ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

SHARP LOWER BOUNDS FOR A GENERALIZED JENSEN INEQUALITY

A.K. RIGLER, S.Y. TRIMBLE AND R.S. VARGA[1]

Dedicated to Professor A. Sharma,
on his retirement from the University of Alberta.

1. Introduction. Our motivation for the research in this paper arose from two recent papers by Beauzamy and Enflo [**2**] and Beauzamy [**3**], which are connected with polynomials and the classical Jensen inequality. To describe their results, let $P(z) = \sum_{j=0}^{m} a_j z^j (= \sum_{j=0}^{\infty} a_j z^j$ where $a_j := 0$ for $j = m+1,\ m+2, \dots)$ be a complex polynomial ($\not\equiv 0$), let d be a number in (0,1), and let k be a nonnegative integer. Then (cf. [**2**, **3**]), $P(z)$ is said to have *concentration d at degrees at most k* if

$$\sum_{j=0}^{k} |a_j| \geq d \sum_{j=0}^{\infty} |a_j|. \tag{1.1}$$

(Later, we shall discuss functions which are *not* polynomials, yet for which (1.1) holds. This accounts for our use of the symbol, ∞, in (1.1).)

Beauzamy and Enflo showed (cf. [**3**, Theorem 1]) that there exists a constant $C_{d,k}$, depending only on d and k, such that, for any polynomial $P(z)$ satisfying (1.1), it is true that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta})|d\theta - \log\Big(\sum_{j=0}^{\infty}|a_j|\Big) \geq C_{d,k}. \tag{1.2}$$

For our purposes here, $C_{d,k}$ will denote the *largest* such constant possible in (1.2), i.e.,

$$C_{d,k} := \inf\Big\{\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta})|d\theta - \log\Big(\sum_{j=0}^{\infty}|a_j|\Big) : P(z) \text{ is a polynomial satisfying } (1.1)\Big\}. \tag{1.3}$$

---

[1] Research of this author supported by the Air Force Office of Scientific Research.

Received by the editors on August 6, 1986 and in revised form on December 19, 1986.

In [**3**], Beauzamy showed that

$$(1.4)\qquad C_{d,k} \geq \tilde{C}_{d,k} := \sup_{1<t<\infty} \left\{ t \log \left(\frac{2d}{(t-1)((\frac{t+1}{t-1})^{k+1} - 1)} \right) \right\},$$

for all $d \in (0,1)$ and all $k = 0, 1, \ldots$. In particular, as $d \in (0,1)$, it follows from (1.4) that

$$(1.5)\qquad \tilde{C}_{d,0} = \log d.$$

It was also shown in [**3**] that, for $d = 1/2$,

$$(1.6)\qquad \lim_{k\to\infty} \frac{\tilde{C}_{1/2,k}}{k} = -2,$$

and that

$$(1.7)\qquad C_{1/2,k} \leq -(2k+1)\log 2 \quad (k = 0, 1, \ldots).$$

It follows from (1.5) and (1.7) that

$$(1.8)\qquad C_{1/2,0} = -\log 2 \quad \text{and } \limsup_{k\to\infty} \frac{C_{1/2,k}}{k} \leq -2\log\ 2.$$

To make a connection between Jensen's inequality and inequality (1.2), let $f(z) = \sum_{j=N}^{\infty} a_j z^j$, with $a_N \neq 0$, be analytic in $|z| \leq 1$. Let $Z_\Delta(f)$ denote the zeros of $f(z)$ in $0 < |z| < 1$, with multiple zeros being repeated. Then, *Jensen's formula* (cf. Ahlfors [**1**, p. 207]) is

$$(1.9)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta = \log|a_N| + \sum_{z_j\in Z_\Delta(f)} \log(1/|z_j|).$$

Since the sum above either is not there (when no zeros exist) or is positive, one obtains the *Jensen inequality*

$$(1.10)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta \geq \log|a_N|.$$

Further, since $f(z)$ is analytic in $|z| < R$ for some $R > 1$, then $\sum_{j=N}^{\infty} |a_j| < \infty$; this means the final sum in (1.1) is finite. Now, suppose that (1.1) is valid for $f(z)$ for some d in (0,1) and for $k = 0$ (so that $N = 0$), even though $f(z)$ is not necessarily a polynomial. Then, Jensen's inequality (1.10) (with $N = 0$) *implies* inequality (1.2) with $C_{d.0} = \log d$ (cf. (1.5)). Conversely, if (1.1) holds with equality for $f(z)$ for the case $k = 0$, then inequality (1.2), with $C_{d.0} = \log d$, *implies* Jensen's inequality (1.10) (with $N = 0$). In this sense, inequality (1.2) can be viewed as a *generalization* of Jensen's inequality.

To go beyond functions analytic in $|z| \leq 1$, let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be analytic in $|z| < 1$, and set

$$(1.11)\qquad \begin{aligned} M_p(r;f) &:= \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p}, \quad \text{for } 0 < p < \infty \\ &\qquad\qquad \text{and } 0 \leq r < 1, \qquad ; \\ M_\infty(r;f) &:= \max_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|, \quad \text{for } 0 \leq r < 1. \end{aligned}$$

As usual (cf. Duren [**5**, p. 2]), for $0 < p \leq \infty$, let

$$(1.12)\qquad \begin{aligned} H^p := \{g(z) : &g \text{ is analytic in } |z| < 1 \\ &\text{and } M_p(r;g) \text{ is bounded as } r \to 1-\}. \end{aligned}$$

If the final sum in (1.1) is finite for $f(z)$, i.e., $\sum_{j=0}^{\infty} |a_j| < \infty$, it is clear that

$$M_\infty(r;f) \leq \sum_{j=0}^{\infty} |a_j| r^j \leq \sum_{j=0}^{\infty} |a_j| < +\infty.$$

