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RATIONAL APPROXIMATION-ANALYSIS OF THE WORK OF PEKARSKIĬ

J. PEETRE AND J. KARLSSON

ABSTRACT. We discuss two great papers by the Soviet mathematician A.A. Pekarskiĭ (Mat. Sb. **124** (1984), 571-588 and **127** (1985), 3-19), where a decisive step is made in the long open problem of describing the space of functions admitting a given rate of best *rational approximation*. We also indicate several directions for further work.

0. Introduction. The main objective of this talk is to report (§4 -§6) on two great papers by A.A. Pekarskiĭ [**16**], [**17**], which, as far as we can see, constitute a real breakthrough in the long open problem concerning the rate of best *rational approximation* in the L_p-metric.

Whereas much work has been devoted to the corresponding problem for *polynomial* approximation (originally only in the case of the uniform or Chebyshev norm; see any monograph on approximation theory), our understanding of *rational approximation* from this point of view has until recently been rather meager (we summarize some previous results in §2). The difficulty comes from the circumstance that the problem (in the case of *rational approximation*) is to some extent *non-linear* in nature (the set of rational functions of degree not exceeding a given number is not a vector space). From the abstract point of view of interpolation of normed Abelian groups [**15**], it is that in this case one has a *non-Archimedean* norm, not an *Archimedean* one.

Approximation theorists usually tend to work on an interval. Pekarskiĭ's result is formulated for the case of the unit circumference $\mathbf{T}$ and all functions are assumed to be analytic in the unit disk D (or rather distributional boundary values of functions analytic there; this latter assumption, however, is not so very restrictive as it sounds) so that function theoretic techniques becomes available. More precisely, his main result:

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THEOREM . $R^{\alpha}_{p\sigma} = B^{\alpha}_{\sigma}$ *where* $1/\sigma = 1/p + \alpha, 1 < p < \infty, \alpha > 0$.

Thus the surprising (?) fact is that the *approximation spaces* $R^{\alpha}_{p\sigma}$ (definition in §1) are exactly certain *Besov spaces* (definition in §1). Approximate results in this sense (within an ε) have been known for quite a time (see §2).

We consider this an excellent opportunity to speak about the work of an outstanding analyst. It is thus hoped that this compilation will serve as introduction to Pekarskiĭ's papers. In the last §7 we indicate some directions for further work.

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1. Definitions. The scale of Besov spaces B^{α}_{pq} (or $B^{\alpha q}_{p}$ or any other combination of the indices), where $-\infty < \alpha < \infty, 0 < p, q \leq \infty$, can be defined for any manifold (without boundary), thus especially for the unit circumference $\mathbf{T}$: in the latter case though, we are here mainly interested in functions (distributions) which are (distributional) boundary values of functions analytic in the unit disk $D(\partial D = \mathbf{T})$. If $0 < p < 1$ they should be thought of as "modelled" on the Hardy space H_p, rather than L_p. It is convenient to put $B^{\alpha}_{p} = B^{\alpha}_{pp}$.

Among the many definitions available the one based on "dyadic" decomposition is the one which is most productive in the present context. Let us recall that - we are now thinking of a group manifold (e.g., $\mathbf{T}$) - that $f \in B^{\alpha}_{pq}$ if and only if f can be written as $f = \Sigma_{\nu}\varphi_{\nu}$ where $\varphi_{\nu} \in L_p(H_p)$ with $\text{supp}\hat{\varphi}_{\nu} \subset \{2^{\nu-1} < |\xi| < 2^{\nu+1}\}$ and

$$\left[\sum_{\nu}(2^{\nu\alpha}||\varphi_{\nu}||_p)^q\right]^{1/q} < \infty.$$

It is important here to bear in mind the "meaning" of the parameters: α measures the smoothness of the function (how may (fractional) derivatives we have control of) and p has to do with the metric (L_p or H_p), the parameter q being connected with the interpolation process.

We denote by L_p^α, where $-\infty < \alpha < \infty$, $1 \leq p < \infty$, the (Bessel) potential or Lipschitz spaces. They are described by the property that the α-th fractional derivative (suitable defined) is *exactly* in L_p. For $0 < p < 1$ we denote by H_p^α the corresponding spaces "modelled" on H_p.

For an introduction to the theory of Besov (and potential) spaces see [**13**] or [**1**, Chapter 6]. For a more scholarly treatment we refer to Triebel's books (e.g., [**34**]).

If X is "any" metric (we are now thinking of $\mathbf{T}$ or else sometimes a bounded interval $\subset \mathbf{R}$) we denote by $\varrho_n(f, X)$ the best approximation of the function f in X by rational functions of degree $\leq n$:

$$\varrho_n(f, X) = \inf \|f - \psi_n\|_X,$$

where ψ_n is rational, $\deg \psi_n \leq n$. If $X = L_p$ (or $X = H_p$) we write $\varrho_n(f, p)$ or, in particular, $p = \infty$ we suppress the last argument, thus writing $\varrho_n(f)$ for $\varrho_n(f, \infty)$. Also in the "analytic" case, according to our convention, the poles of the approximating functions ψ_n should all lie outside the closed disk $\overline{D}$.

