ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

ON IKEBE'S CRITERION

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ABSTRACT. A 0-2 law for the metric projection is shown to hold in most of the common Banach spaces.

Let V be a linear subspace of the normed space E. Denote by P_V the (set-valued) metric projection of E onto V, $P_V x =: \{v \in V : ||x - v|| = d(x, V)\}$. V is called *proximinal* if $P_V x \neq \emptyset\ \forall x \in E$, *semichebyshev* if $|P_V x| \leq 1\ \forall x \in E$, and *Chebyshev* if both, i.e., if $|P_V x| = 1\ \forall x \in E$. If $v \in P_V x$, then $||x - v|| \leq ||x - 0|| = ||x||$, hence $||v|| \leq 2||x||$. For equality to hold, it is necessary that $||x - v|| = ||x||$, i.e., that $0 \in P_V x$. If V is semichebyshev, this implies that $v = 0$, hence $x = 0$. In [**8**], Ikebe showed that if V is a non-Chebyshev finite-dimensional subspace of $E = C[a, b]$, then there are $x \neq 0$ in E and $v \in P_V x$ with $||v|| = 2||x||$, so that

$$(*) \qquad ||v|| < 2||x|| \quad \forall x \in E,\ v \in P_V x$$

characterizes the Chebyshev property in this case.

Ikebe's proof uses the well-known Haar characterization of finite-dimensional Chebyshev subspaces of $C[a, b]$. In Singer's survey [**14**: Proposition 3.2, p. 28] it is observed that Ikebe's result holds also when $E = C(Q)$, Q *any* compact Hausdorff space. In the "added in proof" part of his survey (p. 92), Singer mentions a generalization to $E = C(Q, H)$, H a Hilbert space, due to K.H. Hoffmann [**7**].

Motivated by these results, we say that the normed space E has *Ikebe's property* (Ik) if, in E, every linear subspace satisfying (*) is semichebyshev. We say also that E has (Ik_1) (respectively, (Ik^1)) if this criterion is valid for all 1-dimensional (respectively, 1-codimensional) subspaces. Strictly convex spaces have the (Ik) trivially.

Geometrically, (Ik) (respectively (Ik_1) or (Ik^1)) means that, for every plane (respectively, line or hyperplane) F which supports the unit ball B_E at more than one point, there is a translate of F which supports

Received by the editors on October 20, 1986.

B_E at a set containing a segment of length 2. To see that these three properties are different, consider the following two 3-dimensional spaces:

1. $E = (\ell_1^2 \oplus \mathbf{R})_2$, i.e., $\mathbf{R}^3$ with the norm $||(\xi, \eta, \zeta)|| = ((|\xi| + |\eta|)^2 + \zeta^2)^{1/2}$ (Figure 1.a) has Ik_1 but not Ik^1.

2. E with the unit ball $\{(\xi, \eta, \zeta) : |\zeta| \leq 1,\ \xi^2 + (1 + |\zeta|)^2\eta^2 \leq 1\}$ (Figure 1.b) has Ik^1 but not Ik_1.

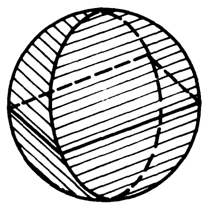

FIGURE 1.a.

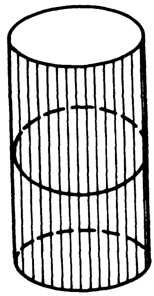

FIGURE 1.b.

In the same "added in proof" part, Singer mentions a paper by W. Pollul ([**13**], unpublished) from which he cites the above characterization of Ik^1 (i.e., For every $f \in E^*$, $||f|| = 1$ with $M_f =: \{x \in B_E : f(x) = 1\}$ containing more than one point, there are $y, z \in M_f$ with $||y - z|| = 2$) as well as the observation that $C(Q)$, $L_1(\mu)$ and also $C[a, b]$ with the L_1-norm satisfy a property stronger than Ik, namely:

(Ik_1^1) Every nontrivial segment in any face has a parallel segment of length 2 in the same face (i.e., if, for every $x, y \in B_E$, $x \neq y$, $f \in E^*$, $||f|| = 1$, and $f(x) = f(y) = 1$, there is $z \in E$ with $f(z) = 1$ and $||z|| = ||z + 2(x - y)/||x - y||\,||$).