Hence, from definition (1.12), $f(z) \in H^\infty$. If $g(z) \in H^p$ for $0 < p \leq \infty$, it is known (cf. [**5**, p. 17]) that $g(z)$ can be extended to $|z| = 1$ by means of a function $\hat{g}(e^{i\theta})$, defined on $[0, 2\pi]$, for which

$$(1.13)\qquad \begin{cases} \hat{g}(e^{i\theta}) = \lim_{r\to 1-} g(re^{i\theta}) \text{ a.e. in } [0,2\pi], \\ \hat{g}(e^{i\theta}) \in L^p[0,2\pi], \text{ and,} \\ \text{if } g(z) \not\equiv 0, \text{ then } \log|\hat{g}(e^{i\theta})| \in L^1[0,2\pi]. \end{cases}$$

With this notation, for any $f(z) = \sum_{j=0}^{\infty} a_j z^j$, ($\not\equiv$), which is analytic in $|z| < 1$ with $\sum_{j=0}^{\infty} |a_j| < \infty$, it follows from (1.13) that

$$(1.14)\qquad J(f) := \frac{1}{2\pi}\int_0^{2\pi} \log|\hat{f}(e^{i\theta})| d\theta - \log\left(\sum_{j=0}^{\infty} |a_j|\right)$$

is well-defined and finite. We now redefine the constants $C_{d.k}$ so that

$$C_{d.k} := \inf\{J(f) : f(z) \in H^\infty \text{ and } f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0) \text{ satisfies (1.1)}\}. \tag{1.15}$$

This is an extension of our previously discussed largest constants, $C_{d.k}$ (cf. (1.3)).

We remark that if $f(z) = \sum_{j=N}^{\infty} a_j z^j$, $a_N \neq 0$, is analytic in $|z| \leq 1$, then it follows from Jensen's formula (1.9) that

$$J(f) = \log\left(|a_N| / \left(\left(\prod_{z_j \in Z_\Delta(f)} |z_j|\right) \sum_{j=N}^{\infty} |a_j|\right)\right). \tag{1.16}$$

This will be used later.

In what follows, we investigate the nature of the constants $C_{d.k}$, as well as the nature of *extremal functions*, i.e., $f(z)$ $(\not\equiv 0)$ satisfying (1.1) and for which

$$J(f) = C_{d.k}. \tag{1.17}$$

Our results are stated in §2, along with additional necessary background and notation, while the proofs of our results are given in §3.

2. Statement of results. As background for our first result, let $f(z) = \sum_{j=N}^{\infty} a_j z^j (a_N \neq 0)$ be in H^p, where $0 < p \leq \infty$, and let $Z_\Delta(f)$ again denote the collection of its zeros in $0 < |z| < 1$, with multiple zeros being repeated. Then

$$B(z) := \begin{cases} z^N \prod_{z_j \in Z_\Delta(f)} \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1 - \bar{z}_j z}\right), & \text{if } Z_\Delta(f) \text{ is not empty,} \\ z^N, & \text{if } Z_\Delta(f) \text{ is empty,} \end{cases} \tag{2.1}$$

is the *Blaschke product* associated with $f(z)$. It is known $B(z) \in H^\infty$ (cf. Rudin [**10**, p. 302]). Next (cf. (1.13) for the definition of $\hat{f}$),

$$F(z) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|\hat{f}(e^{it})| dt\right\} \tag{2.2}$$

is the *outer function* associated with $f(z)$. It is known $F(z) \in H^p$ (cf. [**10**, p. 331]). Continuing, the function

$$S(z) := f(z)/(B(z)F(z)) \tag{2.3}$$

is called the *singular inner function* associated with $f(z)$. We emphasize that the only zeros of $f(z)$ in $0 \leq |z| < 1$ are the zeros of its Blaschke product, $B(z)$, of (2.1). The product, $B(z)S(z)$, is called the *associated inner function* of $f(z)$ (cf. [**5**, §2.4] and [**10**,p. 338]).

Our first result is

THEOREM 1. ($k = 0$). *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$ *be analytic in* $|z| < 1$, *let* $d \in (0,1)$, *and assume that*

$$|a_0| \geq d \sum_{j=0}^{\infty} |a_j|. \tag{2.4}$$

Then, $f(z) \in H^\infty$ *and*

$$J(f) \geq \log d = C_{d.0}. \tag{2.5}$$

Equality holds in (2.5) *if and only if* $f(z)$ *is its own associated outer function multiplied by a constant of modulus one and equality holds in* (2.4). *Consequently, a function which is analytic in* $|z| \leq 1$ (*the closed disk*) *is extremal if and only if it has no zeros in* $|z| < 1$ (*the open disk*) *and equality holds in* (2.4).

Suppose $1/2 \leq d < 1$. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$ be analytic in $|z| \leq 1$ and satisfy $|a_0| = d\sum_{j=0}^{\infty} |a_j|$. If $|z| < 1$, then $|f(z)| \geq |a_0| - |\sum_{j=1}^{\infty} a_j z^j| > |a_0| - \sum_{j=1}^{\infty} |a_j| = (2 - 1/d)|a_0| \geq 0$. So, $f(z)$ has no zeros in $|z| < 1$. It follows from Theorem 1 that $f(z)$ is extremal. This shows there is a very simple mechanism for generating extremal functions if $k = 0$ and $1/2 \leq d < 1$.