We define then the *approximation spaces* R_{Xq}^α, where $\alpha > 0, 0 < q \leq \infty$, by the requirement that $f \in R_{Xq}^\alpha$ if and only if

$$\Big[\sum_\nu (2^{\nu\alpha} \varrho_{2^\nu}(f, X))^q\Big]^{1/q} < \infty.$$

(The connection with the *rate of best approximation* is defined if we take $q = \infty$: $f \in R_{X\infty}^\alpha$ if and only if $\varrho_n(f, X) = O(n^{-\alpha})$.) Again, if $X = L_p$ (or $X = H_p$) we write just R_{pq}^α. If $X = BMO$ we write R_{*q}^α.

2. History. Even though the initial investigations of *rational approximation*, characterization of the best approximant and some model problems were carried out more or less simultaneously with the corresponding work on polynomial approximation by Chebyshev and his student Zolotarev, the subject lay largely dormant for decades. Work in trying to characterize the rate of rational approximation increased in volume in the 1960's, largely triggered off by Newman's paper [**12**] on the approximation of $|x|$. Before that, however, developments in the

Soviet Union had produced an "almost" characterization of the degree of rational approximation, due to Gonchar [**9**].

Here the main idea is to allow exceptional sets. Gonchar proved in 1955 [**9**] that if $\varrho_n(f) = O(n^{-1-\delta})$ for some $\delta > 0$ then f is differentiable a.e. He went on to prove

THEOREM . *Let f be a function such that, for any $\varepsilon > 0$, we can find a perfect set P_ε of almost full measure, $m(I/P_\varepsilon) < \varepsilon$, such that the restriction of f to P_ε is in Lip_α (with the usual convention if $\alpha > 1$). Then there exists, for any $\delta > 0$, a perfect set Q_δ with $m(I/Q_\delta) < \delta$ such that $\varrho_n(f, Q_\delta) = O(n^{-\alpha})$. Conversely, assuming the conclusion we arrive at the premise, but with* Lip_α *replaced by* $\mathrm{Lip}_{\alpha+\eta}, \eta > 0$. *(Of course, $\varrho_n(f, \cdot)$ is now the best uniform approximation on the set in question.)*

Newman's [**12**] exhibiting of a special function admitting a much better order of approximability by rationals than by polynomials led to a lot of work where various consequences are derived. Convex functions and functions of bounded variation were approximated using pasting arguments; Sziisz and Turan proved [**35**] that piecewise analytic functions have order of approximation $O(e^{-c\sqrt{n}})$ and [**33**] that functions whose r-th derivative is of bounded variation have order $O(\log^{2r+2} n/n^{r+1})$. The latter result was improved by Freud [**8**] to $O(\log^2 n/n^{r+1})$ and eventually by Popov [**29**] to $O(n^{-r-1})$. This final result, as observed by Freud, could be used to prove the so-called Newman conjecture: $f \in \mathrm{Lip}_1 \Rightarrow \varrho_n(f) = O(1/n)$. The proven Newman conjecture was an exception to the general belief that nothing sensible can be said about the degree of rational approximation in terms of the smoothness of the function. Restricted classes were studied, e.g., convex functions in Lip_α (Bulanov et al.). A good reference for the history of this and some of the more esoteric pasting work relating local approximating by polynomias and rationals to global *rational approximation* is [**24**].

An interesting almost characterization was obtained by Petrushev [**23**] (see also [**24**]) who, after improving on Popov's results on functions of bounded variations, studied functions with unbounded but manageable variation. Define the modulus of variation

$$\kappa(f;n) = \sup \sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})|$$

where the supremum is taken over all subdivisions of the interval with n points. Then a weak version of Petrushev's result can be stated:

THEOREM . *Suppose* $\kappa(f;n) = O(n^{\beta}), 0 < \varepsilon < 1,\ \alpha > 0,\ f \in \mathrm{Lip}_{\alpha}$. *Then* $\varrho_n(f) = O(n^{\beta+\varepsilon-1})$.

This fits (almost) with a converse result due to Popov [**28**]: $\varrho_n(f) = O(n^{\beta-1}),\ 0 < \beta < 1 \Rightarrow \kappa(f;n) = O(n^{\beta})$.

During these developments people began working systematically with different norms, relations between spline approximation and rational approximation (implicit in much previous work), and using *interpolation spaces.* This "wave of modern real variable methods" resulted, among other things, in an "almost" characterization due to Brudnyĭ [**3**].

THEOREM . $L^{\alpha}_{\sigma+0} \subset R^{\alpha}_{p,\infty} \subset L^{\alpha}_{\sigma-0}$ *where (as in Pekarskiĭ's theorem, see Introduction)* $1/\sigma = \alpha + 1/p,\ \alpha > 0,\ 1 \leq p \leq \infty$.

Here $L^{\sigma}_{\sigma+0} = \cup_{q>\sigma} L^{\alpha}_q$ and $L^{\alpha}_{\sigma-0} = \cap_{q<\sigma} L^{\alpha}_q$, L^{α}_q defined as in §1. The reader might want to refer to the paper by DeVore [**4**] for a nice clean proof of both Popov's and Brudnyĭ's results using certain maximal functions based on local approximation by polynomials.