Although these will follow from more general results, we present direct proofs of Pollul's results (slightly generalized).

PROPOSITION 1. *Let $E = L_1(\mu)$ (μ any measure) or $E = C(Q)_{L_1(\mu)}$ (μ a positive Borel measure on the compact Hausdorff Q). Then E has (Ik_1^1).*

PROOF. Let f, x, y be as above. f can be considered as a norm-1 L_∞-function on the measure space (the proof for the non σ-finite case is almost the same). The assumptions imply that $|x| = fx$ and $|y| = fy$ a.e. Let $z = 2f(|x| - |y|)^-/||x - y||$. Since $0 = \int(fx - fy)d\mu = \int(fx - fy)^+ d\mu - \int(fx - fy)^- d\mu$, we have $||f(fx-fy)^+|| = ||(fx-fy)^+|| = ||(fx-fy)^-|| = ||f(fx-fy)^-||$, while $||x - y|| = ||f(fx - fy)|| = ||f(fx - fy)^+|| + ||f(fx - fy)^-||$. Therefore $||z|| = 1 = f(z)$. Also $z_1 = z+2(x-y)/||x-y|| = 2f(|x|-|y|)^-/||x-y||$ satisfies $||z_1|| = 1 = f(z_1)$. If x, y are continuous, so are z and z_1. □

If Q is a compact Hausdorff space and $\sigma : Q \to Q$ is a continuous involution (i.e., with $\sigma^2 q = q\ \forall q \in Q$), then $C_\sigma(Q)$ denotes the closed subspace $\{x \in C(Q); x(\sigma q) = x(q)\ \forall q \in Q\}$ of $C(Q)$. The class of $C_\sigma(Q)$ spaces contains the class $C(Q)$ and the class of $C_0(T)$ spaces (T locally compact) as special cases. In $C_\sigma(Q)$ we have the "skew Tietze extension theorem": If K is closed in Q with $K \cap \sigma K = \emptyset$, then every $x_0 \in C(K)$ has an extention $x \in C(Q)$ with $||x|| = ||x_0||$ (take $x = (x_1 - x_1 \circ \sigma)/2$, where $x_1 \in C(Q)$ is any norm-preserving extension of x_0).

PROPOSITION 2. *$E = C_\sigma(Q)$ has property* (Ik$_1^1$).

PROOF. Let f, x, y be as above. f is represented by a norm-1 Borel measure μ on Q satisfying $-\mu(A) = \mu(\sigma A)$ for every Borel subset A of Q (cf., e.g., [**1**, Lemma 2]). $f(x) = f(y) = 1$ means that $x(q) = y(q) = 1$ on sptμ^+ and $x(q) = y(q) = -1$ on spt$\mu^- = \sigma(\text{spt}\mu^+)$. We may assume that $||x - y|| = x(q_0)$ for some $q_0 \in Q$. Then $q_0 \notin$ sptμ and there is $h \in C_\sigma(Q)$ with $h(q_0) = h(\text{spt}\mu^+) = 1$, $||h|| = 1$. Let $z = (1-|x-y|/||x-y||)h-(x-y)/||x-y||$, $z_1 = z+2(x-y)/||x-y|| = (1 - |x - y|/||x - y||)h + (x - y)/||x - y||$. Then z, $z_1 \in C_\sigma(Q)$, $||z|| = 1 = (q_0) = -z_1(q_0) = ||z_1||$, and $f(z) = f(z_1) = 1$. □

The $C_\sigma(Q)$ spaces are a subclass of *Lindenstrauss spaces*, i.e., those Banach spaces whose dual is (isometric to) an $L_1(\mu)$ space. An intermediate class is that of Grothendieck spaces, and another subclass is that of affine function spaces on Choquet simplices (cf. [**9**]).