To give an explicit extremal polynomial for the remaining case, namely $0 < d < 1/2$, let n be the positive integer such that $2^{-n} \leq d < 2^{-n+1}$, let $\rho := d2^{n-1}/(1 - d2^{n-1})$, and define $f(z) := (\rho + z)(1+z)^{n-1}$. Calculations based on (1.16) then show that $J(f) = \log d$. Further, (2.4) holds with equality.

For our next result, let $Z(f)$ denote *all* zeros (with multiple zeros being repeated) of $f(z)$, and let

(2.6)
$$\mathcal{H} := \Big\{ f(z) = \prod_{z_j \in Z(f)} (1 - \frac{z}{z_j}) : z_j \neq 0 \text{ for all } j, \sum_{z_j \in Z(f)} \frac{1}{|z_j|} < \infty, \text{ where } z_j \in Z(f) \text{ implies } \mathrm{Re}(z_j) < 0 \text{ and } \overline{z}_j \in Z(f) \Big\}.$$

Each element in $\mathcal{H}$ is an entire function of exponential type 0 (cf. Boas [**4**, p. 29]). If $f(z) \in \mathcal{H}$ and if $Z(f)$ is a finite set, then $f(z)$ is a real polynomial, all of whose zeros lie in $\mathrm{Re}(z) < 0$. Such polynomials are called *Hurwitz polynomials* (cf. Marden [**8**, p. 181]), and this accounts for the use of the symbol $\mathcal{H}$ in (2.6). We also remark that the functional $J(f)$ of (1.16) is well-defined for any $f(z)$ in $\mathcal{H}$. In analogy with (1.15), set

(2.7) $$C^{\mathcal{H}}_{d,k} := \inf \Big\{ J(f) : f(z) = \sum_{j=0}^{\infty} a_j z^j \text{ is in } \mathcal{H} \text{ and satisfies (1.1)} \Big\}.$$

An *extremal function in* $\mathcal{H}$ is a function, $f(z)$, in $\mathcal{H}$ satisfying (1.1) and for which

(2.8) $$J(f) = C^{\mathcal{H}}_{d,k}.$$

We need the following construction. For a (fixed) $d \in (0,1)$ and a (fixed) nonnegative integer k, we claim (cf. Lemma 3 of §3) that there is a unique positive integer n (dependent on d and k) such that

(2.9) $$\frac{1}{2^n} \sum_{j=0}^{k} \binom{n}{j} \leq d < \frac{1}{2^{n-1}} \sum_{j=0}^{k} \binom{n-1}{j}.$$

With this definition of n, set

(2.10) $$\rho := \frac{\binom{n-1}{k}}{\sum_{j=0}^{k} \binom{n-1}{j} - d2^{n-1}} - 1.$$

As we shall see (cf. Lemma 3 of §3), ρ satisfies $1 \leq \rho < \infty$. Note that, if $d = 1/2$, then $n = 2k + 1$ and $\rho = 1$.

THEOREM 2. ($k > 0$). *For d in* (0,1) *and for a positive integer k, let n and ρ be defined from* (2.9) *and* (2.10). *Then, for any $f(z)$ in $\mathcal{H}$ satisfying* (1.1),

$$J(f) \geq \log(\frac{\rho}{(\rho+1)2^{n-1}}) = C^{\mathcal{H}}_{d,k}. \tag{2.11}$$

Set $Q_{n,\rho}(z) := (1 + z/\rho)(1 + z)^{n-1}$. Then, $f(z)$ satisfying (1.1) *is an extremal element in $\mathcal{H}$ if and only if $f(z) = Q_{n,\rho}(z)$.*

Now, (2.9) and (2.10) make sense when $k = 0$. In this case, a computation shows that $\rho/((\rho + 1)2^{n-1}) = d$. Theorem 1 then establishes the truth of (2.11) even when $k = 0$. However, Theorems 1 and 2 also show that the extremal functions in $\mathcal{H}$ for the two cases, $k = 0$ and $k > 0$, are vastly different. There is an infinite number in the former case but precisely one in the latter.

Finally, we turn to the asymptotic behavior of $C^{\mathcal{H}}_{d,k}$, $k > 0$, as either $d \to 0+$ or $k \to +\infty$.

THEOREM 3. *For a fixed positive integer k,*

$$\lim_{d\to 0+} \frac{C^{\mathcal{H}}_{d,k}}{\log d} = 1, \tag{2.12}$$

and, for a fixed d in (0,1),

$$\lim_{k\to\infty} \frac{C^{\mathcal{H}}_{d,k}}{k} = -2\log 2. \tag{2.13}$$

It follows from (1.15) and (2.7) that

$$C_{d,k} \leq C^{\mathcal{H}}_{d,k} \quad (\text{for all } d \in (0,1), k = 0, 1, \dots). \tag{2.14}$$

On applying (2.12) and (2.14), we have, for a fixed k,

$$1 \leq \liminf_{d\to 0+} \frac{C_{d,k}}{\log d}, \tag{2.15}$$

and, on applying (2.13) and (2.14), we have, for a fixed d,

$$\limsup_{k\to\infty} \frac{C_{d.k}}{k} \le -2\log 2. \tag{2.16}$$

In §3, we use these and (1.4) to prove the following

COROLLARY. *For a fixed positive integer k,*

$$\lim_{d\to 0^+} \frac{C_{d.k}}{\log d} = 1, \tag{2.17}$$

and, for a fixed d in (0,1),

$$-2 \le \liminf_{k\to\infty} \frac{C_{d.k}}{k} \le \limsup_{k\to\infty} \frac{C_{d.k}}{k} \le -2\log 2. \text{cf. } (2.16) \tag{2.18}$$

We conjecture that $C_{d.k} = C^{\mathcal{H}}_{d.k}$.

