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CONTRACTION OF THE SCHUR ALGORITHM FOR FUNCTIONS BOUNDED IN THE UNIT DISK

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ABSTRACT. If in the Schur algorithm, some of the parameters γ_n vanish, then successive elements in the sequences (1) $\{D_nF_n\}$ and (2) $\{C_n/E_n\}$ will be equal to each other so that one cannot get continued fractions with the elements of (1) and (2) as approximants. It is also difficult to determine the degree of correspondence of the sequences (1) and (2) to series P and Q at 0 and ∞, respectively. For the contraction, continued fraction expansions can be obtained and the degree of correspondence can be computed, using the contraction, for all elements of the sequences (1) and (2).

1. Introduction. In 1907 Carathéodory investigated functions holomorphic on the unit disk and mapping it into the right half plane $\operatorname{Re} \omega > 0$. In two articles [**4**] in 1917/18 J. Schur studied the related family

$$U := \left\{ f : f(z) \text{ holomorphic and } |f(z)| \leq 1 \text{ for } |z| < 1 \right\}. \tag{1.1}$$

(For further historical remarks and references see [**1**].)

Schur's investigation was based on the algorithm: given $f_n \in U$, determine f_{n+1} by

$$f_{n+1} := t_n^{-1}(z, f_n). \tag{1.2}$$

Here

$$t_n(z, w) := \frac{\gamma_n + zw}{1 + \overline{\gamma}_n zw}, \quad n \geq 0, \tag{1.3}$$

and

$$\gamma_n = f_n(0). \tag{1.4}$$

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Further define

$$T_n(z,w) = T_{n-1}(z, t_n(z,w)),\ n \geq 1,\ \ T_0(z,w) = t_0(z,w). \tag{1.5}$$

Starting with $f \in U$ and setting $f = f_0$ Schur was able to associate with f a sequence of functions $\{f_n\}$ and a sequence of *parameters* $\{\gamma_n\}$ such that $f_n \in U$, $n \geq 0$, and $|\gamma_n| \geq 1$, $n \geq 0$. If $|\gamma_N| = 1$ the sequence terminates, $f_N \equiv \gamma_N$, and

$$f(z) = T_{N-1}(z, \gamma_N).$$

If no $|\gamma_n| = 1$, the sequence continues indefinitely and

$$f(z) = \lim_{n\to\infty} T_n(z, w_n), \quad |w_n| \leq 1,\ |z| < 1.$$

Conversely, given any sequence $\{\gamma_n\}$ with $|\gamma_n| < 1$ $n \geq 0$, the sequence $\{T_n(z, w_n)\}$, $|w_n| \leq 1$, converges, for $|z| < 1$, uniformly on compact subsets to a function $f \in U$.

Since the $t_n(z,w)$ are linear fractional transformations in w, the same is true for their composition $T_n(z,w)$ and hence one can write

$$T_n(z,w) = \frac{C_n(z)zw + D_n(z)}{E_n(z)zw + F_n(z)},\ \ n \geq 0, \tag{1.6}$$

where in view of (1.5) the C_n, D_n, E_n, F_n are polynomials in z satisfying the recursion relations

$$\begin{pmatrix} C_n \\ E_n \end{pmatrix} = z\begin{pmatrix} C_{n-1} \\ E_{n-1} \end{pmatrix} + \overline{\gamma}_n \begin{pmatrix} D_{n-1} \\ F_{n-1} \end{pmatrix}$$
$$\begin{pmatrix} D_n \\ F_n \end{pmatrix} = \gamma_n z\begin{pmatrix} E_{n-1} \\ E_{n-1} \end{pmatrix} + \begin{pmatrix} D_{n-1} \\ F_{n-1} \end{pmatrix}, \tag{1.7a}$$

with the initial conditions

$$C_0 = 1,\ D_0 = \gamma_0,\ E_0 = \overline{\gamma}_0,\ F_0 = 1. \tag{1.7b}$$

It follows from (1.7) that

$$\begin{aligned} C_n &= \gamma_0\overline{\gamma}_n + \cdots + z^n, \\ D_n &= \gamma_0 + \cdots + \gamma_n z^n, \\ E_n &= \overline{\gamma}_n + \cdots + \overline{\gamma}_0 z^n, \\ F_n &= 1 + \cdots + \gamma_0\gamma_n z^n, \end{aligned}$$

