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BERNSTEIN INEQUALITIES IN $\mathbf{L_p}$, $\mathbf{0 \leq p \leq +\infty}$

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1. Introduction. The norm (or quasi-norm) in the space $L_p(T)$ of a function f is defined by

$$(1.1) \qquad ||f||_p = \left(\frac{1}{2\pi}\int_T |f(t)|^p\,dt\right)^{1/p}, \quad 0 < p < \infty.$$

The limiting cases are: for $p \to \infty$ the supremum norm $||f||_\infty$, and for $p \to 0$ (see [**3**, p. 139]) the quasi-norm of L_0,

$$||f||_0 = \exp\frac{1}{2\pi}\int_T \log|f(t)|\,dt.$$

For each of these spaces, one has the inequality

$$(1.2) \qquad \left|\left|\frac{1}{n}T_n'\right|\right|_p \leq ||T_n||_p, \quad 0 \leq p \leq +\infty,$$

where $T_n \in \mathcal{T}_n$, and $\mathcal{T}_n$ is the space of all trigonometric polynomials of degree $\leq n$, with complex coefficients. For $p = \infty$, the relation (1.2) is called the Bernstein inequality; for $1 \leq p < \infty$, it has been established by Zygmund, using an interpolation formula of M. Riesz. This case of (1.2) immediately follows from the Hardy-Littlewood-Pólya order relation $T_n' \prec nT_n$ established in Lorentz [**5**].

For $0 < p < 1$, the inequality (1.2) has been proved by Máté and Nevai [**4**] with an extra factor $(4e)^{1/p}$ on the right. A year later, Arestov [**1**] obtained (1.2) as it stands. The proofs of Máté and Nevai and of Arestov are complicated, and it is desirable to have simple proofs. We do so in §2; as a premium, we obtain a generalization of (1.2), which replaces the map $T_n \to \frac{1}{n}T_n'$ with a map $T_n \to AT_n + \frac{B}{n}T_n'$, where A, B are real numbers with $A^2 + B^2 = 1$. In this way we obtain, for each real α, and each trigonometric polynomial $T_n \in \mathcal{T}_n$ the inequality

$$(1.3) \qquad \left|\left|T_n\cos\alpha + \frac{1}{n}T_n'\sin\alpha\right|\right|_p \leq ||T_n||_p, \quad 0 \leq p \leq \infty.$$

For $p = \infty$ this reads

$$(1.4) \qquad \left|T_n(t)\cos\alpha + \frac{1}{n}T_n'(t)\sin\alpha\right| \le ||T_n||_\infty, \quad t,\alpha \in T.$$

The special case of this, for *real* polynomials T_n, is the inequality of Szegö-van der Corput-Schaake [**6**, p. 70],

$$(1.5) \qquad n^2T_n(t)^2 + T_n'(t)^2 \le n^2||T_n||_\infty^2, \quad t \in T.$$

The simple fact that $\max_{\alpha\in T}(a\cos\alpha + b\sin\alpha) = \sqrt{a^2+b^2}$ shows that (1.4) and (1.5) are equivalent.

To obtain (1.2), Arestov uses subharmonic functions, and Jensen's formula

$$(1.6) \qquad \frac{1}{2\pi}\int_T \log|f(e^{it})|\,dt = \log|f(0)| + \sum{}' \log\frac{1}{|z_k|}.$$

Here $f(z)$ is analytic in $|z| \le 1$, $f(0) \ne 0$, and the sum $\sum'$ extends over all zeros of f with $|z_k| < 1$.