3. Proofs. With the definitions of the spaces, $H^p (0 < p \le \infty)$, in (1.12) and the function, $\hat{f}(e^{i\theta})$, of (1.13), we begin with the

PROOF OF THEOREM 1. Assume $f(z) = \sum_{j=0}^{\infty} a_j z^j$ ($\not\equiv 0$) is analytic in $|z| < 1$, and satisfies

$$|a_0| \ge d \sum_{j=0}^{\infty} |a_j|. \tag{3.1}$$

As previously remarked in §1, the fact that $\sum_{j=0}^{\infty} |a_j|$ is finite implies $f(z) \in H^{\infty}$, as claimed in Theorem 1. Next, it follows from (3.1) that $|a_0| > 0$, since $f(z) \not\equiv 0$. Applying Theorem 17.17 of [**10**, p. 338],

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\hat{f}(e^{i\theta})|d\theta \ge \log|a_0|, \tag{3.2}$$

with equality holding if and only if the associated inner function for $f(z)$ is constant. Finally, using the functional $J(f)$ of (1.14), inequalities (3.1) and (3.2) imply

$$J(f) \ge \log|a_0| - \log\Big(\sum_{j=0}^{\infty} |a_j|\Big) \ge \log d, \tag{3.3}$$

the desired result of (2.5) of Theorem 1. Moreover, equality holds throughout (3.3) if and only if equality holds in both (3.1) and (3.2). If $f(z)$ is analytic in $|z| \le 1$, then Jensen's formula, (1.9), shows that equality in (3.2) is equivalent to there being no zeros of $f(z)$ in $|z| < 1$. From this, Theorem 1 follows. □

It is useful now to list some properties of elements in $\mathcal{H}$. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be in $\mathcal{H}$. Then:

(i) $a_0 = 1$ and $a_j \ge 0$ for all $j = 1, 2, \ldots$.

(ii) If $|Z(f)|$ denotes the cardinality of $Z(f)$, i.e., the number of its elements, then $a_j > 0$ for all $j = 0, 1, \ldots, |Z(f)|$, and $a_j = 0$ for all $j > |Z(f)|$.

(iii) If $f(\rho) = 0$ where $\rho < 0$, then $f(z)\backslash(1 - z/\rho)$ is in $\mathcal{H}$.

(iv) If $\rho < 0$, then $f(z)(1 - z/\rho)$ is in $\mathcal{H}$.

(v) If $f(\rho) = 0$ where ρ is nonreal, then $f(z)/((1 - z/\rho)(1 - z/\bar{\rho}))$ is in $\mathcal{H}$.

Because of (i), we note that $f(1) = \sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty} |a_j|$. Hence (cf. (1.16)),

$$J(f) = -\log(\Pi_{z_j \in Z_\Delta(f)} |z_j| \cdot f(1)) \quad (f(z) \in \mathcal{H}). \tag{3.4}$$

It is convenient to define the numbers

$$\delta_k(f) := \sum_{j=0}^{k} |a_j| \Big/ \sum_{j=0}^{\infty} |a_j|, \qquad \text{for } k = 0, 1, \ldots . \tag{3.5}$$

Note that (1.1) holds if and only if

$$\delta_k(f) \ge d. \tag{3.6}$$

LEMMA 1. *Suppose* $f(z) = \sum_{j=N}^{m} a_j z^j$, *where* $a_m \ne 0$. *(We allow the* a_j *to be complex.) Then*

$$J(f) \ge -m \log 2, \tag{3.7}$$

with equality if and only if $N = 0$ *and* $f(z) = a_0(\zeta - z)^m$, *where* $|\zeta| = 1$.

PROOF. Let $g(z) := \sum_{j=N}^{m} b_j z^j$ be the polynomial obtained from $f(z)$ by requiring that $g(-|\zeta|) = 0$ if and only if $f(\zeta) = 0$ (with matching multiplicities) and by requiring that $b_m = |a_m|$. Since the a_j and b_j are symmetric functions of the zeros of $f(z)$ and $g(z)$, respectively, it follows that $|a_j| \le |b_j|$ for all j. The definition of $g(z)$ and (1.9) imply that $\int_0^{2\pi} \log|f(e^{i\theta})|d\theta = \int_0^{2\pi} \log|g(e^{i\theta})|d\theta$. So, $J(g) \le J(f)$ (cf. (1.14)). Further, if $J(g) = J(f)$, then $|a_j| = b_j$ for all j; in particular, $|a_{m-1}/a_m| = b_{m-1}/b_m$. Since a_{m-1}/a_m and b_{m-1}/b_m are the sums of the zeros of $f(z)$ and $g(z)$, respectively, it follows from the definition of $g(z)$ that, if $J(g) = J(f)$, then the zeros of $f(z)$ must all have the same argument, i.e., they must all lie on a single ray emanating from the origin.

Write $g(z) = b_N z^N \prod_{j=1}^{m-N}(1 - z/z_j)$. It is geometrically evident that $0 < |z_j| < 1$ implies $|z_j| \cdot |1 - 1/z_j| = |z_j - 1| < 2$, and similarly, that $|z_j| \ge 1$ implies $|1 - 1/z_j| \le 2$, with equality if and only if $z_j = -1$. Since $\sum_{j=N}^{m} |b_j| = g(1) = |g(1)| = |b_N| \prod_{j=1}^{m-N} |1 - 1/z_j|$, it follows from (1.16) that

$$\begin{aligned} J(g) &= -\log\Big(\prod_{|z_j|<1} |z_j| \cdot \prod_{|z_j|<1} \Big|1 - \frac{1}{z_j}\Big| \cdot \prod_{|z_j|\ge 1} \Big|1 - \frac{1}{z_j}\Big|\Big) \\ &= -\log\left(\Big(\prod_{|z_j|<1} |z_j| \cdot \Big|1 - \frac{1}{z_j}\Big|\Big)\Big(\prod_{|z_j|\ge 1} \Big|1 - \frac{1}{z_j}\Big|\Big)\right) \\ &\ge -(m - N)\log 2, \end{aligned}$$

with equality if and only if all $z_j = -1$. Since $m - N \le m$, we have (3.7). Further, if $J(f) = -m\log 2$, the preceding remarks show that $f(z)$ must be of the form, $a_0(\zeta - z)^m$, for some $|\zeta| = 1$. A calculation based on (1.16) shows that, in fact, $J(a_0(\zeta - z)^m) = -m\log 2$ if $|\zeta| = 1$. This completes the proof of the lemma. □

