to a characterization of the Leech lattice. In the cases $\Gamma_{32,0}$ and $\Gamma_{40,0}$, at present we cannot say whether the c-sublattice M of L in $\Gamma_{32,0}$ (respectively $\Gamma_{40,0}$) is similar to D_{32} (respectively D_{40}). This time the modular form theory does not help.

Without loss of generality, we may assume that M is generated by $\pm f_i \pm f_j$ which satisfy

$$(\mathbf{f}_i, \mathbf{f}_j) = 2\delta_{ij}, \quad \text{for} \quad 1 \leq i \leq j \leq 40,$$

where δ_{ij} is Kronecker delta. We say that a vector $\mathbf{x}$ in L is a *minimal representative of the equivalence class* $\mathbf{x} + M$ if $\mathbf{x}$ satisfies

$$(\mathbf{x}, \mathbf{x}) \leq (\mathbf{y}, \mathbf{y}), \quad \text{for all} \quad \mathbf{y} \in \mathbf{x} + M. \tag{40}$$

By assumption $(*)$ rank $L =$ rank $M = 40$, and any vector x in L can be written as

$$\mathbf{x} = a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \cdots + a_{40}\mathbf{f}_{40} \quad \text{with} \quad a_i \in \mathbf{Q}, \quad 1 \leq i \leq 40. \tag{41}$$

The following is easy to prove.

LEMMA 3-2. *Let the notations be as above. Then the coefficients a_i of a vector* $\mathbf{x}$ *of the form* (41) *satisfy either*

$$\text{all } a_i \text{ belong to } \mathbf{Z}/4 - \mathbf{Z}/2 \tag{42}$$

or

$$\text{all } a_i \text{ belong to } \mathbf{Z}/2. \tag{43}$$

We call a vector $\mathbf{x}$ of the form (41) with condition (42) (respectively (43)) a vector *of the first kind* (respectively the second kind). The set of the vectors of the second kind in L forms a sublattice J of L, and we see that

$$L \supset J \supset M.$$

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors of the first kind. Then each coefficient of $\mathbf{x} - \mathbf{y}$ belongs to $\mathbf{Z}/2$. This implies that

$$\mathbf{x} \equiv \mathbf{y} (\operatorname{mod} J).$$

We can conclude that

$$L = J + (J + \tau) \quad (\text{ a disjoint union}),$$

where τ is a vector of the first kind.

After a brief consideration it is easy to see that the coefficients of a minimal vector τ of the first kind satisfy one of the following two conditions.

$$\text{all } a_i = \pm 1/4 \quad (i = 1, \ldots, 40) \tag{44}$$

or

$$39\ a_i\text{'s} = \pm 1/4 \text{ and one } a_i = \pm 3/4. \tag{45}$$

We remark that the vector τ satisfying the condition (44) is non-existent because, for such τ, (τ, τ) is not an even integer. It is easy to see that the vector τ satisfying the condition (45) is equivalent modulo M to

$$\sigma = 1/4(\pm\mathbf{f}_1 \pm \cdots \pm \mathbf{f}_{39} \pm 3\mathbf{f}_{40}). \tag{46}$$

Without loss of generality, we may assume that the vector

$$\tau = 1/4(f_1 + \cdots + f_{39} - 3f_{40}) \in L.$$

LEMMA 3-3. *Let J and M be the sublattices of $L \in \Gamma_{40,0}$ defined as above. Then we have*

(i) *A vector $\mathbf{x}$ of the form* (41), *which belongs to J, is a minimal representative of a class in J/M if and only if all a_i equal to zero or $\mathbf{x}$ takes the form*

$$\mathbf{x} = \frac{1}{2}(\rho_{i_1}\mathbf{f}_{i_1} + \cdots + \rho_{i_r}\mathbf{f}_{i_r}), \tag{47}$$

where each $\rho_{i_k} = \pm 1$;