As an example of (1.6) we have, taking $f(z) = z + w$, for any real α independent of s,

$$(1.7) \qquad \begin{aligned} \log^+|w| &= \frac{1}{2\pi}\int_T \log|w + e^{is}|\,ds \\ &= \frac{1}{2\pi}\int_T \log|w + e^{i\alpha}e^{is}|\,ds. \end{aligned}$$

LEMMA 1. *If a real valued analytic function f has $2p$ zeros on T (counting multiplicities), then, for any real A, B, the function $g = Af + Bf'$ has at least $2p$ zeros on T.*

PROOF. We assume that $B \ne 0$. If t is a zero of f of multiplicity l, then it is a zero of g of multiplicity $l-1$. Let now a, b be two zeros of f, b being in the clockwise direction from a, with no zeros of f on the arc a, b. The function f does not change sign on this arc, for example let $f(t) > 0$ there. Then, for small $h > 0, f(a+h) = o(f'(a+h))$, $f'(a+h) > 0$, hence $g(a+h)$ has the sign of B. In the same way, $g(b-h)$ has the sign of $-B$. Thus g changes sign on the arc a, b. □

In §4 we discuss the norm in L_0 of the operator $T_n \to S_n$ for complex A, B, and ascertain the corresponding extremal polynomials. We replace Lemma 1 in this section by a complex variable Lemma 7, which gives the location of zeros of some polynomials. This gives a new proof for the inequality of Arestov and also allows us to find the exact norm on $\mathcal{T}_n$ of some polynomial differential operators (see Theorem 9). For all our inequalities, we also find the cases of equality.

2. Inequalities in $\mathbf{L_0}$. The real case. Let A, B be two real numbers with $B \neq 0$. We consider the operator Λ on $\mathcal{T}_n$ defined by

$$S_n := \Lambda(T_n) := AT_n + B\frac{T_n'}{n}. \tag{2.1}$$

The following theorem contains the statement that

$$||\Lambda T_n||_0 \leq ||T_n||_0, \quad T_n \in \mathcal{T}_n \tag{2.2}$$

(with equality for some T_n of degree n) if and only if $|A - iB| = 1$, that is, if $A^2 + B^2 = 1$.

THEOREM 2. *The norm of the operator* (2.1) *on* L_0 *is equal to* $|A-iB|$. *In other words,*

(2.3)

$$\max_{\substack{T_n \in \mathcal{T}_n \\ n \neq 0}} \left\{\frac{1}{2\pi}\int_T \log|\Lambda T_n(t)|\,dt - \frac{1}{2\pi}\int_T \log|T_n(t)|\,dt\right\} = \log|A - iB|;$$

the maximum is attained for all T_n of degree n that have $2n$ real zeros.

PROOF. Let $T_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$ be a trigonometric polynomial of degree n.

With T_n we associate an algebraic polynomial of degree $\leq 2n$,

$$P(z) := P_{2n}(z) := \mathcal{P}(T_n; z) = \sum_{k=0}^{2n} c_{-n+k} z^k. \tag{2.4}$$

An alternative definition is by $P(e^{it}) = e^{int}T_n(t)$. For given c_{-n}, there is a 1-1 correspondence between the T_n, the P_{2n} and the zeros

$z_1, \ldots, z_{2n}$ of P_{2n}; we put $z_{m+1} = \cdots = z_{2n} = \infty$ if P_{2n} is of degree $m < 2n$. Differentiating (2.4), we obtain

$$T_n'(t) = ie^{-int}[-nP(z) + zP'(z)], \quad z = e^{it}. \tag{2.5}$$

Using this relation, it is easy to prove that $t \in T$ is a zero of T_n of order l exactly when $z = e^{it}$ is a zero of P_{2n} of order l.

Putting $R_{2n}(z) = \mathcal{P}(S_n; z)$ we have $R_{2n}(e^{it}) = e^{int}S_n(t)$ and

$$R(z) = (A - iB)P(z) + \frac{iBz}{n}P'(z). \tag{2.6}$$

We see that the difference under the maximum in (2.3) is equal to

$$\begin{aligned} F(z_1, \ldots, z_{2n}) :&= \frac{1}{2\pi}\int_T \log\left|A - iB + \frac{iBe^{it}}{n}\frac{P'(e^{it})}{P(e^{it})}\right| dt \\ &= \frac{1}{2\pi}\int_T \log\left|A - iB + \frac{iB}{n}\sum_{k=1}^{2n}\frac{e^{it}}{e^{it} - z_k}\right| dt. \end{aligned} \tag{2.7}$$

