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DISCRETELY VALUED FIELDS WITH INFINITE u-INVARIANT: RESEARCH ANNOUNCEMENT

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Herbert Gross (see [**1**, p. 3]) has raised the following problem:

PROBLEM (G). *Is there any commutative field which admits no anisotropic* $\aleph_0$*-quadratic form but which admits, for each* $n \in N$, *some anisotropic form in* n *variables?*

There has been some related work in infinite dimensional anisotropic quadratic forms (see, e.g., Meissner [**2**] and [**3**]), but Gross's problem is still open. In this note we contribute a partial solution to this problem, we prove that if the answer to it is positive then there has to be a field of a very specific kind that fulfills those conditions. Also, we introduce some nonarchimedean analysis techniques to study the discretely real-valued commutative fields for which there is some anisotropic $\aleph_0$-quadratic form. Proofs will be published elsewhere.

Let us first introduce some notations and terminology. The u-invariant of a field F, $u(F)$, as defined by Kaplansky, is the maximum (when it exists) of the set of natural numbers n such that there is an n-dimensional linear space E over F and an anisotropic quadratic form on E. If that maximum does not exist, we say that $u(F)$ is (*weakly*) *infinite*, and in case there is an infinite-dimensional vector space E over F with an anisotropic quadratic form, then we say that $u(F)$ is *strongly infinite.*

We will denote by K any commutative field which is endowed with a nontrivial discrete real valuation (that is, that has as its value group a discrete subgroup of $\mathbf{R}^+$) and is complete for the associated distance. We will call k the residue class field of K, and shall always assume that $\operatorname{char} k \neq 2$, that is to say, that K is nondyadic.

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Embed the field K into a superstructure $\mathcal{X}$ and consider a non-standard model $^*\mathcal{K}$ that is $\aleph_1$-saturated. Denote by $\hat{K}$ the infinitesimal (or non-standard) hull of K, that is, the quotient

$$\text{fin}\, ^*K / \inf\, ^*K$$

of the ring of finite elements of $^*K$ (or elements with finite absolute value) modulo the ideal of infinitesimal elements of $^*K$ (or members of $^*K$ with infinitesimal absolute value); if $t \in {}^*K$, then $\hat{t}$ will be the class $t + \inf {}^*K$.

Then the valuation of K is canonically extended to a valuation on $\hat{K}$; since the value group of K is discrete, it is easy to show that it is the same as the value group of $\hat{K}$. It is well-known that $K = \hat{K}$ if and only if K is locally compact; notice that when K is locally compact, $u(K)$ is necessarily finite.

Our first result will be

THEOREM 1.

(i) *$\hat{K}$ is never a positive answer to problem* (G).

(ii) *No algebraic ultrapower of any commutative field* (*valued or not*) *gives a positive answer to problem* (G).

As a result, for every reduced ultrapower of a discretely valued field, the properties of having strongly infinite u-invariant and weakly infinite u-invariant, are equivalent.

We also obtain the following reduction of H. Gross's problem:

THEOREM 2. *In order to answer the problem* (G), *it is enough to answer it for the discretely valued fields of the type* $F((x))$, *where* F *is any commutative field.*

All these results are consequences of

THEOREM 3. *$u(K)$ is strongly infinite if and only if there is a sequence* (b_n) *of members of* K *of absolute value equal to one such that, for any* *finite set* $\{t_1, \ldots, t_n\} \subset K$,

$$|t_1^2 b_1 + \cdots + t_n^2 b_n| = \max\{|t_1|^2, \ldots, |t_n|^2\}.$$

It is worth mentioning that conditions very similar to the one in the statement of Theorem 3 have been used (in the context of finite dimensional anisotropic quadratic forms) by A. Prestel (cf. [**4**, pp. 90-91]).

We prove Theorem 3 by using the analog of the Gram-Schmidt method of orthogonalizing a linearly independent sequence and then showing that the orthogonal sequence obtained by this procedure is also orthogonal in Birkhoff's sense for normed spaces. The consideration of K-vector spaces with an anisotropic quadratic form as nonarchimedean normed spaces (in the finite dimensional case) is due to T.A. Springer, [**5**].

The proofs of Theorems 1 and 2 are not the only applications of Theorem 3. It seems to be potentially useful in the research on strongly infinite u-invariants in valued fields, since it gives some insight into the structure of the field. For instance, we have also proved

THEOREM 4. *If the square classes group of* K, *i.e.*, $\dot{K}/\dot{K}^2$, *is finite, then* $u(K)$ *is strongly infinite if and only if* K *is formally real and it is (isomorphic to) a field of formal power series.*

THEOREM 5. *If* $u(K)$ *is strongly infinite, so is* $u(k)$. *The converse is also true whenever* $\operatorname{char} K = \operatorname{char} k$.

The last result is related to a consequence of a theorem proved by T.A. Springer more than 30 years ago: $u(K)$ is weakly infinite if and only if $u(k)$ is weakly infinite.

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