ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

ON THE SCHARLAU TRANSFER

BRUNO KAHN

Let F be a field of characteristic $\neq 2$, F_s a separable closure of F and $G_F = \mathrm{Gal}\,(F_s/F)$. The ring $A(F)$ of *monomial representations* of G_F is defined as follows: it is the Grothendieck ring of the category of pairs (E, E'), where E and E' are étale F-algebras and E' is a free E-algebra of rank 2 [**1**, III 2.2]. Such pairs are classified by homomorphisms of G_F in a wreath product $\mathbf{S}_n \int (\mathbf{Z}/2) \simeq O(n, \mathbf{Z})$ (*loc. cit.*), so $A(F)$ really depends only on G_F.

On the other hand, write (here) $W(F)$ for the Witt-Grothendieck ring of F (the Grothendieck ring of the category of non-degenerate quadratic forms over F). In [**1**, III. 2.6] a ring homomorphism $h : A(F) \to W(F)$ was defined; it may be described in (at least) two different ways:

(a) Let (E, E') be a generator of $A(F)$. Since $\mathrm{char}\, F \neq 2$, there is an $a \in E^*$ such that $E' = E[\sqrt{a}]$. Then $h(E, E')$ is the class in $W(F)$ of the quadratic form $q(x) = \mathrm{Tr}_{E/F} ax^2$.

(b) Let $\rho : G_F \to O(n, \mathbf{Z})$ be a homomorphism classifying (E, E'). Then $O(n, \mathbf{Z})$ maps naturally to a Galois-invariant subgroup of $O(n, F_s)$, hence to ρ is associated an element in the nonabelian cohomology set $H^1(G_F, O(n, F_s))$; via [**2**, p.162, Corollary 1], this element corresponds to a quadratic form $h(\rho)$.

Observe that $W(F)$ also depends only on G_F; in [**1**, III.2.7] the question was raised whether or not h depends only on G_F. The aim of this article is to answer this question positively:

THEOREM. *The homomorphism h depends only on G_F, and not on the particular field F.*

Here is a sketch of the proof. One reduces to proving that, for any finite separable extension E of F, the 'Scharlau transfer' $T : W(E) \to W(F)$ given by $T(q)(x) = \mathrm{Tr}_{E/F} q(x)$ depends only on G_F and G_E. To

Received by the editors on October, 1986.

do this, one reduces by dévissage to the case of a quadratic extension; at this stage the proof is reduced to a cohomological lemma which will be proved first. Behind this lemma is a construction imagined by J-P. Serre; I am grateful to him for having let me know about it and allowing me to use it here.

1. A lemma on boundary homomorphisms. Let G be a group and A be a G-module which, as an abelian group, is cyclic of order p^n (p: a prime number). In applications, G will be G_F, a profinite group, so one should think of G-modules as topological G-modules and cohomology of G as cohomology of a profinite group; however the situation is identical in practice so it will be implicit everywhere here.

The action τ of G on A is given by a homomorphism of G in $Aut\,(\mathbf{Z}/p^n) \simeq (\mathbf{Z}/p^n)^*$. Let $0 \subsetneq A' \subsetneq A$ be a subgroup (hence a submodule) of A and $A'' = A/A'$. To the exact sequence $0 \to A' \to A \to A'' \to 0$ are associated boundary homomorphisms:

$$\partial_A : H^i(G, A'') \to H^{i+1}(G, A'), \quad i \geq 0.$$

Another action $\tilde{\tau}$ of G on A will coincide with τ on A' and A'' if and only if, for all $g \in G, \tilde{\tau}(g)\tau(g)^{-1} \in (1 + p^r\mathbf{Z})/p^n\mathbf{Z}$, where $p^r = \max(|A'|, |A''|)$. Then $g \mapsto (\tilde{\tau}(g)\tau(g)^{-1} - 1)/p^r$ defines an element $\chi \in \mathrm{Hom}\,(G, \mathbf{Z}/p^{n-r}\mathbf{Z}) = H^1(G, \mathbf{Z}/p^{n-r})$. Write $\tilde{A}$ for the G-module corresponding to this new action $\tilde{\tau}$.