We note that Mahler [**6**] obtained the inequality, (3.7) (see (4) of his paper). His method of proof was different, and he does not discuss when equality holds in (3.7). For related results, see Mahler [**7**].

LEMMA 2. *Let k be a positive integer, let $f(z)$ be in $\mathcal{H}$, suppose $|Z(f)| \ge k+1$, and suppose that z_1 and z_2 are any two (not necessarily distinct) zeros of $f(z)$, i.e., $z_1, z_2 \in Z(f)$. Unless* z_1 and z_2 are real with $z_1 = -1$ *and* $z_2 \leq -1$ (*or vice-versa*), *there exists an* $h(z) \in \mathcal{H}$ *such that*

$$J(f) > J(h) \tag{3.8}$$

and (*cf.* (3.5))

$$\delta_k(h) > \delta_k(f). \tag{3.9}$$

PROOF. First, suppose that at least one of $\mathrm{Im}(z_1)$ and $\mathrm{Im}(z_2)$ is not zero, say, $\mathrm{Im}(z_1) \neq 0$. From the hypotheses and the definition of $\mathcal{H}$ in (2.6), we know that $f(\bar{z}_1) = 0$. Let $g(z)$ and $h(z)$ be defined by

$$f(z) := \left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{\bar{z}_1}\right)g(z), \text{ where } g(z) := \sum_{j=0}^{\infty} b_j z^j,$$

and

$$h(z) := \left(1 + \frac{z}{\rho}\right)^2 g(z), \text{ where } \rho > 1.$$

From the previously listed properties of $\mathcal{H}$, $g(z)$ and $h(z)$ are in $\mathcal{H}$.

A calculation shows that

$$\delta_k(f) = \Big(\sum_{j=0}^{k-1} b_j + (|z_1|^2 b_k - b_{k-1})/|1 - z_1|^2\Big)/g(1)$$

and

$$\delta_k(h) = \Big(\sum_{j=0}^{k-1} b_j + (\rho^2 b_k - b_{k-1})/(1+\rho)^2\Big)/g(1).$$

Thus, $\delta_k(h) > \delta_k(f)$ if and only if

$$b_{k-1}\left(\frac{1}{|1 - z_1|^2} - \frac{1}{(1+\rho)^2}\right) > b_k\left(\frac{|z_1|^2}{|1 - z_1|^2} - \frac{\rho^2}{(1+\rho)^2}\right). \tag{3.10}$$

With $Z_\Delta(f)$ again denoting the zeros of f of moduli less than 1, set $Z' := Z_\Delta(f)\backslash\{z_1, \bar{z}_1\}$. Then, from (1.16),

$$J(f) = \log\left(\frac{\max\{|z_1|^2, 1\}}{g(1)|1 - z_1|^2 \prod_{\zeta \in Z'} |\zeta|}\right),$$

and

$$J(h) = \log\Big(\frac{\rho^2}{g(1)(1+\rho)^2 \prod_{\zeta \in Z'} |\zeta|}\Big).$$

Thus, $J(f) > J(h)$ if and only if

$$\text{(3.11)} \qquad \frac{\max\{|z_1|, 1\}}{|1 - z_1|} > \frac{\rho}{1+\rho}.$$

If $|z_1| < 1$, then $1/2 < 1/|1 - z_1| < 1$ because $\mathrm{Re}(z_1) < 0$. Hence, there is a $\rho > 1$ such that

$$\frac{1}{|1 - z_1|} > \frac{\rho}{1+\rho} > \frac{|z_1|}{|1 - z_1|}.$$

The left inequality above shows that (3.11) holds and, as $\rho > 1$, also shows that $1/|1 - z_1| > 1/(1 + \rho)$. Thus, the coefficient of b_{k-1} in (3.10) is positive. On the other hand, the right inequality above shows that the coefficient of b_k in (3.10) is negative. Since $|Z(f)| \geq k + 1$ by hypothesis, it follows that $b_{k-1} > 0$. From the previously listed properties of $\mathcal{H}$, it follows that $b_k \geq 0$. So (3.10) is valid.

If $|z_1| \geq 1$, then $1/2 < |z_1|/|1 - z_1| < 1$ because $\mathrm{Re}(z_1) < 0$ and $z_1 \neq -1$. Hence, there is a $\rho_1 > 1$ such that

$$\frac{|z_1|}{|1 - z_1|} = \frac{\rho_1}{1 + \rho_1}.$$

So $1 + 1/\rho_1 = |1 - 1/z_1| < 1 + 1/|z_1|$. This implies that $\rho_1 > |z_1|$ which, in turn, implies that $1/|1 - z_1| > 1/(1 + \rho_1)$. Thus, the right side of (3.10) is zero if $\rho = \rho_1$ and the left side is positive. It follows by continuity that there is some ρ in $(1, \rho_1)$ such that both (3.10) and (3.11) hold.