so that all of these functions are polynomials in z of degree at most n. The degrees are exactly n if all $\gamma_n \neq 0$, $0 \leq m \leq n$. It has also been shown that there are continued fractions having the sequences $\{D_n/F_n\}$ and $\{C_n/E_n\}$ as their nth approximants, respectively, provided all $\gamma_n \neq 0$. The continued fraction for $\{D_n/F_n\}$ is

$$\gamma_0 + (\frac{\gamma_1 - |\gamma_0|^2)z}{1 + \gamma_0\gamma_1 z} + \frac{-\frac{\gamma_2}{\gamma_1}(1 - |\gamma_1|^2)z}{1 + \frac{\gamma_2}{\gamma_1}z} + \frac{-\frac{\gamma_3}{\gamma_2}(1 - |\gamma_2|^2)z}{1 + \frac{\gamma_3}{\gamma_2}z} + \cdots .$$

Define $\Lambda_a(f)$ to be the Taylor series expansion of the function f at a, if it exists, and denote by $\Lambda_\infty(f)$ the Laurent expansion of f at ∞, if it exists, and denote by $\Lambda_\infty(f)$ the Laurent expansion of f at ∞, if it exists. As before let f be the function to which $\{T_n(z, w_n)\}$ converges for $|z| < 1$, $|w_n| \leq 1$. There is also a function g to which the sequence converges for $|z| > 1$, $|w_n| \geq 1$. Set

$$L_0(f) := P = \sum_{n=1}^{\infty} c_n z^n, \quad L_\infty(g) := Q = \sum_{n=0}^{\infty} d_n z^{-n}.$$

The following "correspondence" formulas are known to hold, provided all $\gamma_n \neq 0$.

$$\Lambda_0\Big(\frac{D_n}{F_n}\Big) - P = O(z^{n+1}),$$
$$\Lambda_0\Big(\frac{D_n}{F_n}\Big) - Q = O\Big(\Big(\frac{1}{z}\Big)^n\Big),$$
$$\Lambda_0\Big(\frac{C_n}{D_n}\Big) - P = O(z^n),$$
$$\Lambda_0\Big(\frac{C_n}{D_n}\Big) - Q = O\Big(\Big(\frac{1}{z}\Big)^{n+1}\Big).$$

Thus it becomes of interest to investigate what happens if some $\gamma_n = 0$. It is convenient to consider two cases:

(a) $\gamma_n = 0$ for all $n > N$.

(b) $\exists$ a sequence $\{N_k\}$ of non-negative integers such that $\gamma N_k \neq 0$ and $\gamma_n = 0$ for all $n \neq N_k$.

In case (a) one has

$$T_n(z, w) = \frac{C_N z^{n-N+1} w + D_n}{E_N z^{n-N+1} w + F_n}, \quad n \geq N,$$

in view of

$$t_n(z,w) = zw, \text{ for } n > N.$$

Case (b) will be studied in the remainder of this article. For convenience sake we shall assume that $N_0 = 0$, so that $\gamma_0 \neq 0$. If the first nonvanishing γ_n is γ_m one can write

$$T_n(z,w) = z^m T^{\dagger}_{n-m}(z-w).$$

The results summarized in this section can be found in Schur's original paper and/or in [**1**].

2. Transition to the contraction. We shall, from now on, assume that

$$\gamma_k \neq 0 \Leftrightarrow k = N_n,$$

where $\{N_n\}$ is the sequence of non-negative integers introduced at the end of the last section. As mentioned there, we shall assume $N_0 = 0$. This is equivalent to $\gamma_0 \neq 0$. We now define

$$\beta_n := \gamma N_n, \quad n \geq 0,$$
$$\alpha_n := N_{n+1} - N_n, \quad n \geq 0.$$

It follows that $0 < |\beta_n| < 1$ for all $n \geq 0$ and that α_n is a positive integer for all $n \geq 0$. Further,

$$N_{n+1} = \sum_{k=0}^{n} \alpha_k. \tag{2.1}$$

Next we introduce

$$\hat{t}_n(z,w) := \frac{\beta_n + z^{\alpha_n} w}{1 + \overline{\beta}_n z^{\alpha_n} w}, \quad n \geq 0, \tag{2.2}$$

and

$$\hat{T}_n(z,w) := \hat{T}_{n-1}(z, \hat{t}_n(z,w)), \quad \hat{T}_0(z,w) := \hat{t}_0(z,w).$$