In his memoir [**12**], Lindenstrauss characterized the $L_1(\mu)$-predual spaces by a ball intersection property. We say that a normed E has the n.2.i.p if every family of n mutual intersecting closed balls in E has a nonempty intersection. He proved that the 4.2.i.p implies the n.2.i.p for every n, and that (if E is complete) it is equivalent to E^* being an $L_1(\mu)$-space. Other relevant results from [**12**] are:

(a) If a normed E has n.2.i.p, so does its completion (but the converse is false).

(b) To check n.2.i.p it suffices to consider translates of the unit ball.

(c) If $E, E_1, E_2 \ldots$ have n.2.i.p (for some $n \geq 3$), so do the vector-valued function spaces $(\sum_k \oplus E_k)_{c_0}$, $(\sum_k \oplus E_k)_\infty$ and $C(Q, E)$ (Q any compact Hausdorff), while $(\sum_k \oplus E_k)_1$ and $L_1(\mu, E)$ (μ any measure) have the 3.2.i.p.

Å. Lima [**10, 11**] studies 3.2.i.p and improved some results of Lindenstrauss. He showed that 3.2.i.p is equivalent to the following decomposition property:

(R_3)

$$\forall x, y, \in E \quad \exists z, u, v \in E$$

with

$$x = z + u, \;\; y = z + v, ||x|| = ||z|| + ||u||, \;\; ||y|| = ||z|| + ||v||$$

and

$$||x - y|| = ||u - v|| = ||u|| + ||v||,$$

and that the 3.2.i.p, unlike the 4.2.i.p, is self dual, i.e., a Banach space E has the 3.2.i.p if and only if E^* has the 3.2.i.p.

The finite dimensional spaces with 3.2.i.p are characterized in [**6**] to be the spaces $\mathbf{R} \oplus \mathbf{R} \oplus \cdots \oplus \mathbf{R}$, where the direct sums are in the ℓ_1 or ℓ_∞ sense. Lima [**11**] studies n.2.i.p in operator spaces and proved that:

(a) The space $K(E, F)$ of the compact linear operators from E to F has 3.2.i.p if and only if E and F have 3.2.i.p and either E or F^* is an $L_1(\mu)$ space. If F is a dual space, then the same condition is necessary and sufficient for the space $L(E, F)$ (of bounded linear operators from E to F) to have 3.2.i.p.

(b) $L(L_1(\mu), L_1(\nu))$ has 3.2.i.p. $L(\ell_\infty^3, \ell_1^3)$ does not have 3.2.i.p.

Lima characterized 3.2.i.p by faces of the unit ball (i.e., by sets of the type $M_f = f^{-1}1 \cap S_E, f \in S_{E^*}$) : A real Banach space E has the 3.2.i.p if and only if, for every pair M_1, M_2 of disjoint faces of B_E, there is a face M of B_E such that $M_1 \subset M$, $M_2 \subset -M$.

Fullerton [**5**] defined the (CL) *property* of the normed space E: For every maximal face M of the unit ball B_E, we have $B_E = \text{conv}(M \cup -M)$. From Lima's characterization it follows at once that for real Banach spaces 3.2.i.p $\Rightarrow$ (CL) (if $x \in B_E \backslash \text{conv}(M \cup -M)$, apply the Hahn-Banach theorem to get a face disjoint with both M and $-M$). Since $L(\ell^3_\infty, L^3_1)$ has (CL), the converse implication fails [**11**]. Lindenstrauss observed that Fullerton's results show that (CL) implies a property somewhat weaker than 3.2.i.p, namely:

(3^0.2.i.p) Every 3 mutual intersecting balls, two of which intersect exactly in a single point, have a nonempty intersection.