If a function $f(w)$ is analytic in some region of the ω-plane that may contain the point ∞, then $\log|f(w)|$ is subharmonic in the region [**2**]; it takes the value $-\infty$ at the zeros of f. Sums (with positive coefficients) and integrals with respect to a parameter (for a positive measure) are also subharmonic. Thus, $F(z_1, \ldots, z_{2n})$ is a subharmonic function of each of the variables z_k as long as $z_k \neq e^{it}$. In other words, it is subharmonic with respect to each z_k in the regions $|z| \leq 1$ and $|z| \geq 1$. The function F attains a maximum in each of the regions.

(2.8) *The maximum of* $F(z_1, \ldots, z_n)$ *is attained for some* z_k^* *with* $|z_k^*| = 1$, $k = 1, \ldots, 2n$.

Indeed, let the maximum M of F be attained at $z_1^*, \ldots, z_{2n}^*$. We shall correct this to $|z_k^*| = 1$ for all k. For example, let $|z_k^*| > 1$ for some k. Then $F(\ldots, z_{k-1}^*, z_k, z_{k+1}^*, \ldots)$ is a subharmonic function of z_k in $|z| \geq 1$, with its maximum M at $z_k = z_k^*$. Since z_k^* is an interior point, by the maximum principle, this function is constant in $|z_k| \geq 1$. We can replace z_k^* by any point of $|z| = 1$.

We use a maximal set $z_k^*, |z_k^*| = 1,\ k = 1, \dots, 2n$ in order to determine the value of M. All zeros of P lie on $|z| = 1$ if and only if all $2n$ zeros of T_n are real. Omitting a constant factor, we may assume that T_n is a real polynomial. By Lemma 1, all zeros of S_n are real. But this is in turn equivalent to the assumption that all zeros of R lie on $|z| = 1$. We have $|P(0)| \neq 0$, and from (2.6), $|R(0)| = |P(0)| \cdot |A - iB| \neq 0$. A direct application of Jensen's formula yields

$$\begin{aligned} M &= \frac{1}{2\pi} \int_T \log |R(e^{it})|\, dt - \frac{1}{2\pi} \int_T \log |P(e^{it})|\, dt \\ &= \log |R(0)| - \log |P(0)| = \log |A - iB|. \end{aligned} \tag{2.9}$$

□

We are now able to find all *extremal polynomials* T_n of the operator Λ, that is, $T_n \in \mathcal{T}_n, T_n \neq 0$ for which

$$||\Lambda T_n||_0 = ||\Lambda||_0 ||T_n||_0 = |A - iB|\ ||T_n||_0.$$

This is equivalent to $F(z_1, \dots, z_{2n}) = \log |A - iB|$ for this polynomial. (The following theorem is not needed in §3.)

THEOREM 3. (i) *The extremal polynomials of* Λ *are exactly those* $T_n \in \mathcal{T}_n$ *which are of degree n and for which all the zeros of* $P = \mathcal{P}(T_n)$ *are either in* $|z| \geq 1$, *or in* $|z| \leq 1$; (ii) *the real extremal polynomials are the* T_n *of degree n with* $2n$ *real zeros.*

PROOF. (i) For the points of (2.8), $F(z_1^*, \dots, z_{2n}^*) = \log |A - iB|$. Taking $T_n(t) = e^{int}$ we get $\Lambda T_n = (A + iB)T_n$, $|\Lambda T_n(t)| = |A - iB|$, $t \in T$. Since $\mathcal{P}(T_n, z) = z^{2n}$, we see that $F(0, \dots, 0) = \log |A - iB|$, so that F is constant in the region $|z_k| \leq 1,\ k = 1, \dots, 2n$, and that all T_n with zeros of $\mathcal{P}(T_n)$ in this region are extremal. Likewise, by means of $T_n(t) = e^{-int}$ we find that $F(\infty, \dots, \infty) = \log |A - iB|$, and obtain the corresponding statement for T_n with all $|z_k| \geq 1$.