Now, suppose that $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = 0$. There are three cases. First, suppose one of z_1 and z_2 is in the open interval $(-1, 0)$, e.g., $-1 < z_1 < 0$. Redefine $g(z)$ and $h(z)$ by

$$f(z) : \Big(= (1 - \frac{z}{z_1}\Big) g(z), \text{ where } g(z) := \sum_{j=0}^{\infty} b_j z^j,$$

and

$$h(z) := (1+z)g(z).$$

A calculation shows that

$$\delta_k(f) = \Big(\sum_{j=0}^{k-1} b_j - z_1 b_k/(1-z_1)\Big)/g(1)$$

and

$$\delta_k(h) = \Big(\sum_{j=0}^{k-1} b_j + b_k/2\Big)/g(1).$$

Thus, $\delta_k(h) > \delta_k(f)$ if and only if $1/2 > -z_1/(1-z_1)$, which is always true for z_1 in $(-1,0)$. Redefine $Z' := Z_\Delta(f)\backslash\{z_1\}$. From (1.16),

$$J(f) = \log\Big(\frac{1}{g(1)(1-z_1)\prod_{\zeta\in Z'}|\zeta|}\Big)$$

and

$$J(h) = \log\Big(\frac{1}{2g(1)\prod_{\zeta\in Z'}|\zeta|}\Big).$$

Thus, $J(f) > J(h)$ if and only if $1/(1-z_1) > 1/2$, and the last inequality is certainly true. This completes the first case.

Next, suppose that $\text{Im}(z_1) = \text{Im}(z_2) = 0$ and that both z_1 and z_2 are in the interval $(-\infty, -1)$. In addition, suppose that

$$1 - z_1 - z_2 - z_1 z_2 \geq 0. \tag{3.12}$$

Redefine $g(z)$ and $h(z)$ by

$$f(z) := \Big(1-\frac{z}{z_1}\Big)\Big(1-\frac{z}{z_2}\Big)g(z), \text{ where } g(z) := \sum_{j=0}^{\infty} b_j z^j,$$

and

$$h(z) := (1+z)\Big(1+\frac{z}{\rho}\Big)g(z), \text{ where } \rho > 1.$$

As in the derivation of (3.10), $\delta_k(h) > \delta_k(f)$ if and only if

$$\begin{aligned} b_{k-1}\Big(\frac{1}{(1-z_1)(1-z_2)} - \frac{1}{2(1+\rho)}\Big) \\ > b_k\Big(\frac{z_1 z_2}{(1-z_1)(1-z_2)} - \frac{\rho}{2(1+\rho)}\Big). \end{aligned} \tag{3.13}$$

As in the derivation of (3.11), $J(f) > J(h)$ if and only if

$$\frac{z_1 z_2}{(1-z_1)(1-z_2)} > \frac{\rho}{2(1+\rho)}. \tag{3.14}$$

If equality holds in (3.12), then the left side of (3.14) becomes equal to $1/2$, and (3.14) is true for all $\rho > 1$. Further, since the coefficient of b_k in (3.13) is positive and tends to zero as $\rho \to \infty$ and since, as mentioned before, $b_{k-1} > 0$, it follows that (3.13) can be made true by choosing ρ sufficiently large.

So, suppose strict inequality holds in (3.12). From the fact that z_1 and z_2 are in $(-\infty, -1)$, we have that $1/4 < z_1 z_2 / \big((1-z_1)(1-z_2)\big) < 1/2$. Consequently, there is a $\rho_2 > 1$ such that

$$\frac{z_1 z_2}{(1-z_1)(1-z_2)} = \frac{\rho_2}{2(1+\rho_2)}.$$

In turn, this implies that

$$\frac{1}{2(1+\rho_2)} = \frac{2-(z_1+1)(z_2+1)}{2(1-z_1)(1-z_2)} < \frac{1}{(1-z_1)(1-z_2)}.$$

Thus, the right side of (3.13) is zero if $\rho = \rho_2$ and, since $b_{k-1} > 0$ as before, the left side of (3.13) is positive. It follows by continuity that there is some ρ in $(1, \rho_2)$ such that both (3.13) and (3.14) hold.

Finally, suppose that $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = 0$, that z_1 and z_2 are in $(-\infty, -1)$, but that (3.12) does not hold. Leave $g(z)$ and $\{b_j\}_{j=0}^{\infty}$ as last defined, but redefine $h(z)$ by $h(z) := (1 + z/\rho)g(z), \rho > 1$. Then $\delta_k(h) > \delta_k(f)$ if and only if

$$\frac{b_{k-1}}{(1-z_1)(1-z_2)} > b_k\left(\frac{z_1 z_2}{(1-z_1)(1-z_2)} - \frac{\rho}{1+\rho}\right), \tag{3.15}$$

and $J(f) > J(h)$ if and only if

$$\frac{z_1 z_2}{(1-z_1)(1-z_2)} > \frac{\rho}{1+\rho}. \tag{3.16}$$

It follows from the assumption of the falsity of (3.12) that $1/2 < z_1z_2/\big((1-z_1)(1-z_2)\big) < 1$. So, there is a $\rho_3 > 1$ such that

$$\frac{z_1 z_2}{(1-z_1)(1-z_2)} = \frac{\rho_3}{1+\rho_3}.$$

By continuity, there is some ρ in $(1, \rho_3)$ such that both (3.15) and (3.16) hold. □

LEMMA 3. *For d in (0,1) and for a nonnegative integer, k, there is a unique positive integer, n, dependent on d and k, such that*

$$\frac{1}{2^n}\sum_{j=0}^{k}\binom{n}{j} \le d < \frac{1}{2^{n-1}}\sum_{j=0}^{k}\binom{n-1}{j}. \tag{3.17}$$

Moreover, if the number ρ is defined by

$$\rho := \frac{\binom{n-1}{k}}{\sum_{j=0}^{k}\binom{n-1}{j} - d2^{n-1}} - 1, \tag{3.18}$$

then $\rho \ge 1$.