3^0.2.i.p can be stated in terms of extreme points, e.g., $|f(e)| = 1$ for every $f \in \text{ext}B_{E^*}$, $e \in \text{ext}B_E$, or also: For every $e \in \text{ext}B_E$, $x \in S_E$, at least one of the segments $[e, x]$, $[-e, x]$ lies on the sphere S_E.

It is shown in [**11**] that, if E^* has 3^0.2.i.p., then E has "almost CL", i.e, $B_E = \overline{\text{conv}}(M \cup -M)$ for every maximal face M of B_E. In particular, in the *finite dimensional case* the following are equivalent:

(i) E has (CL),

(ii) E^* has (CL) and

(iii) E has 3^0.2.i.p.

LEMMA 3. *If M is a face of B_E such that $B_E = \text{conv}(M \cup -M)$, then, for every $x, y \in M$, $x \neq y$, there are $u, v \in M$ with $u - v = 2(x - y)/||x - y||$.*

PROOF. For every $z \in M$, $M - z$ spans a maximal subspace F of E whose unit ball is $(M - M)/2$. In particular, $x - y \in F$ and there are $u, v \in M$ with $(x - y)/||x - y|| = (u - v)/2$. □

THEOREM 4. (CL) *spaces have* (Ik^1_1).

PROOF. Immediate, by the last lemma. □

COROLLARY 5. 3.2.i.p *implies* (Ik_1^1). *In particular, all Lindenstrauss spaces have* (Ik_1^1), *hence* (Ik).

Observe that 3^0.2.i.p is satisfied trivially if B_E has no extreme points. Therefore the following example of a space E with $\mathrm{ext}B_E = \emptyset$ which fails (Ik_1) shows that 3^0.2.i.p does not imply (Ik):

EXAMPLE 6. Renorm c_0 by $|||x||| = \max(||x||_\infty, |x_1| + |x_2|/\sqrt{3}, |2x_2|/\sqrt{3})$. The dual space is ℓ_1 with $|||g|||^* = \max(|g_1|, |g_1|/2 + |\sqrt{3}g_2|/2) + \sum_{k=3}^{\infty} |g_k|$. Consider $g = (0, 1/\sqrt{3}, 1/4, 1/8, 1/16, \dots)$, then $|||g|||^* = 1$ and $M_g = \{(t, 1, 1, 1, \dots) : |t| \le 1/2\}$ which has diameter 1. Observe that the dual fails the 3^0.2.i.p If $y = (0, \frac{4}{\sqrt{3}}, 0, \dots)$ and $z = (2, \frac{2}{\sqrt{3}}, 0, 0, \dots)$, then $B(0,1) \cap B(y,1) = \{y/2\}$, $B(0,1) \cap B(z,1) = \{z/2\}$ and $B(y,1) \cap B(z,1) = \{(1, \sqrt{3}, 0, \dots)\}$.

So far we have 2 classes of norms with the Ik - the strictly convex ones and the "very square ones. What about mixing the two?

PROPOSITION 7. *If* (E_k) *is a (finite or infinite) sequence of strictly convex spaces, then* $(\sum \oplus E_k)_{c_0}$ *and* $(\sum \oplus E_k)_1$ *have* Ik_1^1).

PROOF. Let $V \subset \sum \oplus E_k$ be a linear subspace, $x = (x_k) \in P_V^{-1}0$ and $0 \ne v = (v_k) \in P_V x$. We may assume $||x|| = 1$. Then there is $g \in V^\perp$ with $||g|| = 1 = g(x) = \sum g_k(x_k)$. We now apply the representations $(\sum \oplus E_k)_{c_0}^* = (\sum \oplus E_k^*)_1$, $(\sum \oplus E_k)_1^* = (\sum \oplus E_k^*)_\infty$ [2, p. 35].