A polynomial $T_n \in \mathcal{T}_n$ of degree $k < n$ cannot be extremal. For it we can write $\Lambda T_n = AT_n + B^*(T_n'/k)$, $B^* = (k/n)B$ and from Theorem 2 get $||\Lambda T_n||_0 \leq |A - iB^*|\ ||T_n||_0 < |A - iB|\ ||T_n||_0$.

We now have $F(0, z_2^*, \dots, z_{2n-1}^*, \infty) < \log |A - iB|$ for arbitrary $z_2^*, \dots, z_{2n-1}^*$. Indeed, these are zeros of a polynomial $\mathcal{P}(T_n)$ with $c_n = c_{-n} = 0$, and then T_n is of degree $< n$.

Finally, a polynomial T_n cannot be extremal if $P = \mathcal{P}(T_n)$ has zeros both in $|z| < 1$ and in $|z| > 1$. For an extremal T_n, let the zeros z_k^* of P satisfy $|z_1^*| < 1, |z_{2n}^*| > 1$. Then $F(z_1, z_2^*, \ldots, z_{2n}^*)$ is constant for $|z_1| \le 1$. Thus $F(0, z_2^*, \ldots, z_{2n}^*) = \log|A - iB|$. In the same way, $F(0, z_2^*, \ldots, z_{2n-1}^*, \infty) = \log|A - iB|$, a contradiction.

To prove (ii), we note that, for a real polynomial $T_n, c_{-k} = \bar{c}_k,\ k = 0, \ldots, n$. For $P(z) = \mathcal{P}(T_n, z) = z^n \sum_{-n}^{n} c_k z^k$ this implies that, together with a zero z, P has a zero $1/\bar{z}$. If T_n would have zeros other than real, then P would have zeros $|z| \ne 1$ and then it would have zeros both in $|z| < 1$ and in $|z| > 1$, which is impossible if T_n is extremal. □

The class of all T_n with $2n$ real zeros has the important property that it is invariant under operators Λ.

3. Inequalities in $\mathbf{L_p}$. From now on, to have nicest possible formulations, we assume that

$$|A - iB| = 1. \tag{3.1}$$

THEOREM 4. *For all $T_n \in \mathcal{T}_n$, and real A, B satisfying* (3.1)

$$\frac{1}{2\pi}\int_T \log^+ |\Lambda T_n(t)|\, dt \le \frac{1}{2\pi}\int_T \log^+ |T_n(t)|\, dt. \tag{3.2}$$

PROOF. We apply the inequality

$$\frac{1}{2\pi}\int_T \log |\Lambda T_n^*|\, dt \le \frac{1}{2\pi}\int_T \log |T_n^*(t)|\, dt \tag{3.3}$$

to the polynomial

$$T_n^*(t) := T_n^*(t, s) := T_n(t) + e^{is} e^{int}, \tag{3.4}$$

which depends on the real parameter s. Then

$$S_n^*(t) := \Lambda(T_n^*)(t, s) = S_n(t) + e^{is} e^{int} e^{i\alpha}, \quad A + iB = e^{i\alpha},$$

and we obtain

$$\int_T \log|S_n(t) + e^{is} e^{i\alpha} e^{\mathrm{int}}|\, dt \le \int_T \log|T_n(t) + e^{is} e^{int}|\, dt$$

and by means of (1.7),

$$\begin{aligned}\int_T \log^+ |S_n(t)|\, dt &= \int_T dt\, \frac{1}{2\pi} \int_T \log |S_n(t) + e^{is} e^{i\alpha} e^{int}|\, ds \\ &\le \int_T dt\, \frac{1}{2\pi} \int_T \log |T_n(t) + e^{is} e^{int}| ds \\ &= \int_T \log^+ |T_n(t)| dt.\end{aligned}$$