One easily verifies that

$$T_{N_{N+1}-1}(z,w) = \hat{T}_n(z,w), \quad n \geq 0. \tag{2.3}$$

Since $\{\hat{T}_n(z,w)\}$ is thus a subsequence of $\{T_n(z,w)\}$, it follows that

$$\lim_{n\to\infty} \hat{T}_n(z,w_n) = f(z), \text{ for } |z|<1,\ |w_n|\le 1,$$
$$\lim_{n\to\infty} \hat{T}_n(z,w_n) = g(z), \text{ for } |z|>1,\ |w_n|\ge 1.$$

We shall refer to the sequence $\{\hat{T}_n(z,w)\}$ as the *contraction* of the sequence $\{T_n(z,w)\}$.

3. Basic formulas for contractions. Even though our main interest in the sequence $\{\hat{T}_n\}$ is as the contraction of a sequence $\{T_n\}$, we can also think of it as being generated by an arbitrary sequence $\{\beta_n\}$ with $0<|\beta_n|<1$ and an arbitrary sequence of positive integers $\{\alpha_n\}$. Note that it is always possible to reconstruct from a sequence $\{\hat{T}_n\}$ the sequence $\{T_n\}$ of which it is the contraction.

Since it is the composition of linear fractional transformations, $\hat{T}_n(z,w)$ can be written as

$$\hat{T}_n(z,w) =: \frac{\hat{C}_n z^{\alpha_n} w + \hat{D}_n}{\hat{E}_n z^{\alpha_n} w + \hat{F}_n}, \quad n\ge 0. \tag{3.1}$$

Then

$$T_0(z,w) = \frac{\hat{C}_0 z^{\alpha_0} w + \hat{D}_0}{\hat{E}_0 z^{\alpha_0} w + \hat{F}_0} = t_0(z,w) = \frac{z^{\alpha_0} w + \beta_0}{\overline{\beta}_0 z^{\alpha_0} w + 1},$$

so that one can set

$$\hat{C}_0 = 1, \quad \hat{D}_0 = \beta_0, \quad ,\hat{E}_n = \overline{\beta}_0, \quad \hat{F}_0 = 1. \tag{3.2a}$$

For $n\ge 1$ one has, from (2.3),

$$\frac{\hat{C}_n z^{\alpha_n} w + \hat{D}_n}{\hat{E}_n z^{\alpha_n} w + \hat{F}_n} = \frac{\hat{C}_{n-1} z^{\alpha_{n-1}} \left(\frac{\beta_n + z^{\alpha_n} w}{1+\overline{\beta}_n z^{\alpha_n} w}\right) + \hat{D}_{n-1}}{\hat{E}_{n-1} z^{\alpha_{n-1}} \left(\frac{\beta_n + z^{\alpha_n} w}{1+\overline{\beta}_n z^{\alpha_n} w} + \hat{F}_{n-1}\right.}.$$

Hence we can set, for $n \ge 1$,

$$\begin{pmatrix}\hat{C}_n\\ \hat{E}_n\end{pmatrix} = \begin{pmatrix}\hat{C}_{n-1}\\ \hat{E}_{n-1}\end{pmatrix} z^{\alpha_{n-1}} + \overline{\beta}_n \begin{pmatrix}\hat{D}_{n-1}\\ \hat{F}_{n-1}\end{pmatrix},$$
$$\begin{pmatrix}\hat{D}_n\\ \hat{F}_n\end{pmatrix} = \begin{pmatrix}\hat{C}_{n-1}\\ \hat{E}_{n-1}\end{pmatrix} \beta_n z^{\alpha_{n-1}} + \begin{pmatrix}\hat{D}_{n-1}\\ \hat{F}_{n-1}\end{pmatrix}. \tag{3.2b}$$

Using (3.2) and (2.1) one finds, for the functions $\hat{C}_n, \hat{D}_n, \hat{E}_n, \hat{F}_n$, the expressions

$$\begin{aligned} \hat{C}_n &= z^{N_n} + \cdots + \overline{\beta}_n\beta_0, \\ \hat{D}_n &= \beta_n z^{N_n} + \cdots + \beta_0, \\ \hat{E}_n &= \overline{\beta}_0 z^{N_n} + \cdots + \overline{\beta}_n, \\ \hat{F}_n &= \overline{\beta}_0\beta_n z^{N_n} + \cdots + 1. \end{aligned} \tag{3.3}$$

