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u = 4 AND QUADRATIC EXTENSIONS

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1. Introduction. Throughout this paper L will denote a nonformally real field of characteristic $\neq 2$. By $u(L)$ we mean the *u-invariant* of L, i.e., $u(L) = \max\{n \in \mathbf{N}$: there exists an n-dimensional anisotropic quadratic form over $L\}$ (see [**16**, Chapter X]).

The motivation for our work is the following conjecture, which is part of the folklore of quadratic forms theory.

CONJECTURE 1.1. *If $a \in L$, then $u(L(\sqrt{a})) = u(L)$.*

In §2 we will present a couple of examples related to this conjecture in the particular case when $u(L) = 4$. Our strategy is to translate the condition $u(L) = 4$ into the Galois theory of L and then use some well known results on the cohomology of pro - 2 - groups. For the basic concepts and notation we use, the reader may consult [**16**] and [**28**].

One of our main tools will be the following important result. We let $\mathrm{cd}_p(G)$ denote the cohomological p - dimensional of the pro - p - group G (see [**28**, p. I–17]).

THEOREM 1.2. (SERRE [**30**]). *Let G be a pro-p-group that does not contain an element of order p and let H be an open subgroup of G. Then* $\mathrm{cd}_p(G) = \mathrm{cd}_p(H)$.

We now let $L(2) :=$ quadratic closure of L and $G_L := \mathrm{Gal}(L(2)/L)$. Then Theorem 1.2 (with $p = 2$) applies to $G = G_L$ since nonformally real fields L are characterized by the fact that G_L does not contain nontrivial involutions [**7**, Chapter 2, Theorem 3].

To our knowledge the first explicit connection between $\mathrm{cd}_2(G_L)$ and $u(L)$ was found by Ware in [**34**], where it was shown that $u(L) = 2 \Leftrightarrow$

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$\mathrm{cd}_2(G) = 1$. Notice that $u(L) = 1 \Leftrightarrow \mathrm{cd}_2(G_L) = 0$ is trivial since $u(L) = 1 \Leftrightarrow L$ is quadratically closed $\Leftrightarrow G_L = \{1\}$. We can now give a quick proof of the following well known result.

PROPOSITION 1.3. *Let L be a nonformally real field and let $a \in L$. Then*

(1) $u(L) = 2 \Rightarrow u(L(\sqrt{a})) = 2$;

(2) $u(L) > 2 \Rightarrow u(L(\sqrt{a})) > 2$.

Thus $u(L) = 2 \Leftrightarrow u(L(\sqrt{a})) = 2$.

PROOF. We let $G_{L(\sqrt{a})}$ be the subgroup of G_L corresponding to $L(\sqrt{a})$. Then $G_{L(\sqrt{a})}$ is open in G_L, so $\mathrm{cd}_2(G_{L(\sqrt{a})}) = \mathrm{cd}_2(G_L)$ by Theorem 1.2. The result then follows by the preceding comments. □

REMARK. Statement (1) above can easily be proved by standard methods in quadratic forms theory.

We denote by $H^i(G_L)$ the i^{th} cohomology group of G_L with coefficients in $\mathbf{Z}/2\mathbf{Z}$. There exist canonical isomorphisms $\dot{L}/\dot{L}^2 \cong H^1(G_L)$ and $Br_2(L) \cong H^2(G_L)$ given respectively by $a\dot{L}^2 \mapsto (a)$ and $\left[\left(\frac{a,b}{L}\right)\right] \mapsto (a) \cup (b)$ for $a, b \in \dot{L}$ [**29**, Chapter XIV]. Merkurjev's theorem ([**20**]; see also [**2**], [**32**]) states that the classes of quaternion algebras generate $Br_2(F)$, so the latter is indeed a good description of the isomorphism $Br_2(L) \cong H^2(G_L)$.

Our next lemma is a strengthening of [**11**, Theorem 4.7] using Merkurjev's theorem and results in [**3**] and [**4**]. Recall that a field is *linked* if the classes quaternion algebras form a subgroup of $Br_2(L)$. In view of the comments above this is the same as saying that the cup map $H^1(G_L) \times H^1(G_L) \to H^2(G_L)$ is surjective.