Finally, a result that bears comparison to Pekarskiĭ's was obtained by Peller [**20**] in 1980, as a byproduct of his characterization of the symbols of Hankel operators in Schatten classes. This pertains to the limiting case $p = \infty$, but with the uniform metric (L_{∞} or H_{∞}) being replaced by BMO. Thus: $R^{\alpha}_{*1/\alpha} = B^{\alpha}_{1/\alpha}$. (Actually, the result in [**20**] is only for $\alpha < 1$; the extension to $\alpha > 1$ was obtained, apparently simultaneously, by Peller [**21**] and Semmes [**32**] (using Hankel theory) and by Pekarskiĭ [**20**] (directly).)

For the related problem of approximation by *spline functions* see, e.g., [**25**], [**26**], [**19**] (cf. §8).

3. Education: normed Abelian groups. This section briefly recalls the salient facts about the theory of interpolation of (quasi-)normed Abelian groups as developed, around 1970, by one of the authors and G. Sparr in [**15**] (see also Chapter 7 of the book [**1**], and for a more *ad hoc* presentation [**27**]). In fact, this theory was created precisely with applications to "non-linear" approximation in mind. Below we assume that the reader has some previous knowledge of interpolation. Numbers of theorems etc. refer to [**1**].

If A is any Abelian group, a quasi-norm in A is a functional $||\cdot||$ in A satisfying the quasi-triangle inequality: $||a+b|| \leq c(||a|| + ||b||)$ for some constant c independent of a and b in A. We have a norm if $c = 1$ (non-Archimedian case). If we have the stronger inequality $||a+b|| \leq c\max(||a||, ||b||)$ we have an ultranorm (Archimedian case).

LEMMA 3.10.1 (AOKI, ROLEWICZ). *For any quasinorm $||\cdot||$ there exists a norm $||\cdot||^*$ and a number ϱ such that $||a||^\varrho \approx ||a||^*$ (equivalence of quasinorms).*

For "compatible" (definition in [**1**]) pairs $\overline{A} = (A_0, A_1)$ of quasi-normed Abelian groups one can now define K- and J-*spaces* much as in the case of Banach spaces. Especially, one has $a \in \overline{A}_{\theta,q;K} = (A_0, A_1)_{\theta,q;K}$ if and only if $\int_0^\infty (t^{-\theta} K(t,a))^q dt/t < \infty$, where ($K$-*functional*)

$$K(t,a) = \inf_{a=a_0+a_1} (||a_0||_{A_0} + t||a_1||_{A_1}).$$

The usual theorems hold (equivalence theorem, reiteration theorem etc.). (In view of the former we subsequently may drop the K in the notation, thus writing simply $\overline{A}_{\theta,q}$ provided $0 < \theta < 1$.)

The point is that passing to the standpoint of quasi-normed Abelian groups allows additional flexibility: we can take *powers* of quasi-norms without destroying the quasi-norm property. More specifically, if A is any quasi-normed Abelian group and ϱ a positive number, let

$A^{(\varrho)}$ denote the same algebraic structure with the new quasi-norm $||a||A^{(\varrho)} = (||a||_A)^\varrho$.

THEOREM 3.11.6. (POWER THEOREM). *For any compatible pair* $\overline{A} = (A_0, A_1)$ *of quasi-normed Abelian groups,*

$$(A_0^{(\varrho_0)}, A_1^{(\varrho_1)})\eta, r = (A_0, A_1)_{\theta.q}^{(\varrho)}$$

holds with $\theta = \eta\varrho_1/\varrho,\ \varrho = (1-\eta)\varrho_0 + \eta\varrho_1,\ q = \varrho r$ *where* $0 < \varrho < 1, 0 < r \leq \infty$.

Now introduce a new functional (besides the K- and the J-functional), the *E-functional*:

$$E(t,a) = \inf_{||a_0||_{A_0} \leq t} ||a - a_0||_{A_1}.$$

which gives rise to new spaces, the *E-spaces*: $a \in \overline{A}_{\alpha q:E}$ if and only if

$(\int_0^\infty (t^\alpha E(t,a))^q dt/t)^{1/q} < \infty$. Complementing the usual equivalence theorem (connecting K and J, we have

THEOREM 7.1.7. *For any compatible pair* $\overline{A} = (A_0, A_1)$ *of quasi-normed Abelian groups,*

$$\overline{A}_{\alpha r:E}^{(\theta)} = \overline{A}_{\theta.q},$$

where $\theta = 1/(\alpha + 1), r = \theta q$.

Summarizing, approximation spaces are interpolation spaces. In particular, it is clear that in order to "describe" approximation spaces in concrete cases (as the one considered here) it suffices to prove inequalities of Bernstein and Jackson type.

We conclude by listing some of the basic examples to which the theory applies.

EXAMPLES. 1) $A_1 = L_\infty$, $A_0 = L_0$ = all measurable functions with the ultranorm $||a||L_0$ = means supp a. In this case $E(t,a) = a^*(t)$ (decreasing rearrangement).