LEMMA 1. *Assume that the G-module A' is trivial. If $x \in H^i(G, A'')$, one has*

$$\partial_{\tilde{A}} x = \partial_A x + \chi \cdot x,$$

where the cup-product is induced by the pairing $\mathbf{Z}/p^{n-r} \times A'' \to A'$ such that $(n, a'') \mapsto p^r na''$ (with an obvious abuse of notation).

PROOF. For any G-module M, let $C^\cdot(M)$ be the complex of "non-homogeneous cochains" defining the cohomology of M, as in [**2**, p. 121]. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of G-modules, the boundary homomorphism ∂ associated to it is computed as follows on an element $x \in H^i(G, M'')$: choose a representative c of x

in $C^i(M'')$ and lift it to a cochain $\tilde{c} \in C^i(M)$. Then the differential $d\tilde{c}$ defines a cocycle in $C^{i+1}(M')$, whose class in $H^{i+1}(G, M')$ is precisely ∂x. But let d (respectively $\tilde{d}$) be the differential of $C^{\cdot}(A)$ (respectively of $C^{\cdot}(\tilde{A})$): the formula which gives the differential of a cochain c (e.g., *loc. cit.*) shows that

$$\tilde{d}c(g_1, \dots, g_{i+1}) - dc(g_1, \dots, g_{i+1}) = p^r \chi(g_1) c(g_2, \dots, g_{i+1}),$$

hence the lemma. □

2. The symbol (2, d). From now on, we shall simply write $H^i(G)$ for $H^i(G, \mathbf{Z}/2)$, and $H^i(F, -)$ for $H^i(G_F, -)$.

If $a \in F^*$, write (a) or $(a)_F$ for the image of a in $H^1(F)$ via Kummer theory; if $a, b \in F^*$, the cup-product $(a) \cdot (b)$ will often be written (a, b). When $G_F \simeq G_{F'}$, the classes $(-1)_F$ and $(-1)_{F'}$ in $H^1(G_F)$ need not coincide (e.g., take finite fields for F and F'); however, for any $x \in H^1(F)$, one has the formula $(-1)_F \cdot x = x^2$, hence $(-1)_F.x = (-1)_{F'}.x$. The aim of this paragraph is to show that similarly, the map $x \mapsto (2)_F.x$ depends only on G_F (and not on F itself). To prove this we use a construction imagined by Serre (personal communication).

Again let G be any group. If $\alpha \in H^1(G)$, let $\mathbf{Z}(\alpha)$ be the G-module with support $\mathbf{Z}$, with action given by $g \cdot a = (-1)^{\alpha(g)} a$. For $n \geq 2$, set $\mathbf{Z}/n(\alpha) = \mathbf{Z}(\alpha)/n$. Let $\partial_\alpha : H^1(G) \to H^2(G)$ be the boundary associated to the exact sequence $0 \to \mathbf{Z}/2 \to \mathbf{Z}/4(\alpha) \to \mathbf{Z}/2 \to 0$.

LEMMA 2. *The following are equivalent:*

(i) $\partial_\alpha = 0$;

(ii) $H^1(G, \mathbf{Z}/4(\alpha)) \to H^1(G)$ *is onto;*

(iii) *For any* $x \in H^1(G), \alpha \cdot x = x^2$.

PROOF. (i)$\Longleftrightarrow$(ii) is obvious. To see (i)$\Longleftrightarrow$(iii), observe that $\partial_o x = Sq^1 x = x^2$ and use Lemma 1.

Let $\varepsilon \in H^1(G)$ satisfy the equivalent conditions of Lemma 2: e.g., if $G = G_F, \varepsilon = (-1)_F$ will do. Following Serre, we associate to ε a "secondary boundary homomorphism" $\delta_\varepsilon : H^1(G) \to H^2(G)$ as follows.