LEMMA 1.4. *Let L be a nonformally real field. Then $u(L) = 4$ if and only if following two conditions hold:*

(1) $\mathrm{cd}_2(G_L) = 2$;

(2) *the cup map* $H^1(G_L) \times H^1(G_L) \to H^2(G_L)$ *is surjective.*

PROOF. Suppose $u(L) = 4$. Then, by [**11**, Theorem 4.7] (see also [**12**]), L is linked and $I^3L = \{0\}$; also $\mathrm{Br}_2(L) \neq \{1\}$ because $u(L) \geq 4$. From the preceding comments we see that (2) holds; and from $I^3L = \{0\}$ it follows by [**3**, Property 5.16] that $\mathrm{cd}_2(G_L) \leq 2$. Therefore (1) holds.

Now suppose (1) and (2) hold. Then $H^2(G_L) \neq \{0\}$ and $H^3(G_L) = \{0\}$. Since $\mathrm{cd}_2(G_L) \leq 3$ it follows from [**4**, Theorem 3] that $I^2L/I^3L \cong H^2(G_L)$ and $I^3L/I^4L \cong H^3(G_L)$. Therefore $I^2L \neq \{0\}$ and $I^3L = I^4L$; by the Arason - Pfister Korollar 2 [**6**] we see that $I^nL = \{0\}$ for $n \geq 3$. Again by [**11**, Theorem 4.7], we conclude that $u(L) = 4$. □

We remark that there is another conjecture relating the notions of cohomological dimension and the u - invariant:

CONJECTURE 1.5. *If* L *is a nonformally real field, then* $u(L) = 2^{\mathrm{cd}_2(G_L)}$.

(Note added in proof: Recently A.S. Merkurjev showed that $u(L)$ can be any even number. In particular, Conjecture 1.5 is false.)

Notice that Conjecture 1.5 implies Conjecture 1.1. Indeed, we have $\mathrm{cd}_2(G_{L(\sqrt{a})}) = \mathrm{cd}_2(G_L)$ for any $a \in i$ by Theorem 1.2, so, if we assume Conjecture 1.5, we get

$$u(L) = 2^{\mathrm{cd}_2(G_L)} = 2^{\mathrm{cd}_2(G_{L(\sqrt{a})})} = u(L(\sqrt{a})).$$

Conjecture 1.5 is true in the following cases:

(1) $L = F(\sqrt{-1})$ where F is a formally real pythagorean field of finite chain length [**21**], [**22**].

(2) L is a finitely generated extension of a hereditarily quadratically closed field F [**13**].

(3) L is a finite nonformally real 2 - extension of a superpythagorean field. This follows easily from results in [**33**] (see Case 1 in the proof of Proposition 1.6).

We finish this section by giving one more example related to Conjecture 1.1. This example seems to be well known, but since we cannot find a reference, we provide a proof for the reader's convenience.

Recall that an element $a \in \dot{F} \setminus \pm \dot{F}^2$ is *rigid* if $D_F\langle 1, a\rangle = \dot{F}^2 \cup a\dot{F}^2$, *double rigid* if both a and $-a$ are rigid, and that F is a C - *field* if all $a \in \dot{F} \setminus \pm \dot{F}^2$ are rigid. In [**8**] it is shown that in the nonformally real field case the notions of rigid and double rigid coincide (see remarks after the next proposition).