(ii) *The coefficients ρ_{i_k} in* (47) *satisfy $r \equiv 0 \pmod 4$ and*

$$\prod_{k=1}^{r} \rho_{i_k} = 1. \tag{48}$$

PROOF. (i). By the definition of the lattice J, the coefficients a_i of an element $\mathbf{x}$ of J are integers or half-integers. Similar to the above discussion for the vector τ of the first kind, one sees that if $\mathbf{x}'$ is a minimal representative of a class in J/M, then $\mathbf{x}'$ is expressible either as

$$\mathbf{x}' = \frac{1}{2}(\lambda_{i_l}\mathbf{f}_{i_l} + \cdots + \lambda_{i_r}\mathbf{f}_{i_r}) + a_j\mathbf{f}_j \text{ with } \lambda_{i_k}, a_j = \pm 1 \tag{49}$$

$$j \neq \lambda_{i_1}, \ldots, \lambda_{i_k}$$

or is of the form (47). But the vector $\mathbf{x}'$ of the form (49) is not minimal. Conversely, the zero vector or the vector of the form (47) cannot be minimized any more. This completes the proof of (i).

(ii) . We see that

$$(\mathbf{x}, \mathbf{x}) = \frac{1}{2}\sum_{k=1}^{r} \rho_{i_k}^2 = \frac{1}{2}r$$

and

$$(\tau, x) = \begin{cases} \frac{1}{4}(\rho_{i_l} + \cdots + \rho_{i_r}) & \text{if } i_r < 40 \\ \frac{1}{4}(\rho_{i_l} + \cdots + \rho_{i_{r-1}} - 3\rho_{i_r}) & \text{if } i_r = 40. \end{cases} \tag{50}$$

Since L is an even lattice, r must satisfy $r \equiv 0 (\mathrm{mod}\, 4)$. Since L is integral, from (50) we have

$$\rho_{i_l} + \cdots + \rho_{i_r} \equiv 0 (\mathrm{mod}\, 4). \tag{51}$$

Under the condition $r \equiv 0(\mathrm{mod}\, 4)$, the condition (51) is equivalent to (48). This completes the proof of (ii). □

By Lemma 3-3, we can take $\mathbf{x}$ of the form (47) satisfying $r \equiv 0(\mathrm{mod}\, 4)$ and (48). Any such vector is equivalent modulo M to a unique vector

$$w = 1/2(\mathbf{f}_{i_1} + \mathbf{f}_{i_2} + \cdots + \mathbf{f}_{i_r}). \tag{52}$$

DEFINITION. We call a minimal representative of the form (52) *the canonical representative of* J/M. We take the zero vector as the canonical representative of M.

We define a mapping φ from J/M to the 40 dimensional vector space $\mathbf{F}_2^{40}$ over the field of 2 elements $\mathbf{F}_2 = GF(2)$. Let $\mathbf{x}$ be a non-zero canonical representative of the form (52), then $\varphi(x) = \text{supp}\,(\mathbf{x}) = (x_i) \in \mathbf{F}_2^{40}$ (the support of $\mathbf{x}$) is the binary vector whose coordinates are given by

$$x_i = \begin{cases} 1 & \text{if } i = i_l, \cdots, i_r \\ 0 & \text{if } i \neq i_l, \cdots, i_r \end{cases} \quad 1 \leq i \leq 40.$$

To the zero vector $\mathbf{x}_0$, we define $\varphi(\mathbf{x}_0) = 0 \in \mathbf{F}_2^{40}$. The weight $wt(\text{supp}\,(\mathbf{x}))$ of the binary vector $\text{supp}\,(\mathbf{x})$ is defined to be the number of non-zero coordinates of $\text{supp}\,(\mathbf{x})$.

Let $\mathbf{x}$ and $\mathbf{y}$ be two non-zero canonical different representatives of J/M, then $\mathbf{x}+\mathbf{y}$ is not necessarily minimal.

DEFINITION. We define $\text{supp}\,(\mathbf{x}) * \text{supp}\,(\mathbf{y})$ as the number of the coordinates of $\varphi(\mathbf{x})$ and $\varphi(\mathbf{y})$ taking the value 1 in common. Let $\mathbf{w}$ be the canonical representative of the class to which $\mathbf{x}+\mathbf{y}$ belongs, then we easily see that

$$(53)\quad wt(\text{supp}\,(w)) = wt(\text{supp}\,(x)) + wt(\text{supp}\,(\mathbf{y})) - 2\text{supp}\,(\mathbf{x}) * \text{supp}\,(\mathbf{y})$$

and

$$(54)\qquad \text{supp}\,(\mathbf{x}) \cdot \text{supp}\,(\mathbf{y}) \equiv \text{supp}\,(\mathbf{x}) * \text{supp}\,(\mathbf{y}) (\text{mod}\, 2),$$

where $\text{supp}\,(x) \cdot \text{supp}\,(y)$ is the inner product of the space $\mathbf{F}_2^{40}$. We define $\varphi(\mathbf{x}+\mathbf{y})$ by $\varphi(\mathbf{w})$. Then we can verify that

$$(55)\qquad \varphi(\mathbf{x}+\mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y})$$

holds.