We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/4(\varepsilon) & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/8(\varepsilon) & \longrightarrow & \mathbf{Z}/4(\varepsilon) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbf{Z}/2 & &
\end{array}$$

in which the unique column is a short exact sequence, whence a commutative diagram in cohomology (solid arrows):

$$\begin{array}{ccccc}
H^1(G,\mathbf{Z}/4(\varepsilon)) & \longrightarrow & H^1(G) & \xrightarrow{0} & H^2(G) \\
\downarrow & & \downarrow & & \| \\
H^1(G,\mathbf{Z}/8(\varepsilon)) & \longrightarrow & H^1(G,\mathbf{Z}/4(\varepsilon)) & \xrightarrow{\delta} & H^2(G) \\
 & & \downarrow & \nearrow_{\delta_\varepsilon} & \\
 & & H^1(G) & &
\end{array}$$

Let $x \in H^1(G)$ and $\tilde{x}$ be a lift of x in $H^1(G,\mathbf{Z}/4(\varepsilon))$. Since the column in the above diagram is exact, $\delta\tilde{x}$ only depends on x: this defines $\delta_\varepsilon x$.

Let $\omega \in H^1(G)$ be another element; write $\mathbf{Z}/8(\varepsilon,\omega)$for the G-module supported by $\mathbf{Z}/8$, with action given by $g \cdot x = (-1)^{\varepsilon(g)}5^{\omega(g)}x$.

LEMMA 3. *The following are equivalent:*

(iv) $H^1(G,\mathbf{Z}/8(\varepsilon,\omega)) \to H^1(G,\mathbf{Z}/4(\varepsilon))$ *is onto*;

(v) *for all* $x \in H^1(G), \delta_\varepsilon x = \omega \cdot x$.

Once again this is a simple application of Lemma 1. For example, if $G = G_F$, then $\omega = (2)_F$ satisfies (iv) and (v).

Let $\chi \in H^1(G) = \mathrm{Hom}\,(G,\mathbf{Z}/2)$; to χ corresponds its kernel H_χ. If $\chi \cdot x = 0$ for all $x \in H^1(G)$, we shall call H_χ a *ghost subgroup* of G. The main lemma in this article is

MAIN LEMMA. *Assume that there is* $\varepsilon_o \in H^1(G)$ *such that, for any ghost subgroup* $H \subseteq G, \operatorname{Res}^G_H \varepsilon_o$ *satisfies conditions* (i)-(ii) *of Lemma* 2 *for* H. *Then the boundary* δ_ε *constructed above does not depend on the choice of* ε *satisfying* (i)-(iii) (*for* G).

PROOF. It is enough to show $\delta_\varepsilon = \delta_{\varepsilon_o}$. Condition (iii) shows that $\chi = \varepsilon - \varepsilon_o$ has kernel a ghost subgroup H of G. We may assume $\chi \neq 0$, hence $(G : H) = 2$; then there is a long exact sequence

$$\to H^{i-1}(G) \xrightarrow{\cdot\chi} H^i(G) \xrightarrow{\operatorname{Res}} H^i(H) \xrightarrow{\operatorname{Cor}} H^i(G) \to \cdots$$

For $i = 2$, this shows that $\operatorname{Res} : H^2(G) \to H^2(H)$ is injective. By construction, $\operatorname{Res}\varepsilon = \operatorname{Res}\varepsilon_o$, hence $\mathbf{Z}(\varepsilon)$ and $\mathbf{Z}(\varepsilon_o)$ are H-isomorphic. Therefore, for all $x \in H^1(G)$, $\operatorname{Res}\delta_\varepsilon x = \delta_{\operatorname{Res}\varepsilon_o}(\operatorname{Res} x) = \operatorname{Res}\delta_{\varepsilon_o} x$, and $\delta_\varepsilon x = \delta_{\varepsilon_o} x$.

COROLLARY. *For a field* F, *the map* $x \to (2)_F \cdot x$ *only depends on* G_F.

Indeed, $\varepsilon_o = (-1)_F$ satisfies the hypothesis of the Main Lemma.

3. The Scharlau transfer in a separable extension. Recall that $W(F)$ may be defined by generators and relations:

generators: $\langle a \rangle, a \in H^1(F)$;

relations: $\langle a \rangle + \langle b \rangle = \langle c \rangle + \langle d \rangle$ if $a + b = c + d$ and $a \cdot b = c \cdot d$.