PROPOSITION 1.6. *Let L be a nonformally real field and $a \in \dot{L}$ a rigid element. Then $u(L(\sqrt{a})) = u(L)$.*

PROOF. Let B be the set of non rigid elements of L. In particular $a \notin B$. We divide our proof into two cases:

Case 1. $B = \dot{L}^2 \cup -\dot{L}^2$. In this case L is a C - field and therefore so is $L(\sqrt{a})$ by [**33**, Corollary 2.8]. The square class exact sequence [**16**, Theorem VII. 3.4] shows that $|\dot{L}/\dot{L}^2| = |\dot{L}(\sqrt{a})/\dot{L}(\sqrt{a})^2|$ if $|\dot{L}/\dot{L}^2| < \infty$ and that if $|\dot{L}/\dot{L}^2| = \infty$, then $|\dot{L}(\sqrt{a})/\dot{L}(\sqrt{a})^2| = \infty$, too. In the first case we have $u(L) = |\dot{L}/\dot{L}^2| = u(L(\sqrt{a}))$ [**33**, Example 1.11(iii)]. In the second case we get $u(L) = u(L(\sqrt{a})) = \infty$ since it is easy to see that a C - filed with infinitely many square classes has anisotropic forms of arbitrarily large dimension.

Case 2. $B \neq \dot{L}^2 \cup -\dot{L}^2$. By [**35**, Theorem 2.16] there exists a 2 - Henselian valuation v on L such that $v(a)$ is not 2 - divisible in the value group Γ_v of v. By [**5**, Lemma 4.4] it follows that v is non - dyadic. Denote by L_v the residue field of L with respect to v. Since v is 2 - Henselian, there exists a unique extension w of v to $K := L(\sqrt{a})$. We denote by Γ_w and K_w the value group and residue field of w respectively, and as usual we consider $\Gamma_v \subset \Gamma_w$ and $K_v \subset K_w$.

The fundamental inequality for extension of valuations [**27**, Proposition G.4] shows that

$$[K_w : L_v][\Gamma_w : \Gamma_v] \leq 2$$

and, in particular,

$$[\Gamma_w : \Gamma_v] \leq 2.$$

Now $w(a) \in \Gamma_w$ is 2 - divisible since $w(a) = 2w(\sqrt{a})$. Since $v(a) \in \Gamma_v$ is not 2 - divisible, we conclude that $\Gamma_v \subsetneq \Gamma_w$, and hence

$$[\Gamma_w : \Gamma_v] = 2 \quad \text{and} \quad [K_w : L_v] = 1.$$

The second equality above shows that $L_v \cong K_w$. From the first one we see that $2\Gamma_w \subset \Gamma_v$. Furthermore, $w(a) \in 2\Gamma_w \backslash 2\Gamma_v$ which shows that $2\Gamma_v \subsetneq 2\Gamma_w$, i.e.,

$$1 \neq [2\Gamma_w : 2\Gamma_v].$$

Since

$$[2\Gamma_w : 2\Gamma_v] \leq [\Gamma_w : \Gamma_v] = 2$$

we conclude that

$$[2\Gamma_w : 2\Gamma_v] = 2.$$

Hence

$$\begin{aligned}\dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_w/2\Gamma_w] &= \dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_w/\Gamma_v] + \dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_v/2\Gamma_w] \\ &= \dim_{\mathbf{Z}/2\mathbf{Z}}[2\Gamma_w/2\Gamma_v] + \dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_v/2\Gamma_w] \\ &= \dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_v/2\Gamma_v].\end{aligned}$$

Since v is nondyadic we can use Springer theory (see [**17**, p. 36]). Set $m = \dim_{\mathbf{Z}/2\mathbf{Z}}[\Gamma_w/2\Gamma_w]$; then

$$\begin{aligned}u(L) &= 2^m u(L_v) \\ u(K) &= 2^m u(K_w)\end{aligned}$$

or $u(L) = u(K) = \infty$ if $m = \infty$. Since we have seen that $L_v = K_w$, we conclude that $u(L) = u(K)$. □

REMARK 1.7. In [**8**] rigidity is defined as follows: an element $a \in \dot{L}$ is rigid if $D_L\langle 1, a\rangle = R(L) \cup aR(L)$ where $R(L)$ denotes the Kaplansky radical of F. $R(L)$ can be defined as

$$R(L) = \cap_{b \in \dot{L}} D_L\langle 1, b\rangle.$$

It is then easy to check that the existence of a rigid element (according to our previous definition) forces the two definitions of "rigid" to coincide.