In the $(\sum \oplus E_k)_{c_0}$ case, we have $\max_k ||x_k|| = \max ||x_k - v_k|| = 1$, $\sum ||g_k|| = 1$. Therefore $g_k(x_k) = g_k(x_k - v_k) = ||g_k||\ \forall k$. If $g_k \ne 0$, then $g_k(x_k) = g_k(x_k - v_k) = ||g_k||$ implies by strict convexity that $x_k = x_k - v_k$, i.e., $v_k = 0$. Therefore we must have $g_m = 0$ for some m. Let $z_1 = x - x_m + v_m/||v_m||$, $z_2 = x - x_m + v_m/||v_m||$. Then $||z_j|| = 1 = g(z_j)$ for $j = 1, 2$, which shows (Ik_1^1).

In the $(\sum \oplus E_k)_1$ case, we have $\sum ||x_k|| = \sum ||x_k - v_k|| = 1$ and $\max ||g_k|| = 1$. Therefore $g_k(x_k) = ||x_k||$ and $g_k(x_k - v_k) =$

$\|x_k - v_k\|\ \forall k$. Strict convexity implies that x_k and $y_k = x_k - v_k$ are nonnegatively proportional. If $x_k \neq 0$ or $y_k \neq 0$, then there are $\alpha_k, \beta_k \geq 0$ and $\mu_k \in E$ with $\|u_k\| = 1 = g_k(u_k)$, $x_k = \alpha_k u_k$, $y_k = \beta_k u_k$. If $x_k = y_k = 0$, take $\alpha_k = \beta_k = 0$ and u_k arbitrary. Let $z_k = 2(\alpha_k - \beta_k) - u_k/\|x - y\|$. Since $\sum \alpha_k = \sum \beta_k = 1$, we have $\sum(\alpha_k - \beta_k) = 0$ hence $\sum(\alpha_k - \beta_k)^+ = \sum(\alpha_k - \beta_k)^-$, so that $g(z) = \|z\| = \sum_k 2(\alpha_k - \beta_k)/\sum_j |\alpha_j - \beta_j| = 1$, as well as

$$\left\| x + 2\frac{x-y}{\|x-y\|} \right\| = \sum_k \frac{|2(\alpha_k - \beta_k)^- + 2(\alpha_k - \beta_k)|}{\sum |\alpha_j - \beta_j|}$$
$$= \sum_k \frac{2(\alpha_k - \beta_k)^+}{\sum_j |\alpha_j - \beta_j|} = 1.$$

□

REMARK 8. A completely analogous computation shows that, if E is strictly convex and if $L_1(\mu, E)^* = L_\infty(\mu, E^*)$ (e.g., when μ is finite and E^* has the Radon-Nikodym property with respect to μ, [**3** p. 98]), then $L_1(\mu, E)$ has (Ik_1^1).

Similarly, we can consider $C_0(Q, E)$ where E is strictly convex. The dual space is $M(Q, E^*)$, the space of regular Borel E^*-valued measures on Q with finite total variation [**4**, p. 387]. $Q_0 =: \{q \in Q : x(q) \neq y(q)\}$ is a nonempty open set, and the variation of the E^*-valued measure g on Q_0 must be 0 (by strict convexity of E). Taking an Urysohn function φ supported in Q_0, $z_1 = (1 - \varphi)x + \varphi(x - y)/\|x - y\|$, $z_2 = (1 - \varphi)x - \varphi(x - y)/\|x - y\|$ shows (Ik_1^1).

On the other hand, the other way of combining strict convexity with (CL) may fail. E.g.:

EXAMPLES 9. We already saw that $(\ell_1^2 \oplus \mathbf{R})_2$ has Ik_1 but fails Ik^1, $E = (\ell_1^2 \oplus \ell_2^1)_2$, i.e., $\mathbf{R}^4$ with the norm $\|(\omega, \xi, \eta, \zeta)\| = \Big((|\omega| + |s|)^2 + (|\eta| + |\zeta|)^2\Big)$, fails even Ik_1 (consider the segment $1/\sqrt{2}\{(t,\ 1 - t, t/2,$

$1 - t/2) :\ 0 \leq t \leq 1\})$.