2) Analogous example with operators in a Hilbert space. $A_1 =$ bounded operators, $A_0 = S_0 =$ finite rank operators T with $||T||S_0 =$ rank T.

3) $A_1 = L_p, A_0 =$ polynomials with $||f||A_0 =$ degree of f.

4) $A_1 = L_p$, $A_0 =$ rationals with again $||f||A_0 =$ degree of f (the number of poles).

Notice that we are in the Archimedean case in example 3) and in the non-Archimedean case in example 4), as foreseen in §0.

4. Converse (the Bernstein inequality). In [**16**] Pekarskiĭ establishes four inequalities of Bernstein type:

(1) $||r||_{H_\sigma^\alpha} \leq c_1(\alpha, p)n^\alpha ||r||_{H_p}$.

(2) $||r||_{B_\sigma^\alpha} \leq c_2(\alpha, p)n^\alpha ||r||_{H_p}$.

(3) $||r||_{H_{1/\alpha}^\alpha} \leq c_3(\alpha, p)n^\alpha ||r||_{BMOA}$.

(4) $||r||_{B_{1/\alpha}^\alpha} \leq c_1(\alpha, p)n^\alpha ||r||_{BMOA}$.

Here, as before, $1/\sigma = \alpha + 1/p$ (with $1 < p \leq \infty$) and r stands for a rational function with all its poles off $\overline{D}, n$ being their number (the degree of r). By what was said in §3, it is clear that his main theorem is a consequence of (2). Now we proceed to the proof of (1).

A. *Pekarskiĭ's Proof.* In the definition of the Hardy-Sobolev spaces H_σ^α it is, in the present context, expedient to to use the Riemann-Liouville derivative, not, as usual, the Weyl derivative. Indeed, for any f (analytic in D) we set (see [**16**], formula (1))

$$f^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{|\zeta|=\varrho} f(\zeta)\left(1 - \frac{z}{\zeta}\right)^{-1-\alpha} \zeta^{-1-[\alpha]} d\zeta,$$

$$|z| < \varrho < 1 \quad ([\alpha] = \text{ integer part of } \alpha).$$

Then the main step in Pekarskiĭ's proof of (1) is the modified formula [**16**, formula (31)]

$$(5) \qquad r^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{T} r(\zeta)\left(1 - \frac{B(z)}{B(\zeta)}\right)^{1+\alpha} \left(1 - \frac{z}{\zeta}\right)^{-1-[\alpha]} d\zeta,$$

valid for all rational functions r with poles outside $\overline{D}$. Here B is the Blaschke product

$$B(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z},$$

where a_0 and $1/\overline{a}_1, \ldots, 1/\overline{a}_n$ are the poles of r. The idea is that, although the factor $(1 - z/\zeta)^{-1-\alpha}$ blows up as z approaches the boundary point ζ, this is compensated by the skillfully chosen factor $(1 - B(z)/B(\zeta))^{-1-\alpha}$, which is small there. The proof of (5) is immediate; just use Cauchy's formula.

The rest of the proof of (1) is now embodied in four lemmas, which we for convenience state *in extenso*, although it is only lemma 4 that is needed to understand the line of thought.

Set

$$Q(z, \zeta) = \frac{B(z) - B(\zeta)}{z - \zeta},$$

$$\lambda(z, \beta) = \sum_{k=0}^{n} \left(\frac{1 - |a_k|}{|z - a_k|}\right)^{\beta} \frac{1}{|z - a_k|} (\beta > 0).$$

LEMMA 1. *Let* $z \in \mathbf{T}$ *and* $1 \in \mathbf{N}$. *Then*

$$\frac{(2l - 1)!}{2\pi} \int_{\mathbf{T}} |Q(z, \zeta)|^{2l} |d\zeta|$$
$$= z^1 \sum_{j=1}^{1} \binom{2l}{j} (-1)^{1-j} B^{-j}(z) [B^j(z) z^{l-1}]^{(2l-1)}.$$

LEMMA 2. *Let* $z \in \mathbf{T}$ *and* $s \in \mathbf{N}$. *Then*

$$|B^{(s)}(z)| \leq 2^s s! \lambda^s(z, 1/s).$$

LEMMA 3. *Let* $z \in \mathbf{T}$ *and* $\alpha > 0$. *Then*

$$||Q(\cdot, z)||_{1+\alpha} \leq c(\alpha) \lambda^{\frac{\alpha}{\alpha+1}} \left(z, \frac{1}{\alpha + 2}\right).$$

LEMMA 4. *For* $f \in L^p(\mathbf{T})$, $p \in [1, \infty]$, *and* $\alpha > 0$ *set*

$$g(z) = \int_{\mathbf{T}} |Q(\zeta, z)|^{1+\alpha} f(\zeta)\ |d\zeta|.$$

Then $g \in L_\sigma$ *and*

$$||g||_\sigma \leq c(\alpha, p) n^\alpha ||f||_p$$

where $1/\sigma = 1/p + \alpha$.