Hence $W(F)$ only depends on G_F. Let E/F be a finite, separable extension. To a quadratic form q over E, associate the quadratic form over F defined by $T(q)(x) = \operatorname{Tr}_{E/F} q(x)$: this defines a homomorphism $T_{E/F} : W(E) \to W(F)$. In this section, we shall prove

PROPOSITION 1. *The map* $T_{E/F}$ *only depends on* G_E *and* G_F.

PROOF. *Step* 1. Proposition 1 is true for quadratic extensions.

Indeed, suppose E/F quadratic, hence $E = F(\sqrt{d})$; the class $(d) \in H^1(F)$ is the character of G_F with kernel G_E. Let $a \in H^1(E)$: we have to prove that $T\langle a\rangle$ only depends on G_F. But there are $x, y \in H^1(F)$ such that $T\langle a\rangle = \langle x\rangle + \langle y\rangle$; by [**1**, II.2.1] one has

$$x + y = \mathrm{Cor}_{E/F} a + (d);$$
$$x \cdot y = \mathbf{N}(a) + (2, d),$$

where $\mathbf{N}$ is the multiplicative transfer.

Obviously $x + y$ depends only on G_E and G_F; by the Main Lemma this is the same for $x \cdot y$. Therefore $T_{E/F}\langle a\rangle$ only depends on G_E and G_F.

Step 2. The general case. We will proceed with a standard dévissage argument, as in [**1**, II.] The following lemma is well-known [**3**].

LEMMA 4. *Let K/F be a finite extension of odd degree. Then* Res : $W(F) \to W(E)$ *is injective.*

Let F' be another field such that $G_F \simeq G_{F'}, \phi : G_{F'} \to G_F$ an isomorphism and E' the extension of F' corresponding to $\phi^{-1}(G_E)$. If $x \in W(E)$, we wish to show that $T_{E'/F'}\phi^* x = \phi^* T_{E/F} x$, where ϕ^* is the isomorphism induced by ϕ^* on Witt groups. Let $\tilde{E}$ be the Galois closure of E over $F, K \subseteq E$ the fixed field of some Sylow 2-subgroup of Gal(E/F) and K' the extension of F' which corresponds to K via ϕ : K/F and K'/F' have the same odd degree. By Lemma 4, it will be enough to show that

$$(*) \qquad \mathrm{Res}_{K'/F'} T_{E'/F'} \phi^* x = \phi^* \mathrm{Res}_{K/F} T_{E/F} x.$$

The étale K-algebra $K \otimes_F E$ is a direct product of extensions K_i/K which are filtered by successive quadratic extensions. By repeatedly applying Proposition 1 for a quadratic extension, we find that $T_{K_i'/K'} \mathrm{Res}_{K_i'/E'} \phi^* x = \phi^* T_{K_i/K} \mathrm{Res}_{K_i/E} x$ for all i, and therefore that $(*)$ holds. □

4. Proof of the theorem. Keep the notations as just above. We have to prove that the diagram

$$\begin{array}{ccc} A(F) & \xrightarrow{h} & W(F) \\ \phi^* \downarrow & & \phi^* \downarrow \\ A(F') & \xrightarrow{h'} & W(F') \end{array}$$

commutes. Let $x = (E, E[\sqrt{a}]) \in A(F)$ be a generator. The image $h(x)$ is the class in $W(F)$ of $\mathrm{Tr}_{E/F} ax^2$. Up to splitting E into its minimal ideals, we may assume that E is a field; then Proposition 1 shows that $\phi^* T_{E/F}\langle a\rangle = T_{E'/F'}\langle \phi^*, a\rangle$, hence $\phi^* h(x) = h'(\phi^* x)$. □

REFERENCES

1. B. Kahn, *Classes de Stiefel-Whitney de forms quadraticques et de représentations galoisiennes réelles,* Invent. Math. **78** (1984), 223-256.

2. J.-P. Serre, *Corps Locaux,* Hermann, Paris, 1968.

3. T.A. Springer, *Sur les formes quadratiques d'indice zéro,* C.R. Acad. Sci. Paris **234** (1952), 1517-1519.

Université de Paris 7, UFR de Mathématiques, 5e Étage, tour 45-55, 75251 Paris Cedex 05, France