The characterization of the 2-dimensional spaces with (Ik) follows immediately from the following two obvious observations:

PROPOSITION 10. *In any normed space E, if $[u,v]$ is a segment of length 2 on the unit sphere, then the 2-dimensional subspace $F =$ span(u,v) has the parallelogram unit ball $B_F =$ conv$(\pm u, \pm v)$.*

PROOF. $|| \pm u|| = || \pm v|| = || \pm uv||/2 = 1$ determines the sphere S_F. □

PROPOSITION 11. *If E has (Ik^1), then, for every supporting hyperplane H of the unit sphere S_E which is not semichebyshev, $H \cap S_E$ contains a segment $[u,v]$ of length 2.*

COROLLARY 12. *Among the 2-dimensional spaces, those having* (Ik) are exactly the strictly convex ones, and $\ell_1^2 \cong \ell_\infty^2$.

COROLLARY 13. *The property* (Ik) *is not inherited by subspaces, quotient spaces or dual spaces. Also, the* 4.3.i.p *does not imply* (Ik).

PROOF. The 2-dimensional space whose unit ball is a square with 2 semicircles (Figure 2.a) does not have (Ik), although its dual (Figure 2.b) does. The rest follows from Propositions 1 and 2 and from the fact that, by Helly's theorem, every 2-dimensional space has 4.3.i.p. □

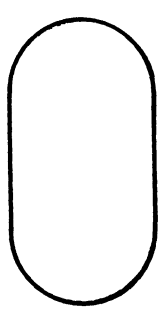

FIGURE 2.a.

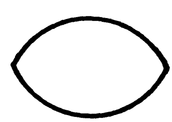

FIGURE 2.b.

Among the 3-dimensional spaces, besides the strictly convex ones, ℓ_1^3 and ℓ_∞^3, we have (by Proposition 7) also the spaces whose unit balls are "double cones" (Figure 3.a) or "tomato cans" with strictly convex bases (or, more generally, of the type $\operatorname{conv}(A \cup -A)$, A strictly convex (Figure 3.b)).

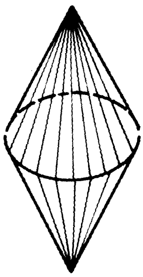

FIGURE 3.a.

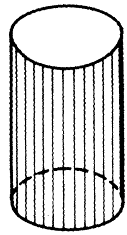

FIGURE 3.b.

REFERENCES

1. Y. Benyamini, *Separable G spaces are isomorphic to $C(K)$ spaces*, Israel Jour. Math. **14** (1973), 387-293.

2. M.M. Day, *Normed Linear Spaces*, 3rd edition, Springer, 1973.

3. J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys **15**, Amer. Math. Soc. 1977.

4. N. Dinculeanu, *Vector Measures*, Pergamon, 1967.

5. R.E. Fullerton, *Geometrical characterization of certain function spaces*, Proc. Inter. Symp. Linear Spaces, Jerusalem, 1961, 227-236.

6. A.B. Hansen and Å. Lima, *The structure of finite-dimensional Banach spaces with the* 3.2.i.p. Acta Math. **146** (1981), 1-23.

7. K.H. Hoffmann, *Über ein Eindeutigkeitskriterium bei der Tscheby scheff Approximation mit regulären Funktionensystemen*, Funktional analytische Methoden der Numerische Mathematik, Oberwolfach (Birkhäuser Verlag), 1967, 71-79.

8. C. Y. Ikebe, *A characterization of Haar subspaces in $C[a,b]$*, Proc. Japan Acad. **44** (1968), 219-220.

9. A. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. **126** (1971), 165-193.

10. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1088), 1-62.

11. ———, *Intersection properties of balls in spaces of compact operators*, Annal. Inst. Fourier Grenoble **28** (1978), 35-65.

12. J. Lindenstrauss, *Extension of compact operators*, Memoirs Amer. Math. Soc. **48** (1964).

13. W. Pollul, *Über ein Eindeutigkeitskriterium für besten Approximationen*, unpublished.

14. I. Singer, *The theory of best approximation and functional analysis*, SIAM regional conference series in applied mathematics, No. 13, 1974.

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