PROOF. Given any nonnegative integer k, consider the sequence

$$\left\{\frac{1}{2^l}\sum_{j=0}^{k}\binom{l}{j}\right\}_{l=k}^{\infty}, \tag{3.19}$$

whose initial term is unity. We claim that this sequence is strictly decreasing and has limit zero. To see this, for convenience set

$$a_l := \frac{1}{2^l}\sum_{j=0}^{k}\binom{l}{j} \quad (l = k, k+1, \dots). \tag{3.20}$$

Since

$$\binom{l+1}{j} = \binom{l}{j} + \binom{l}{j-1}, \tag{3.21}$$

it follows from (3.20) that

$$a_{l+1} = a_l - \frac{1}{2^{l+1}}\binom{l}{k} \quad (l = k, k+1, \dots),$$

which implies that (3.19) is strictly decreasing. Next, as a consequence of the Central Limit Theorem (cf. Patel and Read [**9**, pp. 169-170]), we have

$$\left| a_l - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(2k+1-l)/\sqrt{l}} e^{-t^2/2} dt \right| < \frac{0.28}{\sqrt{l}} \tag{3.22}$$

for all $l \geq \max\{k; 1\}$. As k is fixed, (3.22) shows that $a_l \to 0$ as $l \to \infty$. (It is certainly the case that there are simpler ways of showing $a_l \to 0$ than by using (3.22). However, (3.22) is used in an important way later to establish the falsity of (3.32) and (3.33).)

So, for d in $(0,1)$, the strictly decreasing nature of the a_l of (3.20) implies there is a unique positive integer n, with $n \geq k+1$, such that (3.17) is satisfied. It follows directly from (3.17) and (3.21) that ρ, defined in (3.18), satisfies $\rho > 1$. □

PROOF OF THEOREM 2. Since the right side of (2.11) is monotone increasing in $\rho \geq 1$, and bounded above by $-(n-1)\log 2$, it follows from Lemma 1 that there is no need to consider polynomials in $\mathcal{H}$ of degree less than n. It follows (cf. (3.19)) from the definition of n in (2.9), that $n \geq k+1$. Lemma 2 then implies that it is sufficient to suppose that $f(z) = (1 + {}^{z}\!/_{\rho})(1+z)^{m-1}$, where $m \geq n$ and $\rho' \geq 1$. Since this $f(z)$ must satisfy (1.1), it can be shown that $m \leq n$, and if $m = n$, then $\rho \leq \rho'$, where ρ is defined now in (2.10). Thus, we need only consider the case when $m = n$ and $\rho \leq \rho'$. A computation based on (3.4) shows that

$$J\left(\left(1 + \frac{z}{\rho'}\right)(1+z)^{n-1}\right) = \log\left(\frac{\rho'}{(1+\rho')2^{n-1}}\right).$$

We note that the quantity inside the logarithm is a strictly increasing function of ρ'. Consequently, with

$$Q_{n,\rho}(z) := \left(1 + \frac{z}{\rho}\right)(1+z)^{n-1},$$

we have that

$$J(Q_{n,\rho}) = \min\{J(f) : f(z) \in \mathcal{H} \text{ and } f(z) \text{ satisfies}(1.1)\}.$$

This establishes (2.11) and completes the proof of Theorem 2. □

PROOF OF THEOREM 3. We first prove (2.12). Let k be a fixed positive integer. For each d in $(0,1)$, let n and ρ be defined from (2.9) and (2.10). From Theorem 2, we have (cf. (2.11))

$$C^{\mathcal{H}}_{d,k} = \log\left(\frac{\rho}{1+\rho}\right) - (n-1)\log 2. \tag{3.23}$$

Since $1 \leq \rho < \infty$, it follows that $-\log 2 \leq \log(\rho/(1+\rho)) < 0$. So

$$-n\log 2 \leq C^{\mathcal{H}}_{d,k} < -(n-1)\log 2.$$

Write $n = n(d)$ to denote the dependence of n on d. Then the above inequalities become

$$\frac{-(n(d)-1)\log 2}{\log d} < \frac{C^{\mathcal{H}}_{d,k}}{\log d} \leq \frac{-n(d)\log 2}{\log d}. \tag{3.24}$$

Thus, to prove $\lim_{d\to 0^+}(C^{\mathcal{H}}_{d,k}/\log d) = 1$, i.e., (2.12), it suffices to show

$$\lim_{d\to 0^+} \frac{-n(d)\log 2}{\log d} = 1. \tag{3.25}$$

From the definition of a_l in (3.20) and from (3.17), we have that

$$\log a_{n(d)} \leq \log d < \log a_{n(d)-1}. \tag{3.26}$$

Short calculations based on the definition of a_l establish both

$$\lim_{l\to\infty} \frac{\log a_{l+1}}{\log a_l} = 1 \tag{3.27}$$

and

$$\lim_{l\to\infty}\left(\frac{\log a_l}{-l\log 2}\right) = 1. \tag{3.28}$$

It follows from (3.26) and (3.27) that

$$\lim_{d \to 0^+} \frac{\log a_n(d)}{\log d} = 1. \tag{3.29}$$

Combining (3.28) and (3.29) then gives (3.25).