REMARK 1.8. Note that, in the proof of Proposition 1.6, we actually proved that $W(L(\sqrt{a})) \cong W(L)$. Indeed $W(L(\sqrt{a})) \cong W(K_w)[\Gamma_w/2\Gamma_w] \cong W(L_v)[\Gamma_v/2\Gamma_v] \cong W(L)$, cf. [**35**]. (Added in proof: This was done earlier by L. Berman, see Pacific J. Math. **89** (1980), Corollary **26**, p. 261.

2. Examples.

DEFINITION 2.1. Let F be a field and $b \in \dot{F}$. We say that F is *uniformly linked with slot* b if every quaternion algebra over F is equivalent to $(\frac{b,x}{F})$ for some $x \in \dot{F}$.

The preceding definition was suggested by B. Jacob. To him we also owe the statement of the next proposition, which is an improvement of our previous version. Recall that $R(L)$ denotes the Kaplansky radical of L (cf. Remark 1.7).

PROPOSITION 2.2. *Let L be a nonformally real field such that*

(1) $\mathrm{cd}_2 G_L = 2$,

(2) *L is uniformly linked with slot b.*

Then $u(L) = 4$. If $a \in R(L)$ then (1) *and* (2) *also hold for $L(\sqrt{a})$ (in particular* (2) *holds with the same slot* (b) *and therefore $u(L(\sqrt{a})) = 4$.*

PROOF. From Lemma 1.4 and Merkurjev's theorem, it follows immediately that $u(L) = 4$. The fact that (1) holds for $L(\sqrt{a})$ follows from Theorem 1.2.

Now let $A \in \mathrm{Br}_2(L(\sqrt{a}))$ and let $\mathrm{cor} : \mathrm{Br}_2(L(\sqrt{a})) \to \mathrm{Br}_2(L)$ denote the corestriction homomorphism [**29**, Chapter VII.7]. From (2) we conclude that $\mathrm{cor}\, A = [(\frac{b,c}{L})]$ for some $c \in \dot{L}$. Since $a \in R(L)$ there exists $d \in \dot{L}(\sqrt{a})$ such that $c = N_{L(\sqrt{a})/L}(d)$. By the well known projection formula [**32**, (1.4)] we see that

$$\mathrm{cor}\left[\left(\frac{b,d}{L(\sqrt{a})}\right)\right] = \left[\left(\frac{b,c}{L}\right)\right]$$

and thus

$$\operatorname{cor} A\left[\left(\frac{b, d}{L(\sqrt{a})}\right)\right] = 1.$$

From Arason's long exact sequence [**1**, Satz 4.5], it then follows that

$$A\left[\left(\frac{b, d}{L(\sqrt{a})}\right)\right] = \left[\left(\frac{b, e}{L(\sqrt{a})}\right)\right]$$

for some $e \in \dot{L}$. Therefore $A = \left[\left(\frac{b, ed}{L(\sqrt{a})}\right)\right]$ and we are done. □

REMARK 2.3. Conditions (1) and (2) of Proposition 2.2 can be rephrased as follows:

(1)′ $I^2L \neq \{0\}$, $I^3L = \{0\}$;

(2)′ $I^2L = \langle\langle -b\rangle\rangle IL$ for some $b \in \dot{L}$.

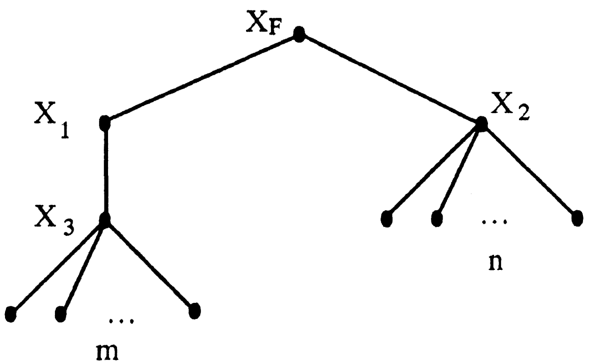
One can actually give a completely analogous proof by using Scharlau's transfer and reciprocity [**16**, Chapter VII]. We leave this to the interested reader.