In (3.4), we could have replaced e^{int} by e^{-int}. □

In order to get from $\log^+$ to the function $(\cdot)^p$ we can use the formula

$$u^p = p^2 \int_0^\infty s^{p-1} \log^+ \frac{u}{s} ds\ , \quad p > 0.$$

We can even slightly generalize it. Let $\Phi(u)$, $\Phi(0) = 0$ and $\Psi(u) = u\Phi'(u)$ be continuous positive increasing functions defined for $u \ge 0$. Functions u^p, $\log^+ u$, $\log(1 + u^p)$, $p > 0$ are examples. Then

$$\Phi(u) = \int_0^{+\infty} \log^+ \frac{u}{s} d\Psi(s). \tag{3.5}$$

Indeed, for $0 < v < u$,

$$\begin{aligned}\Phi(u) - \Phi(v) &= \int_v^u \Phi'(s) ds = -\int_v^u s\Phi'(s) d \log \frac{u}{s} \\ &= v\Phi'(v) \log \frac{u}{v} + \int_v^u \log \frac{u}{s} d\Psi(s).\end{aligned} \tag{3.6}$$

The last integral is majorized by $\Phi(u) - \Phi(v)$, which has the limit $\Phi(u)$ for $v \to 0$. Hence the integral $\int_0^u \log(u/s) d\Psi(s)$ converges. Moreover, the first term on the right has a limit $C \ge 0$. Assumption $C > 0$ leads to contradiction, since then $\Phi'(v) \geq \text{Const}/v \log(u/v)$, and $\int_0^u \Phi'(v)dv$ diverges. Making $v \to 0$ in (3.6), we obtain (3.5).

The following theorem is due to Arestov [**1**] if $A = 0,\ B = 1$.

THEOREM 5. *For each function Φ of the described type and for each $T_n \in \mathcal{T}_n$,*

$$\int_T \Phi(|S_n(t)|)dt \leq \int_T \Phi(|T_n(t)|)dt. \tag{3.7}$$

PROOF. By means of (3.2)

$$\begin{aligned} \int_T \Phi(|S_n(t)|)dt &= \int_0^\infty d\Psi(s) \int_T \log^+ \left|\frac{S_n(t)}{s}\right| dt \\ &\leq \int_0^\infty d\Psi(s) \int_T \log^+ \left|\frac{T_n(t)}{s}\right| dt \\ &= \int_T \Phi(|T_n(t)|)dt. \end{aligned} \tag{3.8}$$

□

This argument allows to find all extremal polynomials T_n, at least if Ψ is strictly increasing.

THEOREM 6. *If $B \neq 0$ and if Ψ is strictly increasing, then the equality*

$$\int_T \Phi(|S_n|)dt = \int_T \Phi(|T_n|)dt \tag{3.9}$$

holds if and only if $T_n(t) = C_1 e^{-int} + C_2 e^{int}$ with some complex C_1, C_2.

PROOF. From (3.8) we see that (3.9) is equivalent to

$$\int_T \log^+ \left|\frac{S_n(t)}{s}\right| dt = \int_T \log^+ \left|\frac{T_n(t)}{s}\right| dt$$

for all $s > 0$. Let $\mathcal{U}_n$ be the set of all $T_n \in \mathcal{T}_n$ with this property. Clearly, $\mathcal{U}_n$ does not depend on Φ. On the other hand, Parceval's formula shows that (3.9) holds with $\Phi(u) = u^2$ if and only if T_n has the required form.

4. Complex operators Λ. We begin with some remarks about the linear function $A + Bz$, $B \neq 0$ with complex A, B. Its zero $z = -A/B$ lies in the upper half plane $\operatorname{Im} z \geq 0$, if and only if $\operatorname{Im}(-A\overline{B}) \geq 0$, or if $\Delta \geq 0$, where

$$\Delta := \Delta(A, B) := \operatorname{Re}A \operatorname{Im}B - \operatorname{Im}A \operatorname{Re}B. \tag{4.1}$$

The zero z is in the lower half plane if $\Delta \leq 0$, it is real if $\Delta = 0$. In the latter case, $A = e^{i\alpha}A_1$, $B = e^{i\alpha}B_1$ with some real A_1, B_1, α. Theorems of §2, §3 apply to the operator (2.1) also in this case. We concentrate now on the cases $\Delta \neq 0$.