COMMENT. Lemmas 1 and 2 are straightforward, especially Lemma 2. Lemma 3 follows from Lemma 1 and Lemma 2. Lemma 4 follows from Lemma 3 essentially by interpolation. We shall discuss this in detail in subsection B. It is clear that the basic inequality (1) follows from Lemma 4 using Pekarskiĭ's main formula (5). Inequality (2) again follows readily from (1) using an interpolation inequality. Again (3) and (5) are easy limiting cases and will not be discussed here.

B. *Schur interpolation.* At least as long as $\sigma \geq 1$, Lemma 4 can be derived from Lemma 3 by complex interpolation. However, there is a more direct route by adapting a classical argument originating from I. Schur's classical paper [**31**] (1911). This we will set forth now.

Recall first that the (Riesz-)Thorin theorem says that if T is a linear operator on some measure space such that

$$T : L_{p_0} \to L_{q_0}, \qquad T : L_{p_1} \to L_{q_1},$$

then

$$T : L_p \to L_q$$

for appropriate intermediate values of the exponents

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \qquad (\theta \in (0,1)).$$

As Riesz himself acknowledges in [**30**], the special case $p_0 = q_0 = 1$, $p_1 = q_1 = \infty$ is elementary and in principle contained in Schur [**31**]. We shall now show that the same type of argument can be used in an even somewhat more general situation. We consider only integral operators

$$Tf(x) = \int Q(x, y) f(y)\, dy.$$

THEOREM. *We have*

$$||Tf||_{q_1} \leq \left|\left| \int |Q(x,y)|\, dx \right|\right|_{p_0'}^{1-\theta} \cdot \left|\left| \int |Q(x,y)|\, dy \right|\right|_{q_1}^{\theta} ||f||_p$$

where

$$\frac{1}{q} = 1 - \theta + \frac{\theta}{q_1}, \ \frac{1}{p} = \frac{1-\theta}{p_0}, \ \theta \in [0,1].$$

Explanation. The finiteness of the expression $|| \int f|Q(x,y|\, dx||p_0'$ entails that $T : L^{p_0} \to L^1$ is a bounded map. Similarly, the finiteness of the second expression entails that $T : L^\infty \to L^{q_1}$ is bounded.

PROOF. We convert the problem into one of estimating the bilinear expression $\langle Tf, g\rangle$, $f \in L^{p_0}$, $g \in L^{q_1'}$. A clever application Hölder's inequality then completes the argument.

Actually, the present formulation of the theorem is yet insufficient. We have to add a version where T is allowed to vary. (In the case of complex interpolation this corresponds to interpolation of an "analytic" family $\{T_z\}_{0<\mathrm{Re}z<1}$.) In the present context we can do with a "convex" family $\{T_\theta\}_{0<\theta<1}$ instead. Indeed, it suffices to impose a condition of the type

$$|Q_\theta(x,y)| \leq |Q_0^{1-\theta}(x,y)||Q_1^\theta(x,y)| \tag{6}$$

on the kernels, to produce an analogous result involving

$$\left|\left| \int |Q_0(x,y)|\, dx \right|\right|_{p_0'} \text{ and } \left|\left| \int |Q_1(x,y)|\, dy \right|\right|_{q_1}. \tag{7}$$

We leave the details to the reader. □

C. *An interpolation inequality.* To get inequality (2) from inequality (1) one can use the general interpolation inequality

$$||f||_{B_\sigma^\alpha} \leq c||f||_{H_p}^{1-\theta}||f||_{H_\tau^\beta}^\theta, \tag{8}$$

where

$$\frac{1}{\sigma} = \frac{1}{p} + \alpha, \quad \frac{1}{\tau} = \frac{1}{p} + \alpha, \quad \alpha = \beta\theta, \quad \frac{1}{\sigma} = \frac{1-\theta}{p} + \frac{\theta}{\tau} \quad (\theta \in (0,1)).$$

See, e.g., [**34**] for a general background. Pekarskiĭ gives (in [**16**]) an *ad hoc* approach. However, his proof can be interpreted from a more general point of view. Let us briefly elaborate this.

There are classically two ways of characterizing Hardy classes: 1° via Littlewood-Paley theory (area functions and all that), 2° using suitable (non-tangential etc.) maximal functions. The traditional proof of (8) goes via 1°, whereas Pekarskiĭ's approach uses 2°. On the interpolation level it amounts to the following result, implicit in [**16**].

LEMMA. *Let K be a measurable function of two variables x, t (where $0 < t < \infty$). Then*

$$\| \, \|t^{-\alpha} K(x,t)\|_\sigma \|_\sigma \leq \text{const.} \| \sup_t |K| \, \|_p^{1-\theta} \| \sup_t t^{-\beta} |K| \, \|_\tau^\theta,$$

where $1/\sigma = (1-\theta)/p + \theta/\tau$, $\alpha = \theta\beta (\theta \in (0,1))$. (*The t integration is with respect to the measure dt/t.*)

5. Direct part (the Jackson inequality). We wish to establish the following statement, which, again by §3, is all we need to complete the proof of the main theorem.

CLAIM. For every function $f \in B_\sigma^\alpha$ there exists a rational function ψ with deg $\leq n$ such that

$$\|f - \psi^{(m)}\|_{H_p} \leq c(\alpha, p, m) n^{-\alpha} \|f\|_{B_\sigma^\alpha}.$$

(Here m is any integer such that $\sigma(m+1) > 1$.)