EXAMPLE 2.4. We now construct a field which satisfies the conditions of Proposition 2.2. Let F be a formally real pythagorean field with order space $X_F = X_1 \oplus X_2$, where

(1) $X_1 = X_3 \times H$ where $H = \mathbf{Z}/2\mathbf{Z}$ and X_3 is a direct sum of m 1 - element spaces, $m > 0$.

(2) X_2 is the direct sum of n 1 - element spaces, $n > 0$.

For the concepts and notation needed above, see [**17**], [**18**]. We can also see X_F by its graph ([**24**, Definition 0.3]; see also [**10**]).



Now let $L = F(\sqrt{-1})$. Note that $\mathrm{st}(F) = 2$ [**19**] and therefore $u(L) = 2^{\mathrm{st}(F)} = 4$ by [**22**], so $\mathrm{cd}_2(G_L) = 2$. If $\{b\}$ is a basis of H over $\mathbf{Z}/2\mathbf{Z}$, it is easily seen by the methods of [**23**] that L is uniformly linked with slot b and that $n > 0$ implies that the Kaplansky radical $R(L)$ of L is nontrivial.

We now recall some important facts that will be needed in the proof of the next proposition.

Let G be a finitely generated pro - 2 - group and let $n = \dim_{\mathbf{Z}/2\mathbf{Z}} H^1(G)$; then n is the cardinality of any minimal set of generators of G. Let J be the free pro - 2 - group in n generators. We then get an exact sequence

$$1 \to R \to J \to G \to 1$$

where R is the group of relations of G. We can make $H^1(R)$ into a J - module by defining, for $j \in J, u \in H^1(R)$ and $r \in R$,

$$(j \cdot u)(r) = u(j^{-1}rj).$$

Then there is an exact sequence (via spectral sequences)

$$0 \to H^1(G) \xrightarrow{\inf} H^1(J) \xrightarrow{\mathrm{res}} H^1(R)^J \xrightarrow{\mathrm{tg}} H^2(G) \to 0$$

where inf = inflation, res = restriction and tg = transgression. $H^1(R)^J$ is the set of elements in $H^1(R)$ invariant under the action of J. Since inf is an isomorphism in our case, $\mathrm{tg} : H^1(R)^J \to H^2(G)$ is also an isomorphism.

Fix a set $\{x_1, \dots, x_n\}$ of generators of J. Each x_i induces a character $\chi_i \in H^1(J)$ by $\chi_i(x_j) = \delta_{ij}$ (δ_{ij} is the Kronecker delta; $\delta_{ij} = 1$ if $i = j$ and 0 otherwise). Since $\inf : H^1(G) \to H^1(J)$ is an isomorphism, we can look at the χ_i's as characters in $H^1(G)$. Then, for $1 \le i, j \le n, \chi_i \cup \chi_j \in H^2(G)$. We want to compute $[\mathrm{tg}^{-1}(\chi_i \cup \chi_j)](r)$ for $r \in R$.

We define a filtration $\{J_i\}$ of J by

$$J_1 = J, \qquad J_{i+1} = J_i^2[J_i, J] \quad (i \ge 1)$$

where $[J_i, J]$ denotes the commutator of J_i and J, i.e., the closure of the subgroup of J generated by elements of the form $g^{-1}j^{-1}gj$ for $g \in J_i, j \in J$. Each $r \in R$ can be written in the form

$$r \equiv x_1^{2a_1} \dots x_n^{2a_n} \prod_{i<j} [x_i, x_j]^{b_{ij}} \pmod{J_3} \tag{1}$$

where the a_i's and b_{ij}'s are in $\mathbf{Z}/2\mathbf{Z}$. Then

$$[\mathrm{tg}^{-1}(\chi_i \cup \chi_j)](r) = \begin{cases} a_i & \text{if } i = j \\ b_{ij} & \text{if } i \neq j. \end{cases} \tag{2}$$

All these results can be found in [**31**]; see also [**15**] for an excellent exposition and explicit computations with tg.