To compare $A - iB$ and $A + iB$ we have

$$|A - iB|^2 - |A + iB|^2 = 4\Delta,$$

hence $|A - iB|$ is $=$, or $<$ or $>$ than $|A + iB|$ when $\Delta = 0$, $\Delta < 0$, and $\Delta > 0$, respectively. The distance from the point A to a point of the interval $[-Bi, Bi]$ is maximal at one of the endpoints. It follows for example that

$$|A + ixB| \leq |A + iB|, \quad -1 \leq x \leq 1, \text{if } \Delta < 0. \tag{4.2}$$

We shall assume that

$$B \neq 0, \ |A \pm iB| \neq 0. \tag{4.3}$$

Lemma 7 below replaces Lemma 1 of the real case.

Let $P(z) := P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$ be any polynomial of degree $\leq 2n$ with zeros $z_1, \ldots, z_{2n}$, and let $R := R_{2n}$ be related to P by (2.6), that is, let

$$R_{2n}(z) = \sum_{k=0}^{2n} \left[A - iB\left(1 - \frac{k}{n}\right)\right] a_k z^k. \tag{4.4}$$

We denote by $y_1, \ldots, y_{2n}$ the zeros of R.

LEMMA 7. (i) *If* $\Delta \leq 0$, *then* $\max |y_k| \leq \max |z_k|$. (ii) *If* $\Delta \geq 0$, *then* $\min |y_k| \geq \min |z_k|$.

PROOF. Let y be a zero of R; we have $y \neq 0$. If we replace the coefficients a_k of P by $a_k e^{-i\alpha k}$, the same will happen to the coefficients of z^k in (4.4). If z, y were some zeros of P, R respectively, then the modified polynomials will have zeros $e^{i\alpha} z$, $e^{i\alpha} y$. Hence, in proving (i) or (ii), we may assume that $y > 0$.

From (2.6), $A - iB = -\frac{iBy}{n}\frac{P'(y)}{P(y)}$, hence

$$\sum_{k=1}^{2n} \frac{1}{y - z_k} = \frac{n}{y}\,\frac{iB - A}{iB},$$

$$\sum_{k=1}^{2n} \left(\frac{1}{y - z_k} - \frac{1}{2y}\right) = \frac{inA}{yB}.$$

Taking real parts,

$$\sum_{k=1}^{2n} \left(\frac{y - \mathrm{Re} z_k}{|y - z_k|^2} - \frac{1}{2y}\right) = \mathrm{Re}\frac{inA \cdot \overline{B}}{y|B|^2},$$

$$\frac{1}{2y}\sum_{k=1}^{2n} \frac{y^2 - |z_k|^2}{|y - z_k|^2} = \frac{n\Delta}{y|B|^2}.$$

From this we obtain both (i) and (ii). □

The following theorem gives the norm $||\Lambda||_0$ of the operator (2.1) and also describes all extremal polynomials. We shall show that the norm of the operator Λ of (2.1) on the subspace $\mathcal{T}_n$ of L_0 is equal to

$$||\Lambda||_0 = \lambda := \max(|A - iB|,\ |A + iB|). \tag{4.5}$$

THEOREM 8. (i) *If* $\Delta < 0$, *then* $||\Lambda||_0 = |A + iB|$. *The extremal polynomials of* Λ *are those* T_n *of degree* n *for which the zeros* z_k *of* $P = \mathcal{P}(T_n)$ *satisfy* $|z_k| \leq 1$, $k = 1, \ldots, 2n$. (ii) *If* $\Delta > 0$, *then* $||\Lambda||_0 = |A - iB|$; *the extremal* T_n *are of degree* n *and satisfy* $|z_k| \geq 1$, $k = 1, \ldots, 2n$.