It is convenient to break up the argument into several steps.

Step 1. This involves a new characterization of Besov space. We write $f(\in B_\sigma^\alpha)$ in the form

$$f = \sum_{k=2}^{\infty} \varphi_k^{(m)}$$

where each φ_k is analytic in a concentric disk of radius $\mu_k = 1 + 2^{-k}$ and - this is the crucial part, of course - in addition

$$\left[\sum_{k=2}^{\infty} (2^{k\alpha} 2^{km} \|\varphi_k^{(m)}(\mu_k \cdot)\|_{H_p})^\sigma\right]^{1/p} < \infty.$$

This is not hard to prove. In one direction one can use the usual "dyadic" decomposition.

In the sequel it is convenient to put

$$a_k = 2^{km} \| \psi_k^{(m)}(\mu_k \cdot) \|_{H_p}.$$

Thus we have

$$\left[\sum_{k=2}^{\infty} (2^k a_k)^\sigma \right]^{1/\sigma} < \infty. \tag{1}$$

Step 2. One shows that for each k there exists a rational function ψ_k with $\deg \psi_k \leq n$ such that

$$\| \varphi_k^{(m)} - \psi_k^{(m)} \|_{H_p} \leq c 2^{k\alpha} n_k^{-\alpha} a_k.$$

Here a_k are the numbers appearing in (1) and n_k are any numbers such that $n = \Sigma n_k$. (Notice that we do *not* assume n_k, nor n, to be integers. If $n_k < 1$ we adopt the convention that $\psi_k \equiv 0$.)

This is also quite easy, in principle. The idea is as follows. One writes each ψ_k as a Cauchy integral over a circle "halfway" between the circles $|z| = 1$ and $|z| = \mu_k$ (the latter thus being the "boundary for analyticity"). Next one writes this "big" Cauchy integral as a sum of "small" Cauchy integrals extended over arcs $\gamma_{j,k}$ of length comparable to the number 2^{-k}. Finally, the approximating rational function ψ_k is obtained by putting together suitable finite segments of the Laurent developments of the small Cauchy integrals about the center of the arcs $\gamma_{j,k}$. Thus it is really only the elementa of function theory that participate.

Step 3. We have to show that with a clever choice of n_k the rational function $\psi = \Sigma \psi_k$ will do the job. (As $n_k < 1$ except for a finite set of indices this is really a finite sum.)

First we rearrange the a_k in decreasing order, denoting the rearranged sequence by b_k. Then (1) takes the form

$$\left[\sum (2^{t_k} b_k)^\sigma \right]^{1/\sigma} < \infty, \tag{2}$$

where thus t_k too is just some enumeration of the indices k.

We now invoke the following interpolation lemma, only implicit in [**17**].

LEMMA. *Let g_k be measurable functions such that*

1° $$\|g_k\|_{L_\sigma} \le b_k,$$

2° $$\|g_k\|_{L_\infty} \le b_k 2^{(t_k - s_k)/\sigma}$$

and (crucial if $p \ge 1$!)

3° $$2^{t_k^\alpha} b_k \approx (n_k/n)^{-1/\sigma} \text{ with } n = \sum n_k,\ n_k = 2^{s_k}.$$

Then

(3) $$\Big\|\sum g_k\Big\|_{L_p} \le C\Big(\sum (b_k 2^{\alpha(t_k - s_k)})^p\Big)^{1/p},$$

where C is a suitable constant, with the usual relations between the parameters, viz. $1/\sigma = \alpha + 1/p,\ 0 < p < \infty, \alpha > 0$.

Notice that
$$(b_k 2^{\alpha(t_k - s_k)})^p \approx n^{-\alpha p}(b_k 2^{t_k \alpha})^\sigma$$
so that the right hand side of (3) can also be written as

$$\Big[\sum (b_k 2^{t_k^\alpha})^\sigma\Big]^{1/\sigma}.$$

From this it is easy to complete the proof of the claim. (One chooses ψ_k such that 3° holds with $g_k = \varphi_m^{(m)} - \psi_k^{(m)}$; this fixes the choice of n_k.)

PROOF OF THE LEMMA. $0 < p \le 1$. This is easy (and does not require 3°). From a trivial limiting case of Hölder's inequality, viz.

$$\|g_k\|_{L_p} \le \|g_k\|_{L_\sigma}^{1-\theta} \|g_k\|_{L_\infty}^{\theta}$$

with (in general) $1/p = (1-\theta)/\sigma + \theta/\infty$, that is (in our case) $\theta = \alpha\sigma$, we infer that

$$\|g_k\|_{L_p} \le b_k 2^{\alpha(t_k - s_k)}.$$

Now use that (if $0 < p \le 1$)

$$\left\|\sum g_k\right\|_{L_p} \le \left(\sum \|g_k\|^p\right)^{1/p}. \tag{4}$$

$1 < p < \infty$. 3° helps us to circumvent the fact that (4) in general is not true if $p > 1$. We require, however, two more lemmata which are more or less well-known.