PROPOSITION 2.5. *Let $G_1, \dots, G_m$ be finitely generated nontrivial pro - 2 - groups and let $G = *_{i=1}^m G_i$ be the free product of the G_i's in the category of pro - 2 - groups. Suppose that, for all $1 \leq i \leq n$, we have*

(i) $\mathrm{cd}_2 G_i \leq 2$ (*and hence G_i has not nontrivial involution*);

(ii) *if H is any open subgroup of G_i, then the cup map*

$$H^1(H) \times H^1(H) \to H^2(H)$$

is surjective. Then conditions (i) *and* (ii) *hold for G in place of G_i. In particular we have* $\mathrm{cd}_2 G = \max_{1 \leq i \leq m}\{\mathrm{cd}_2 G_i\}$.

PROOF. Let H be an open subgroup of G. By [**9**] we can write $H = M * (*_{i=1}^m H \cap G_{ij})$ where M is a free finitely generated pro - 2 - group and, for each $1 \leq i \leq n$, the G_{ij}'s run over a certain finite set of conjugates of G_i (more precisely, G_{ij} runs over the set $\{G_i^{\sigma_{i,\alpha}}\}_\alpha$ where $\{\sigma_{i,\alpha}\}_\alpha$ is a complete set of representatives of the double coset decomposition $G = \cup_\alpha (H\sigma_{i,\alpha} G_i)$. We will not need this specific description). Since M is free, we have $\mathrm{cd}_2 M = 1$; from [**25**, Satz 4.1] we see that $\mathrm{cd}_2 H = \max_{1 \leq i,j \leq m}\{\mathrm{cd}_2(H \cap G_{ij})i\}$. Now each $H \cap G_{ij}$ is an open subgroup of G_{ij}. Since $G_{ij} \cong G_i$, Theorem 1.2 gives $\mathrm{cd}_2(H \cap G_{ij}) = \mathrm{cd}_2(G_{ij}) = \mathrm{cd}_2(G_i) \leq 2$. Hence $\mathrm{cd}_2 H \leq 2$ and $\mathrm{cd}_2 G = \max_{1 \leq i \leq m} \mathrm{cd}_2 G_i \leq 2$.

For each $1 \leq i \leq n$, fix a minimal set $X_i = \{x_{i1}, \dots, x_{in_i}\}$ of generators of G_i. Since $H^1(G) = \oplus_{i=1}^m H^1(G_i)$ [**25**, Satz 4.1], it follows that $X = \{x_1, \dots, x_n\} = \cup_{i=1}^m X_i$ is a minimal set of generators of G ($n = \sum_{i=1}^m n_i$). Similarly, if $\tilde{R}_i$ is a minimal set of relations for G_i then $\tilde{R} = \cup_{i=1}^m \tilde{R}_i$ is a minimal set of relations for G; this follows from $H^2(G) = \oplus_{i=1}^m H^2(G_i)$ [**25**, Satz 4.1] and from [**28**, Corollary, p. I.41] and subsequent remarks.

Now let J be the free pro -2 - group on X. Then $G \cong J/R$ where R is the closed normal subgroup of J generated by $\tilde{R}$. We are now in the situation described in the remarks preceding this proposition, and we use the notations explained therein. For each $i, 1 \leq i \leq n$, we get characters $\chi_{ij} \in H^1(G_i)$, $1 \leq j \leq n_i$. We now claim that $\chi_{ij} \cup \chi_{kl} = 0$ if $i \neq k$. Suppose the claim proved for the moment. Since $H^2(G) = \oplus_{i=1}^m H^2(G_i)$, any element $\chi \in H^2(G)$ can be written as

$$\chi = \sum_{i=1}^{m} \Big[\Big(\sum_{1 \leq r \leq n_i} a_{ir}\chi_{ir}\Big) \cup \Big(\sum_{1 \leq s \leq n_i} b_{is}\chi_{is}\Big)\Big]$$

for some $a_{ir}, b_{is} \in \mathbf{Z}/2\mathbf{Z}$. The claim then allows us to write

$$\chi = \left[\sum_{\substack{i=1 \\ 1 \leq r \leq n_i}}^{m} a_{ir}\chi_{ir}\right] \cup \left[\sum_{\substack{i=1 \\ 1 \leq s \leq n_i}}^{m} b_{is}\chi_{is}\right] := \chi_1 \cup \chi_2$$

where $\chi_1, \chi_2 \in H^1(G) = \oplus_{i=1}^m H^1(G_i)$.