The next theorem connects J/M with a code. For the notion and the standard notations in the coding theory one may refer [**3**] or [**10**]. We prove

THEOREM 2. *Let L be an element of $\Gamma_{40,0}$. Assume that the c-sublattice M of L is similar to D_{40}. J is sublattice defined as above.*

Then the mapping φ defines an isomorphism between J/M and a doubly even binary $[40, 20]$ code C with non-zero minimal weight 8.

PROOF. Let $\mathbf{x}$ be a canonical representative of a class $\neq M$ in J/M. By Lemma 3-3, (ii), we see that

$$wt(\operatorname{supp}(\mathbf{x})) = r \equiv 0 (\operatorname{mod} 4) \tag{56}$$

and

$$(\mathbf{x}, \mathbf{x}) = r/2 \tag{57}$$

Let $\mathbf{x}$ and $\mathbf{y}$ be two canonical representatives of different classes in J/M, and $\mathbf{w}$ be the canonical representative of the class to which $\mathbf{x}+\mathbf{y}$ belongs. By (53) and (56) we see that

$$\operatorname{supp}(\mathbf{x}) * \operatorname{supp}(\mathbf{y}) \equiv 0 (\operatorname{mod} 2). \tag{58}$$

By appealing to Theorem 4 in [**10**], the properties (54), (56) and (58) imply that C, the image of J/M under φ, is a doubly even self-orthogonal binary code of length 40. By (57) and that L does not contain a 2-vector, we get

$$wt(\operatorname{supp}(\mathbf{x})) \geq 8.$$

It remains to prove that C is self-dual. We let $M^{\#}$ denote the dual lattice of M. Since M is similar to D_{40} with scaling factor $\sqrt{2}$, the determinant $d(M)$ of the lattice is given by

$$\begin{aligned} d(M) &= 2^{40} d(D_{40}) \\ &= 2^{42}. \end{aligned}$$

It is known (e.g., [**4**]) that

$$\begin{aligned} d(M) = [M^{\#} : M] & \\ &= [M^{\#} : J^{\#}][J^{\#} : L][L : J][J : M], \\ [J : M] &= [M^{\#} : J^{\#}] \end{aligned}$$

and

$$[J^{\#} : L] = [L : J].$$

Since we see that $[L : J] = 2$, we get

$$[J : M] = 2^{20}. \tag{59}$$

Since φ is an injective map from J/M into $\mathbf{F}_2^{40}$, the self-duality of C follows from (59). □

4. Concluding remarks. In the course of §3, although we restricted our consideration of the first kind vector to

$$\tau = \frac{1}{4}(\mathbf{f}_1 + \cdots \mathbf{f}_{39} - 3\mathbf{f}_{40}),$$

it is provable that any vector of the first kind leads to a unique isomorphism class in $\Gamma_{40,0}$. With this fact, we can prove that two isomorphic lattices L_1 and L_2 in $\Gamma_{40,0}$ with their c-sublattices similar to D_{40} induce equivalent $[40, 20, 8]$ binary codes. In [**8**], we give three inequivalent such codes. From these codes three non-isomorphic lattices L_1, L_2 and L_3 in $\Gamma_{40,0}$ arise (Conf. [**9**]) with the property that $L_i = \mathcal{L}_{4+6}(L_i)$ and $L_i \neq \mathcal{L}_4(L_i)$ $(i = 1, 2, 3)$.

Perhaps $\Gamma_{40,0}$ is the last stage where the binary codes play a role in constructing the even unimodular extremal lattices. In the construction (or the understanding) of even unimodular extremal lattices of rank 48 (respectively 56 and 64), some ternary codes may play a role. However, the notion of c-sublattice can be applied even to these cases by adjusting $1/\sqrt{3}$ as the scaling factor instead of $1/\sqrt{2}$.

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DEPARTMENT OF INFORMATION SCIENCE, HIROSAKI UNIVERSITY, 3, BUNKYO-CHO, HIROSAKI AOMORI, JAPAN