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ON WEAKLY ANALYTIC SUBSETS OF ℓ^∞

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Let $\Delta_0 = \{-1, 1\}^{\mathbf{N}}$, considered as a subset of ℓ^∞. In [4], M. Talagrand shows that there exists a non weakly Borel subset of Δ_0. This result answered the following question of G. Edgar [2]: Does there exist a Banach space X such that the norm Borel sets are not equal to the weak Borel sets?

The purpose of this paper is to show that, although there exist non weakly Borel subsets of Δ_0, every subset of Δ_0 is weakly analytic. That is, every subset of Δ_0 is the continuous image of a weakly Borel subset of Δ_0.

Our notation and terminology follow [1]. By the Cantor set, we mean the set $\Delta = \{-1, 1\}^{\mathbf{N}}$ with the topology of coordinate convergence. For each n there is a natural partition π_n of Δ into 2^n open subsets: the elements of π_n are obtained by fixing the first n coordinates. This gives rise to 2^{2^n} different continuous functions $\varphi : \Delta \to \{-1, 1\}$ which are constant on the members of π_n. The totality of all such functions φ (over all $n = 1, 2, 3, \ldots$) forms a countable set. Let $(\varphi_n)_{n=1}^\infty$ be an enumeration of these functions.

A subsequence of $(\varphi_n)_{n=1}^\infty$ is the sequence $(r_n)_{n=1}^\infty$ of Rademacher functions, defined by

$$r_n(\varepsilon) = \varepsilon_n \quad \text{for } \varepsilon \in \Delta.$$

We shall assume for later that the enumeration above was chosen so that

$$\varphi_{2n} = r_{2n}, \quad \text{for every } n.$$

The following two observations are easily proved.

(1) If $\mu \in C(\Delta)^*$, then the variation norm of μ is given by

$$||\mu|| = \sup_n |\langle \mu, \varphi_n \rangle|.$$

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(2) The sequence $(r_n)_{n=1}^{\infty}$ is isometrically an ℓ^1-basis in $C(\Delta)$. That is, for any finite sequence of scalars $(\alpha_1)_{i=1}^{N}$, we have

$$\left\| \sum_{i=1}^{N} \alpha_i r_i \right\|_{\infty} = \sum_{i=1}^{N} |\alpha_i|.$$

For each ε in Δ, let $\mu_\varepsilon \in C(\Delta)^*$ be defined by

$$\langle \mu_\varepsilon, f \rangle = f(\varepsilon) \text{ for } f \in C(\Delta),$$

and let

$$\Delta_M = \{\mu_\varepsilon : \varepsilon \in \Delta\}.$$

The space $C(\Delta)^*$ has three natural topologies: the norm, weak, and weak* topologies. We note that $(\Delta_M, \text{weak}^*)$ is homeomorphic to Δ, and (Δ_M, norm) is discrete. Moreover, (Δ_M, weak) is also discrete. (For each $\varepsilon \in \Delta$, let F_ε in $C(\Delta)^{**}$ be given by$F_\varepsilon(\mu) = \mu(\{\varepsilon\})$. Then $\{\mu_\varepsilon\} = \{\mu = \Delta_M : F_\varepsilon(\mu) \neq 0\}$.) Thus

(3) Δ_M and all its subsets are weakly closed in$C(\Delta)^*$.

Again we consider the set $\Delta_0 = \{-1, 1\}^{\mathbf{N}}$ as a subset of the Banach space ℓ^∞. The space ℓ^∞ also has three natural topologies: The norm, weak and weak* (considering ℓ^∞ as the dual of ℓ^1). Again, $(\Delta_0, \text{weak}^*)$ is homeomorphic to Δ and(Δ_0, norm) is discrete. However, the space (Δ_0, weak) is a somewhat unknown beast. It is easy to see that it is not homeomorphic to Δ. For if $\mathcal{F}$ is free ultrafilter in the natural numbers $\mathbf{N}$, then the set

$$\{\chi_F - \chi_{(\mathbf{N}/F)} : F \in \mathcal{F}\}$$

is clearly closed in (Δ_0, weak), but is well-known not to be a Borel set in Δ [**3**]. On the other hand, (Δ_0, weak) cannot be discrete, since Talagrand's theorem of [**4**] states that there are non weakly Borel subsets of Δ_0.

Although there exist non weakly Borel subsets of Δ_0, the following is true:

THEOREM. *Every subset of Δ_0 is weakly analytic. (That is, every subset of Δ_0 is the continuous image of a weakly Borel set of Δ_0.)*

In fact, if $L : \ell^\infty \to \ell^\infty$ is the continuous linear operator given by $L(\{\alpha_n\}_{n=1}^\infty) = \{\alpha_{2n}\}_{n=1}^\infty$, then if B is a subset of Δ_0, there exists a weakly closed subset A of Δ_0 such that $L(A) = B$.

PROOF. Let $T : \ell^1 \to C(\Delta)$ be the bounded linear operator satisfying

$$T(e_n) = \varphi_n, \quad \text{for all } n,$$

where $(\varphi_n)_{n=1}^\infty$ is as above and $(e_n)_{n=1}^\infty$ is the usual basis for ℓ^1. By (1), the adjoint $T^* : C(\Delta)^* \to \ell^\infty$ is an isometry of $C(\Delta)^*$ into ℓ^∞. Hence T^* is a homeomorphism of $(C(\Delta)^*$, weak) onto a weakly closed subspace of $(\ell^\infty$, weak). It is clear that the set $D = T^*(\Delta_M)$ is contained in Δ_0. We have

(4) D and all its subsets are weakly closed in ℓ^∞.

Let E denote the even integers and $U : \ell^1(E) \to \ell^1$ be the natural injection. Then

$$T \circ U(e_{2n}) = T(e_{2n}) = \varphi_{2n} = r_{2n},$$

and, thus, by (2), $T \circ U$ is an isometry of $\ell^1(E)$ into $C(\Delta)$. Therefore $(T \circ U)^* = U^* \circ T^*$ maps $C(\Delta)^*$ onto $\ell^\infty(E)$. It is easily checked that $(T \circ U)(\Delta_M) = \{-1, 1\}^E$. Therefore

(5) $U^*(D) = \{-1, 1\}^E$.

Let S denote the natural isomorphism from $\ell^\infty(E)$ onto ℓ^∞. If B is contained in Δ_0, then $S^{-1}(B)$ is a subset of $\{-1, 1\}^E$. Hence, by (5), there exists a subset A of D such that $U^*(A) = S^{-1}(B)$. Since $L = S \circ U^*$, we have $B = L(A)$. But, by (4), A is weakly closed. Since L is clearly weakly continuous, B is weakly analytic.

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