Since any open subgroup H of G has the form $M * \left(*_{i=1}^m H \cap G_{ij}\right)$ (see beginning of this proof) and $H \cap G_{ij}$ is an open subgroup of $G_{ij} \cong G_i$, the proof above also works for H and therefore (ii) holds for G.

We now prove the claim. Choose $\tilde{r} \in \tilde{R}_k$. Expression (1) can be written as

$$\tilde{r} \equiv x_{11}^{2a_{11}} \cdots x_{ni_n}^{2a_{ni_n}} \prod_{\substack{i<j \\ r,s}} [x_{ir}, x_{js}]^{b_{irjs}} \prod_{\substack{i \\ t<v}} [x_{it}, x_{iv}]^{b_{itv}} (\mathrm{mod} J_3).$$

Our choice of X_k and $\tilde{R}_k$ show that all the b_{irjs} are 0. Hence, for $i \neq j$ and any $1 \leq r \leq n_i, 1 \leq s \leq n_j$, we have

$$[\mathrm{tg}^{-1}(\chi_{ir} \cup \chi_{js})](\tilde{r}) = 0.$$

Since $\tilde{R} = \cup \tilde{R}_k$ generates R we conclude that $\mathrm{tg}^{-1}(\chi_{ir} \cup \chi_{js}) = 0 \in H^1(R, 2)^J$ for $i \neq j$. But $\mathrm{tg} : H^1(R)^J \to H^2(G)$ is an isomorphism and hence $\chi_{ir} \cup \chi_{js} = 0 \in H^2(G)$ for $i \neq j$. □

COROLLARY 2.6. *Let L be a nonformally real field such that $G_L = *_{i=1}^m G_i$ where the G_i's are finitely generated pro - 2 - groups satisfying conditions* (i) *and* (ii) *of Proposition* 2.7 *and such that* $\mathrm{cd}_2(G_i) = 2$ *for at least one i (here $*$ denotes free product in the category of pro - 2 - groups). Then $u(K) = 4$ for any finite 2 - extension K of L.*

PROOF. This is an immediate consequence of Propositions 2.5 and 1.4. □

We finish by exhibiting an example to which Corollary 2.6 can be applied.

EXAMPLE 2.7. Let F be a formally real pythagorean field with $\mathrm{cl}(F) < \infty$. In [**23**] it is shown that $G_{F(\sqrt{-1})}$ is a Demushkin pro - 2 - group if and only if F is of the type $(4, 2^3)$ (i.e., $|X_F| = 4$ and $|\dot{F}/\dot{F}^2| = 2^3$). Since any open subgroup of a Demushkin group is again a Demushkin group [**28**, Corollary p. I-51] we see that the conditions of Proposition 2.5 hold for $G_{F(\sqrt{-1})}$.

Now let $K = \cap_{i=1}^m F_i$, where all F_i's are of type $(4, 2^3)$ and assume $X_F = \oplus X_{F_i}$. Let $L = K(\sqrt{-1})$. From [**14**, Lemma 9] it follows that $G_F \cong {*}_{i=1}^m G_{F_i}$, and from [**9**] we get $G_L \cong M * [\, {*}_{i=1}^m (G_L \cap G_{F_i})]$ where M is some finitely generated free pro - 2 - group and $*$ denotes free product in the category of pro - 2 - groups. Since $G_L \cap G_{F_i} \cong G_{F_i(\sqrt{-1})}$ we get $G_L \cong M * [*_{i=1}^m G_{F_i(\sqrt{-1})}]$ and therefore G_L satisfies the hypothesis of Corollary 2.6. □

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