It follows that these expressions are all polynomials of exact degree N_n in z. Since $\hat{T}_n(z,w) = \hat{t}_1 \circ \cdots \circ \hat{t}_n(z,w)$, the determinant of the transformation on the left is the product of the determinants on the right, that is

$$z^{\alpha_n}(\hat{C}_n\hat{F}_n - \hat{D}_n\hat{E}_n) = \prod_{k=0}^{n}((|\beta_k|^2 - 1)z^{\alpha_k}).$$

Hence

$$\hat{C}_n\hat{F}_n - \hat{D}_n\hat{E}_n = (-1)^{n+1}\hat{p}_n z^{N_n}, \tag{3.4}$$

where

$$\hat{p}_n := \prod_{k=0}^{n}(1 - |\beta_k|^2). \tag{3.5}$$

From (3.2b) one obtains

$$\hat{D}_n\hat{F}_{n-1} - \hat{F}_n\hat{D}_{n-1} = z^{\alpha_{n-1}}\beta_n(\hat{C}_{n-1}\hat{F}_{n-1} - \hat{E}_{n-1}\hat{D}_{n-1}),$$

so that

$$\hat{D}_n\hat{F}_{n-1} - \hat{F}_n\hat{D}_{n-1} = (-1)^n\beta_n\hat{p}_{n-1}z^{N_n}. \tag{3.6}$$

Similarly one proves

$$\hat{C}_n\hat{E}_{n-1} - \hat{E}_n\hat{C}_{n-1} = (-1)^{n+1}\overline{\beta}_n\hat{p}_{n-1}z^{N_{n-1}}. \tag{3.7}$$

Using (3.6) we get

$$\begin{aligned} H_n &= \frac{\hat{D}_n}{\hat{F}_n} - \frac{\hat{D}_{n-1}}{\hat{F}_{n-1}} \\ &= \frac{(1)^n\beta_n\hat{p}_{n-1}z^{N_n}}{(1 + \cdots + \beta_0\beta_n z^{N_n})(1 + \cdots + \beta_0\beta_n z^{N_n})(1 + \cdots + \beta_0\beta_{n-1}z^{N_{n-1}})} \end{aligned}$$

so that

$$P = \beta_0 + \sum_{k=1}^{\infty} \Lambda_0(H_k) \text{ and } Q = \beta_0 + \sum_{k=n+1}^{\infty} \Lambda_\infty(H_k)$$

are well defined power series satisfying

$$P - \Lambda_0(\hat{D}_n/\hat{F}_n) = \sum_{k=n+1}^{\infty} \Lambda_0(H_k), \quad Q - \Lambda_\infty(\hat{D}_n/\hat{F}_n) = \sum_{k=n+1}^{\infty} \Lambda_\infty(H_k).$$

From this

$$\Lambda_0\left(\frac{\hat{D}_n}{\hat{F}_n}\right) - P = O(z^{N_{n+1}}), \tag{3.8}$$

$$\Lambda_\infty\left(\frac{\hat{D}_n}{\hat{F}_n}\right) - Q = O\left(\left(\frac{1}{z}\right)^{N_n}\right), \tag{3.8}$$

follows. Using (3.7) an analogous arguments leads to

$$\Lambda_0\left(\frac{\hat{C}_n}{\hat{E}_n}\right) - P = O(z^{N_n}), \tag{3.9}$$

$$\Lambda_\infty\left(\frac{\hat{C}_n}{\hat{E}_n}\right) - Q = O\left(\left(\frac{1}{z}\right)^{N_{n+1}}\right).$$

Since the sequences $\{\hat{D}_n/\hat{F}_n\}$ and $\{\hat{C}_n/\hat{E}_n\}$ converge for $|z| < 1$ and $|z| > 1$ to the functions f and g, respectively, it follows from [**3**, Theorem 5.11] that

$$P = \Lambda_0(f), \quad Q = \Lambda_\infty(g).$$

We shall return to the implications of the formulas (3.8) and (3.9) for "correspondence" and membership in certain Padé tables (see [**3**, Sections 5.1 and 5.5]) in §5.