PROOF. (i) If $T_n = e^{int}$, then $\Lambda(T_n) = (A + iB)T_n$ and $z_1 = \cdots = z_{2n} = 0$. This shows that, for the function F of (2.7), $F(0, \ldots, 0) = \log|A + iB|$. Its maximum M is achieved at some z_k^* of (2.8). Here

$P(0) = a_0 \neq 0$, $R(0) = (A - iB)a_0 = (A - iB)P(0)$. Jensen's formula and Lemma 7 yield

$$M = \log|R(0)| - \log|P(0)| - \log\prod_1^{2n}|y_k^*| + \log\prod_1^{2n}|z_k^*| \tag{4.6}$$

where P corresponds to the zeros z_k^*, and y_k^* are the zeros of R. Since

$$\prod_1^{2n}|z_k^*| = \left|\frac{a_0}{a_{2n}}\right|, \quad \prod_1^{2n}|y_k^*| = \left|\frac{A - iB}{A + iB}\frac{a_0}{a_{2n}}\right|$$

we have $M = \log|A + iB|$.

Since $F(\infty, \ldots, \infty) = \log|A - iB| < M$, the function $F(z_1, \ldots, z_{2n})$ is $< M$ if $|z_k| \geq 1$, $k = 1, \ldots 2n$ with at least one strict inequality. That an extreme polynomial cannot have one of its z_k in $|z| < 1$ and some other of the z_k in $|z| > 1$, and that its degree must be exactly n, follows as in the proof of Theorem 3.

The proof of (ii) is similar but easier, since instead of (4.6) one uses (2.9). □

One has now also the analogues of the results of §3. Instead of $|A - iB| = 1$ one assumes that $\lambda = 1$; in the proof of Theorem 4, in case that $\Delta > 0$, one replaces the last term in (3.4) by $e^{is}e^{-int}$.

Let now Λ be the differential operator $\Lambda(D) = \prod_{j=1}^q (A_jI + B_jD/n)$. We want to estimate its norm in terms of the characteristic polynomial $\Lambda^*(z) = \prod_{j=1}^q (A_j + B_jz)$. We have the trivial estimate

$$\max(|\Lambda^*(i)|,\ |\Lambda^*(-i)|) \leq ||\Lambda||_p \leq \prod_{j=1}^q \lambda_j,\ 0 \leq p \leq \infty, \tag{4.7}$$

if λ_j corresponds to A_j, B_j by means of (4.5). The lower estimate follows because

$$||\Lambda e^{\pm int}||_p = ||\Lambda^*(\pm i)e^{\pm int}||_p = |\Lambda^*(\pm i)|.$$

If all zeros of Λ^* are in the upper (lower) half plane, one gets a precise formula.

THEOREM 9. *If all zeros of the polynomial* Λ^* *are in the upper half plane, then*

$$||\Lambda||_p = |\Lambda^*(-i)|, \; 0 \le p \le +\infty. \tag{4.8}$$

This follows from (4.7), because $\lambda_j = A_j - iB_j$, $j = 1, \ldots, q$.

REFERENCES

1. V.V. Arestov, *On integral inequalities for trigonometric polynomials and their derivatives*, Izv. Akad. Nauk SSSR, Ser. Mat. **45** (1981), 3-22.

2. J.B. Conway, *Functions of One Complex Variable*, 2nd edition, Springer, New York 1978.

3. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, 2nd edition, Cambridge University Press, 1952.

4. A. Máté and P. Nevai, *Bernstein inequality in* L^p *for* $0 < p < 1$ *and* $(C, 1)$ *bounds of orthogonal polynomials*, Ann. of Math. **111** (1980), 145-154.

5. G.G. Lorentz, *Convergence theorems for polynomials with many zeros*, Math. Z. **186** (1984), 117-123.

6. Q.I. Rahman et G. Schmeisser, Les inégalités de Markoff et de Bernstein, Les Presses de l'Univ. de Montréal, Montréal, 1983.

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