To establish (2.13) of Theorem 3, fix d in $(0,1)$ and consider $C^{\mathcal{H}}_{d,k}$ as $k \to \infty$. Again, let n and ρ be defined by (2.9) and (2.10), and write $n = n_k$ to denote the dependence of n on k. Then (3.23) can be written as

$$C^{\mathcal{H}}_{d,k} = \log\left(\frac{2\rho}{1+\rho}\right) - n_k \log 2, \tag{3.30}$$

and (3.26) becomes

$$a_{n_k} \leq d < a_{n_k - 1}.$$

Thus,

$$\limsup_{k \to \infty} a_{n_k} \leq d \leq \liminf_{k \to \infty} a_{n_k - 1}. \tag{3.31}$$

Now, suppose that

$$\liminf_{k \to \infty} \frac{n_k}{2k} < 1. \tag{3.32}$$

Then, there is an $\varepsilon > 0$ and a sequence of positive integers $\{k_l\}_{l=1}^{\infty}$ with $\lim_{l \to \infty} k_l = \infty$ such that

$$\frac{n_{k_l}}{2k_l} \leq 1 - \varepsilon \quad (l = 1, 2, \dots).$$

For ease of notation, write $n(k_l) = n_{k_l}$. Then the above inequality implies that

$$\frac{2k_l + 1 - n(k_l)}{\sqrt{n(k_l)}} \geq \frac{1 + 2\varepsilon k_l}{\sqrt{2(1-\varepsilon)k_l}} \to +\infty, \text{ as } l \to \infty.$$

With l replaced by $n(k_l)$ in (3.22), (3.22) can be used to show that $a_{n(k_l)} \to 1$, which contradicts (3.31). Thus, (3.32) is false. Similarly, assuming that

$$\limsup_{k \to \infty} \frac{n_k}{2k} > 1, \tag{3.33}$$

(3.22) can now be used to show that $a_{n(k_l)} \to 0$, again contradicting (3.31). Hence, (3.33) is also false. This proves

$$\lim_{k\to\infty} \frac{n_k}{2k} = 1. \tag{3.34}$$

Now, divide by k in (3.30). Noting that $0 \le \log(2\rho/(1+\rho)) < \log 2$ and using (3.34), it follows that

$$\lim_{k\to\infty} \frac{C^{\mathcal{H}}_{d,k}}{k} = -2\log 2,$$

the desired result, (2.13), of Theorem 3. □

PROOF OF COROLLARY. To establish (2.17), it follows from (2.15) that it is enough to show that

$$\limsup_{d\to 0^+} \frac{C_{d,k}}{\log d} \le 1. \tag{3.35}$$

Let $t_0 > 1$. Using (1.4),

$$\frac{C_{d,k}}{\log d} \le \inf_{1<t<\infty} \left\{ t + \frac{(t\log 2) - t\log\left((t-1)\left(\left(\frac{t+1}{t-1}\right)^{k+1} - 1\right)\right)}{\log d} \right\}$$
$$\le t_0 + \frac{(t_0\log 2) - t_0\log\left((t_0-1)\left(\left(\frac{t_0+1}{t_0-1}\right)^{k+1} - 1\right)\right)}{\log d}.$$

Hence,

$$\limsup_{d\to 0^+} \frac{C_{d,k}}{\log d} \le t_0.$$

Since the only restriction on t_0 was that $t_0 > 1$, it follows that (3.35) must hold.

To establish (2.18), it follows from (2.16) that it is enough to show that

$$\liminf_{k\to\infty} \frac{C_{d,k}}{k} \ge -2. \tag{3.36}$$

Let $t_1 > 1$. Using (1.4),

$$\frac{C_{d,k}}{k} \geq \sup_{1<t<\infty} \left\{ \frac{t \log\left(2d/(t-1)\right)}{k} - \frac{t \log\left(\left(\frac{t+1}{t-1}\right)^{k+1} - 1\right)}{k} \right\}$$
$$\geq \frac{t_1 \log\left(2d/(t_1-1)\right)}{k} - \frac{t_1 \log\left(\left(\frac{t_1+1}{t_1-1}\right)^{k+1} - 1\right)}{k}.$$

Hence,

$$\liminf_{k\to\infty} \frac{C_{d,k}}{k} \geq -t_1 \lim_{k\to\infty} \log\left(\left(\frac{t_1+1}{t_1-1}\right)^{k+1} - 1\right)^{1/k}$$
$$= -t_1 \log\left(\frac{t_1+1}{t_1-1}\right).$$

Letting $t_1 \to \infty$, we get (3.36). □

REFERENCES

1. L.V. Ahlfors, *Complex Analysis*, third edition, McGraw-Hill Book Co., New York, 1979.

2. B. Beauzamy and P. Enflo, *Estimations de produits de polynômes*, J. Number Theory **21** (1985), 390-412.

3. ———, *Jensen's inequality for polynomials with concentration at low degrees*, Numer. Math. **49** (1986), 221-225.

4. R.P. Boas, *Entire Functions*, Academic Press, Inc., New York, 1954.

5. P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

6. K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika **7** (1960), 98-100.

7. ——— , *On two extremum properties of polynomials*, Ill. J. Math. **7** (1963), 681-701.

8. M. Marden, *Geometry of Polynomials*, Mathematical Surveys Number 3, American Mathematical Society, Providence, R.I. 1966.

9. J.K. Patel and C.B. Read, *Handbook of the Normal Distribution*, Marcel Dekker, New York, 1982.

10. W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Co., New York, 1966.

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF MISSOURI-ROLLA, ROLLA, MO 65401

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF MISSOURI-ROLLA, ROLLA, MO 65401 AND DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409

INSTITUTE FOR COMPUTATIONAL MATHEMATICS, KENT STATE UNIVERSITY, KENT, OH 44242