4. Continued fraction expansions. The following lemma is useful in deriving continued fraction expansions for certain sequences.

LEMMA. *Let sequences* $\{X_n\}$ *and* $\{Y_n\}$ *satisfy the recursion relations*

$$X_n = a_n X_{n-1} + b_n Y_{n-1}, \quad n \geq 1,$$
$$Y_n = c_n X_{n-1} + d_n Y_{n-1}, \quad n \geq 1,$$

where $b_n \neq 0, \quad c_n \neq 0$ *for all* $n \geq 1$. *Then, for* $n \geq 2$,

$$X_n = \left(a_n + \frac{b_n}{b_{n-1}} d_{n-1}\right) X_{n-1} - \frac{b_n}{b_{n-1}} (a_{n-1} d_{n-1} - b_{n-1} c_{n-1}) X_{n-2},$$
$$Y_n = \left(d_n + \frac{c_n}{c_{n-1}} a_{n-1}\right) Y2_{n-1} - \frac{c_n}{c_{n-1}} (a_{n-1} d_{n-1} b_{n-1} c_{n-1}) Y_{n-2}.$$

The proof rests on the fact that

$$X_n = a_n X_{n-1} - b_n c_{n-1} X_{n-1} + b_n d_{n-1} Y_{n-2}$$

and

$$b_{n-1} Y_{n-2} = X_{n-1} - a_{n-1} X_{n-2}.$$

Substituting Y_{n-2} from the second expression into the first yields the formula. □

Applying this result to the sequences $\{\hat{C}_n\}$ and $\{\hat{D}_n\}$ leads to

$$\hat{C}_n = \left(z^{\alpha_{n-1}} + \frac{\overline{\beta}_n}{\overline{\beta}_{n-1}}\right) \hat{C}_{n-1} - \frac{\overline{\beta}_n}{\overline{\beta}_{n-1}} z^{\alpha_{n-2}} (1 - |\beta_{n-1}|^2) \hat{C}_{n-2},$$
$$\hat{D}_n = \left(1 + \frac{\beta_n}{\beta_{n-1}} z^{\alpha_{n-1}}\right) \hat{D}_{n-1} - \frac{\beta_n}{\beta_{n-1}} z^{\alpha_{n-1}} (1 - |\beta_{n-1}|^2) \hat{D}_{n-2},$$

for $n \geq 2$. Similarly, for $\{\hat{E}_n\}$ and $\{\hat{F}_n\}$ and $n \geq 2$, and one obtains

$$\hat{E}_n = \left(z^{\alpha_{n-1}} + \frac{\overline{\beta}_n}{\overline{\beta}_{n-1}}\right) \hat{E}_{n-1} - \frac{\overline{\beta}_n}{\overline{\beta}_{n-1}} z^{\alpha_{n-2}} (1 - |\beta_{n-1}|^2) \hat{E}_{n-2},$$
$$\hat{F}_n = \left(1 + \frac{\beta_n}{\beta_{n-1}} z^{\alpha_{n-1}}\right) \hat{F}_{n-1} - \frac{\beta_n}{\beta_{n-1}} z^{\alpha_{n-1}} (1 - |\beta_{n-1}|^2) \hat{F}_{n-2}.$$

Now $\hat{C}_0 = 1, \quad \hat{E}_0 = \beta_0$ and thus

$$\hat{C}_1 = z^{\alpha_0} + \beta_0 \overline{\beta}_1 = (z^{\alpha_0} + \beta_0 \overline{\beta}_1) \hat{C}_0 + \overline{\beta}_1 (1 - |\beta_0|^2) \hat{C}_{-1},$$
$$\hat{E}_1 = \beta_1 z^{\alpha_0} + \overline{\beta}_1 = (z^{\alpha_0} + \beta_0 \overline{\beta}_1) \hat{E}_0 + \overline{\beta} (1 - |\beta_0|^2) \hat{E}_{-1},$$

provided one sets $\hat{C}_{-1} = 0, \quad \hat{E}_{-1} = 1$. Similarly, $\hat{D}_0 = \beta_0, \quad \hat{F}_0 = 1$ and

$$\hat{D}_1 = \beta_1 z^{\alpha_0} + \beta_0 = (\overline{\beta}_0 \beta_1 z^{\alpha_0} + 1)\hat{D}_0 + \beta_1 z^{\alpha_0}(1 - |\beta_0|^2)\hat{D}_{-1},$$
$$\hat{F}_1 = \overline{\beta}_0 \beta_1 z^{\alpha_0} + 1 = (\beta_0 \beta_1 z^{\alpha_0} + 1)\hat{F}_0 + \beta_1 z^{\alpha_0}(1 - |\beta_0|)^2)\hat{F}_{-1},$$

provided one sets $\hat{D}_{-1} = 1, \quad \hat{F}_{-1} = 0$. It follows that $\{\hat{D}_n \hat{F}_n\}$ is the sequence of approximants of the continued fraction