LEMMA 2.4 (in [**17**].)

$$\left(\int_0^\infty \left(\sum_{-\infty}^{\infty} d_j \Delta_j^{\alpha+1/p}(x)\right)^p dx\right)^{1/p} \le c\left[\sum_{-\infty}^{\infty} (2^{\alpha j} d_j)^p\right]^{1/p},$$

where $1 \le p \le \infty$, $\Delta_j(x) = \min(2^j, 1/x)$ *and the* d_j *are arbitrary positive numbers.*

LEMMA 2.3 (in [**17**].)

$$\int_0^\infty \sum_j \lambda_j(x))^p \, dx \le \int_0^\infty \left(\sum_j \lambda_j^*(x)\right)^p dx,$$

where $1 \le p < \infty$ *and the* λ_j *are positive measurable functions,* λ_j^* *being their increasing rearrangements.*

We notice that Chebyshev's inequality gives

$$g_k^*(x) \le b_k(\Delta_{t_k - s_k}(x))^{1/\sigma}.$$

Now write

$$\sum g_k = \sum_j \sum_{k \in G_j} g_k + \sum_{s_k < 0} g_k \equiv S' + S'',$$

where each set G_j consists of those indices k such that $s_k - t_k =$ const. $= j$. The sum S'' is easy (easier) to handle, so below we concentrate on S'. Our two lemmata show that

$$S' \leq c\Big(\sum_j \Big| \sum_{k \in G_j} 2^{\alpha(t_k - s_k)} b_k \Big|^p\Big)^{1/p}$$
$$\leq c\Big(\sum_k (2^{\alpha(t_k - s_k)} b_k)^p\Big)^{1/p},$$

where the last inequality depends on the following auxiliary reasoning (it is only here that condition 3° comes up). Within each set G_j the index s_j is essentially uniquely determined by t_k. Therefore the numbers $2^{\alpha(t_k - s_k)} b_k$ can be treated as a *geometric* progression. In a geometric series the largest term is the dominant one. Hence we can replace the sum with just one term.

6. Further comments. In this section we briefly review some "postliminary" cases of the main result discussed in the previous sections.

First of all, if $p = \infty$ there is a natural substitute for the usual (uniform) metric, namely, BMO. With this change, the main result formally carries over: $R^{\alpha}_{*1/\alpha} = B^{\alpha}_{1/\alpha}$, as $\sigma = \alpha$ if $p = \infty$. As already told in §2 this result was first obtained via the theory of Hankel operators. For the uniform metric itself (i.e., $X = L_\infty$ or rather H_∞) only results "within ε" are available, for time being, at least (for details see [**17**], especially p. 17-18, where an exact result with $\alpha > 1$ can be found). The case of the "Bloch metric" has been considered by Semmes [**32**].

Again, if $0 < p \leq 1$ a curious thing takes place. Let us introduce a *"modified" approximation space* $\overline{R}^{\alpha}_{pq}$, obtained exactly in the same way as R_{pq} upon replacing the quantity $\varrho_n(f,p)$ by the "modified" best rational approximation $\overline{\varrho}_n(f,p)$, where for a general metric,

$$\overline{\varrho}_n(f,X) = \inf ||f - \psi_n^{(m)}||_X,$$

where ψ_n is rational, deg $\psi_n \leq n$. Here m is an integer which (in the case at the hand) has to be adjusted so that $1/p - 1 < m \leq 1/p$. Then again the main result carries over: $\overline{R}^{\alpha}_{p\sigma} = B^{\alpha}_{\sigma}$ in the usual hypothesis $1/\sigma = \alpha + 1/p$ [**17**]. (With R in place of $\overline{R}$ the result is not true.)

What happens if $\alpha \to \infty$? Let us remark that Newman's result [**12**] formally agrees with Pekarskiĭ's in the sense that the function $|x|$ obviously belongs to all spaces $B^{\alpha}_{1/\alpha}$ (we are thinking of the case of the uniform metric). So the question arises of giving a description of the space of functions admitting a prescribed order of best rational approximation which is not necessarily a power.

Finally, let us remark that we have found $R^{\alpha}_{p\sigma}$ only in a case when the parameters are coupled in a special way (viz. $1/\sigma = \alpha + 1/p$). Really, what one would like to have had instead is the case $\sigma = \infty$, which, as we have seen (§1), is directly related to the original problem. From the "abstract nonsense" point of view (§3) there is a simple way out: Just introduce the space formally obtained by real interpolation from the "diagonal" Besov scale B^{α}_{σ} with $1/\sigma - \alpha =$ const. and we are in business. These new spaces, let us denote them by $G^{\alpha}_{\sigma q}$, where q is the interpolation parameter, are, apparently, very important in analysis, as they also arise in Hankel theory. They are no longer Besov spaces. There arises therefore the question of studying their properties, especially, trying to find a more concrete representation. One such description of "G-*spaces*" is briefly indicated in [**14**] and another one can be found in [**22**].

7. Perspective. Concluding, we would like to indicate some more directions for subsequent work. (Pekarskiĭ's methods being of such a general character, it is tempting to try to push the results as far as possible...)