(4.1)
$$\beta_0 + \frac{\beta_1(1 - |\beta_0|^2)z^{\alpha_0}}{1 + \overline{\beta}_0 \beta_1 z^{\alpha_0}} + \frac{-\frac{\beta_2}{\beta_1}(1 - |\beta_1|^2)z^{\alpha_1}}{1 + \frac{\beta_2}{\beta_1} z^{\alpha_1}} + \frac{-\frac{\beta_3}{\beta_2}(1 - |\beta_2|^2)z^{\alpha_2}}{1 + \frac{\beta}{\beta_2} z^{\alpha_2}} + \cdots .$$

Similarly $\{\hat{E}_n / \hat{C}_n\}$ is the sequence of approximants of the continued fraction

$$\overline{\beta}_0 + \frac{\overline{\beta}_1(1 - |\beta_0|^2)}{z^{\alpha_0} + \beta_0 \overline{\beta}} + \frac{-\frac{\overline{\beta}_2}{\overline{\beta}_1}(1 - |\beta_1|^2)z^{\alpha_0}}{z^{\alpha_1} + \frac{\overline{\beta}_2}{\overline{\beta}_1}} + \frac{-\frac{\overline{\beta}_3}{\overline{\beta}_2}(1 - |\beta_2|^2)z^{\alpha_1}}{z^{\alpha_2} + \frac{\overline{\beta}_3}{\overline{\beta}_2}} + \cdots .$$

From these two expansions it follows that

$$\frac{\hat{E}_n(\frac{1}{z})}{\hat{C}_n(\frac{1}{z})} = \overline{\left[\frac{\hat{D}_n(\overline{z})}{\hat{F}_n(\overline{z})}\right]}.$$

For $|z| < 1$ the sequence on the left converges to $1/g(\frac{1}{z})$ and the sequence on the right to $\overline{f}(z)$. We thus have

$$g(z) = \frac{1}{\overline{f}(\frac{1}{z})}, \quad |z| > 1. \tag{4.3}$$

We can summarize some of the results we have obtained as follows.

THEOREM 4.1. *Let $f \in U$ be such that its parameter sequence $\{\gamma_n\}$ satisfies $\gamma \neq 0$ and $\gamma_n \neq 0$ if and only if $n = N_k$ and there are infinitely many integers N_k. Further let $\beta_n = \gamma N_n, \quad \alpha_n = N_{n+1} - N_n$. Then the continued fraction (4.1) converges to $f(z)$ and the continued fraction (4.2) converges to $1/g(1/z)$ for $|z| < 1$.*

There is a similarity between continued fractions (4.1) and (4.2) and C-fractions $K(a_n z^{\alpha_n}/1)$. However this similarity does not go as far as one might at first suspect.

5. Correspondence in the general case. In this section we determine the correspondence of C_n/E_n and D_n/F_n to $P = \Lambda_0(f)$ and $Q = \Lambda_\infty(g)$ in the general case (provided an infinite number of $\gamma_n \neq 0$ and $\gamma_0 \neq 0$). We use the results of the preceeding sections.

From the recursion relations (1.7a), one obtains for $m \neq N_k$, that is, $\gamma_m = 0$,

$$(5.1)\qquad \begin{pmatrix} C_m \\ E_m \end{pmatrix} = z \begin{pmatrix} C_{m-1} \\ E_{m-1} \end{pmatrix}, \quad \left(\frac{D_m}{F_m} = \begin{pmatrix} D_{m-1} \\ F_{m-1} \end{pmatrix}\right).$$

Iteration of these formulas leads to

$$(5.2)\qquad \begin{pmatrix} C_{N_n+k} \\ E_{N_n+k} \end{pmatrix} = z^k \begin{pmatrix} C_{N_n} \\ E_{N_n} \end{pmatrix}, \quad \begin{pmatrix} D_{N_n+k} \\ F_{N_n+k} \end{pmatrix} \begin{pmatrix} D_{N_n} \\ F_{N_n} \end{pmatrix},$$

for $0 \leq k \leq N_{n+1} - N_n - 1 = \alpha_n - 1$. From the identity (2.4), noting that the normalizations are the same, one deduces

$$(5.3)\qquad \begin{pmatrix} C_{N_{n+1}-1} \\ E_{N_{n=1}-1} \end{pmatrix} = z^{\alpha_n - 1} \begin{pmatrix} \hat{C}_n \\ \hat{E}_n \end{pmatrix}, \quad \begin{pmatrix} D_{N_{n+1}-1} \\ F_{N_{n+1}-1} \end{pmatrix} = \begin{pmatrix} \hat{D}_n \\ \hat{F}_n \end{pmatrix}.$$

Combining (5.2) for $k = N_{n+1} - N_n - 1$ and (5.3) one gets

$$\begin{pmatrix} C_{N_{n+1}-1} \\ E_{N_{n+1}-1} \end{pmatrix} = z^{\alpha_n - 1} \begin{pmatrix} C_{N_n} \\ E_{N_n} \end{pmatrix} = z^{\alpha_n} - 1 \begin{pmatrix} \hat{C}_n \\ \hat{E}_n \end{pmatrix},$$

$$\begin{pmatrix} D_{N_{n+1}-1} \\ F_{N_{n+1}-1} \end{pmatrix} = \begin{pmatrix} D_{N_n} \\ F_{N_n} \end{pmatrix} = \begin{pmatrix} \hat{D}_n \\ \hat{F}_n \end{pmatrix}.$$

From this

$$(5.4) \qquad \hat{C}_n = C_{N_n}, \quad \hat{D}_n = D_{N_n}, \quad \hat{E}_n = E_{N_n}, \quad \hat{F}_n = F_{N_n}$$

follows. Since constant terms as well as coefficients of z^{N_n} in $\hat{F}_n\hat{E}_n$ do not vanish, it follows from (3.8) and (3.9) that

$$P\hat{F}_n - \hat{D}_n = O(z^{N_{n+1}}), \qquad Q\hat{F}_n - \hat{D}_n = O\left(\left(\frac{1}{z}\right)^0\right),$$

$$P\hat{E}_n - \hat{C}_n = O(z^{N_n}), \qquad Q\hat{E}_n - \hat{C}_n = O\left(\left(\frac{1}{z}\right)^{\alpha_n}\right).$$

Hence, for $0 \leq k < \alpha_n$,

$$PF_{N_n+k} - D_{N_n+k} = O(z^{N_{n+1}}), \qquad QF_{N_n+k} - D_{N_n+k} = O\left(\left(\frac{1}{z}\right)^0\right).$$

Thus (considering *actual* degrees) D_{N_n+k}/F_{N_n+k} is the *weak* (N_n, N_n) *two point Padé approximant for* (P, Q) *of order* (N_{n+1}, N_n) (for definitions see [**2**]). Further,

$$PE_{N_n+k} - C_{N_n+k} = O(z^{N_n+k}), \qquad QE_{N_n+k} - C_{N_n+k} = O\left(\left(\frac{1}{z}\right)^{\alpha_n - k}\right).$$

Now $2N_n + 2k + 1 \leq N_n + k + N_{n+1}$, since $k \leq \alpha_n - 1$, and hence C_{N_n+k}/E_{N_n+k} is the *weak* $(N_n + k, N_n + k)$ *two point Padé approximant of order* $(N_n + k, N_{N+1})$ *for* (P, Q).

For the ordinary two point Padé approximants one has: D_{N_n+k}/F_{N_n+k} is the (N_n, N_n) two point approximant of order (N_{n+1}, N_n) for (P, Q). However C_{N_n+k}/E_{N_n+k} is not always a two point Padé approximant for (P, Q) since $N_{n+1} + N_n \geq 2N_n + 2k + 1$ holds only for $\alpha_n \geq 2k + 1$. If k satisfies this condition, then C_{N_n+k}/E_{N_n+k} is the $(N_n + k, N_n + k)$ two point Padé approximant of order (N_{n+1}, N_n) for (P, Q).

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