1. First of all one may ask what can be said about *other metrics* (than L_p or H_p). Especially, what about the Lipschitz metric? Some results in this direction (based on Hankel operators) were indicated in [**14**].

2. Next, what about *other domains*, say, multiply connected domains Ω with a smooth boundary (so that the Besov spaces are defined)? If the domain is simply connected, by conformal mapping we are essentially back in the previous case of the unit disk D. We conjecture that Pekarskiĭ's theorem is valid also in this more general situation.

3. In this connection it is useful to notice the fact that a rational function as a quotient of polynomials is never used. What is really more important is the partial fraction expansion of rational functions.

This suggests that in more general situations, instead of rationals (or m-th derivatives of rationals), one should use finite sums of *reproducing kernels* of one kind or other. For instance, to every Ω there corresponds a natural reproducing kernel, namely the Bergman kernel $B_a (a \in \Omega)$. Define a new kernel K_a by the requirement $K'_a = B_a$, $K_a(a) = 0$. Thus we are led to approximate with finite sums $\Sigma c_\nu K_{a_\nu} (a_\nu \in \Omega)$.

4. *Connection with Hankel forms.* As we have already mentioned, the first definitive results on the rate of best rational approximation were, in the case of the BMO-metric, obtained by Peller [**20**], using Hankel operators. This connection with Hankel operators is, to some extent, still a mystery. In [**11**] a theory of Hankel *forms* (rather than operators) over quite general "homogeneous" domains (in any number of (complex) variables) has been developed and especially applied to the case of Fock space. In the latter case, one also has an approximation byproduct. Consider the space $F_p^\alpha(\mathbf{C}^n)$ of entire functions f in $\mathbf{C}^n$ such that $f(z) \cdot \exp(-\frac{1}{2} \cdot \alpha |z|^2) \in L_p(\mathbf{C}^n)$, where $1 \leq p < \infty$ and $\alpha > 0$. Then it is a question of approximation with finite sums $\Sigma c_\nu \exp(-z\bar{a}_\nu)$. It turns out that the corresponding approximation spaces (for approximation in the F_∞^α-metric) can be identified using the scale of spaces $F_p^\alpha(\mathbf{C}^n)$.

5. Somehow the above is also tied up what has been called *Bol's theorem* (cf. [**2**]), which is basic in function theory for, for instance, such things as *Eichler cohomology* [**7**]. Consider the group G of unimodular 2×2 matrices ($\varphi \in G$ if and only if $\varphi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$). Let the holomorphic function $f(z)$ transform under G according to the rule $f(z) \to f(\varphi z)(cz + d)^{\mu - 1}, \mu$ a fixed integer ≥ 0. Then Bol's theorem states that its μ-th derivative $f^{(\mu)}(z)$ changes by the rule $f^{(\mu)}(z) \to f^{(\mu)}(z)(cz + d)^{-\mu - 1}$. As a generalization of the Schwarz derivative, a basic object in conformal mapping, one can now form the expression $D^\lambda([f^{(\mu)}]^{(\lambda - 1)/(\mu + 1)})$ (where $D = d/dz$, λ another integer) which likewise changes "convariantly" under G. The nullspace of this differential operator consists of *Abelian functions.* Thus one is lead to an approximation problem with linear combinations of Abelian functions.

6. These functions are not quite unrelated to the previous reproducing kernels.

7. Since so much function theory has been involved in the previous

treatment, it is natural to ask what can be said about the case of *several complex variables.* A first major concern is then what are the natural objects with which we should try to approximate? In the light of what was said above, the following suggests itself quite naturally. Let $z = (z_0, \dots, z_n)$ be homogeneous coordinates in an n-dimensional space, restricted by the requirement $|z_0|^2 - |z_1|^2 - \cdots |z_n|^2 > 0$; in other words, if we project we get the complex unit ball. As approximating functions we now propose linear aggregates of rational functions of the special form $P_\mu(z)/L(z)$, where P_μ is a homogeneous polynomial (form) of degree μ and $L(z)$ a linear form not vanishing in the domain under consideration, so that, in particular, the pole divisor is a hyperplane. A first question is then whether such functions are dense in any of the usual function spaces, for instance, the ball algebra.

8. Latest developments. (added July 1986). In a recent paper [**19**] Pekarskiĭ has extended his techniques in [**16**] (cf. §4) to prove a Bernstein type inequality in the case of *rational approximation* on an interval. Combining this result with earlier results by Petrushev [**25**], [**26**] on spline approximation and recent work by DeVore and Popov [**36**], this gives a rather complete picture of what is going on in that case also. In a lecture at Lund (June 25, 1986) Peller gave an outline of how to reduce *rational approximation* on a circle (disk) to the case of an interval, and vice versa, based on the very interesting transformation theory of the scale of spaces B^α_σ as developed by Dyn'kin [**5, 6**].

9. Note (added Dec. 1986). This is an *abridged* version (all jokes omitted). The authors will send copies of the original, *uncensored* manuscript on request.

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MATEMATISKA INSTITUTIONEN, BOX 6701, S-11385, STOCKHOLM, SWEDEN CHALMERS UNIV. OF TECH., S-41296, GOTEBORG, SWEDEN