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A PROBLEM OF RUBEL CONCERNING APPROXIMATION ON UNBOUNDED SETS BY ENTIRE FUNCTIONS

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Let F be a closed subset of the finite complex plane $\mathbf{C}$. We call two functions f and g, defined on F, *equivalent*, in which case we write $f \sim g$, provided f and g are bounded (equivalently unbounded) on the same sequences. We shall consider only continuous functions and so we may restrict our attention to sequences $\{z_n\}$ such that $z_n \to \infty$. Thus $f \sim g$ if and only if: for any sequence $\{z_n\}$ in $F, f(z_n) \to \infty$ if and only if $g(z_n) \to \infty$.

By $H(F)$ we denote the set of functions holomorphic on (a neighborhood of) F, and we set $A(F) = C(F) \cap H(F^0)$. The closed set F is called an *Arakelyan set* if, for each $f \in A(F)$ and each constant $\varepsilon > 0$, there exists an entire function g such that $|f - g| < \varepsilon$ on F. In this terminology the celebrated Arakelyan Theorem [**2**] states that F is an Arakelyan set if and only if $\overline{\mathbf{C}} \backslash F$ is both connected and locally connected. For further results related to Arakelyan's theorem see [**4**] and [**5**].

Let us call a closed set F a *Rubel set* if, for each $f \in A(F)$, there exists an entire function g such that $f \sim g$. Clearly an Arakelyan set is always a Rubel set.

The notion of Rubel set was introduced by L.A. Rubel who called them weak Arakelyan sets. At the 1976 Symposium on Potential Theory at Durham, Rubel posed the problem of characterizing Rubel sets. This problem also appears to be related to another problem posed by Anderson and Rubel [**1**].

Goldstein [**6**] has given a condition which, in case $F^0 = \emptyset$, is necessary in order for F to be a Rubel set. If $F^0 \neq \emptyset$ the condition is no longer necessary. Nor is it sufficient, even for $F^0 = \emptyset$.

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In the present note, we give a necessary condition which, in case $F^0 = \emptyset$, is also sufficient. We also discuss Goldstein's condition in light of our own.

Following Stray, we denote by $\Omega(F)$ the set of all $z \in \mathbf{C}\backslash F$ which cannot be arcwise connected to ∞ by a path in $\mathbf{C}\backslash F$. That is, there is no continuous map $\gamma_z(t)$, $0 \leq t < +\infty$, with $\gamma_z(0) = z$, $\gamma_z(t) \in \mathbf{C}\backslash F$, and such that $\gamma_z(t) \to \infty$ as $t \to +\infty$. We denote

$$\hat{F} = F \cup \Omega(F).$$

DEFINITION. (CONDITION-Ω). For each compact K, there is a compact $\tilde{K}$ such that

$$\partial F\backslash \tilde{K} \subset \partial\{(\widehat{F \cup K})\}.$$

THEOREM. *In order for a closed set $F \subset \mathbf{C}$ to be a Rubel set, condition-Ω is necessary: and if $F^0 = \emptyset$, it is also sufficient.*

PROOF (NECESSITY). Suppose condition-Ω is not satisfied. Then there is a compact K and a sequence of discs $D_n = D(z_n, r_n)$ with $z_n \in \partial F, z_n \to \infty$, and $D_n \subset (\widehat{F \cup K})^0$. One may clearly assume $\overline{D}_n \cap \overline{D}_m = \emptyset$ if $n \neq m$ and $D_n \cap K = \emptyset$. Put $F' = F \backslash \cup_n D_n$. Then also

$$\Omega(F' \cup K) = \Omega(F \cup K) \cup (\cup_{n=1}^{\infty} D_n).$$

Choose $\eta_n \in \Omega(K \cup F)$ close to the center z_n of D_n, and form a series

$$f(z) = \sum_n \frac{C_n}{(z - \eta_n)^{m_n}}$$

such that $|f| < 1$ on $F' \cup K$ and $f(z_n) \to \infty$ as $n \to \infty$. If $h \in H(\mathbf{C})$ and $h \sim f$, then h is bounded on $F' \cup K$ and, hence [**7**], on $\Omega(F' \cup K)$ which includes all z_n. This is the desired contradiction which proves the necessity. □

In the proof of sufficiency, we shall invoke the following result from potential theory.

LEMMA 1. *Let U be an unbounded open set in $\mathbf{C}$ all of whose boundary points in $\mathbf{C}$ are regular for the Dirichlet problem. Then there exists a positive continuous unbounded function ϕ on ∂U such that, for any s continuous on ∂U with $0 \leq s \leq \phi$, the generalized solution H_s^U of the Dirichlet problem for s on U is a classical solution in the sense that*

$$\lim_{\substack{z \to \zeta \\ z \in U}} H_s^U(z) = s(\zeta), \ \forall \zeta \in \partial U.$$

PROOF. Let K be a compact set in $\mathbf{C}$. We show the existence of a function $\phi_K \nearrow \infty$ on $\mathbf{R}^+$ such that, if s is positive and continuous and $s \leq \phi_K$, then

$$\lim_{z \to p} H_s^U(z) = s(p)$$

at each regular point $p \in K^\circ \cap \partial U$. By Lemma 2 in [3] it is enough to ensure that H_s^U is bounded on $K \cap U$. Now

$$H_s^U(z) = \int_{\partial U} s(\zeta) d\omega_z(\zeta),$$

where ω_z denotes harmonic measure. If $\omega_z(\partial U \backslash Q) = 0$, for some compact Q, clearly H_s^U is bounded.

Suppose that $\omega_z(\partial U \backslash Q) > 0$, for all compact Q and some $z = z(Q)$, $z \in K \cap U$. By Dini's theorem, we have

$$\lim_{r \nearrow \infty} \omega_z(\partial U \backslash (|\zeta| < r)) = 0$$

uniformly on $K \cap U$. Thus, we may choose $r_n \nearrow \infty$ such that

$$n\omega_z(\partial U \backslash (|\zeta| < r_n)) < 2^{-n}, \ \forall z \in K \cap U.$$

Hence, for all $z \in K \cap U$,

$$\begin{aligned} H_s^U(z) &= \int_{\partial U \cap (|\zeta| \leq r_1)} s(\zeta) d\omega_z(\zeta) + \sum_{n=1}^{\infty} \int_{\partial U \cap (r_n < |\zeta| < r_{n+1})} s(\zeta) d\omega_z(\zeta) \\ &\leq \int_{\partial U \cap (|\zeta| \leq r_1)} s(\zeta) d\omega_z(\zeta) + \sum_{1}^{\infty} \frac{1}{2^n}, \end{aligned}$$

which is bounded if $s \leq \phi_K$, where ϕ_K is chosen so that

$$\phi_K \leq n, \quad \text{for } r_n < |\zeta| \leq r_{n+1}.$$

Thus, we have shown that, for any compact K, there exists $\phi_K \nearrow \infty$ on $\mathbf{R}^+$ such that, if s is eventually dominated by ϕ_K, then H_s^U is bounded on K.

Now let $K_n = (|\zeta| \leq n)$ and $\phi_n = \phi_{K_n}$. Finally, set

$$\phi = \inf_{1 \leq j \leq n} \phi_j, \text{ on } n \leq |\zeta| < n+1.$$

Then, ϕ has the properties required to prove Lemma 1. □

PROOF (SUFFICIENCY). We assume $F^0 = \emptyset$ and $f \in C(F)$. Also, we may assume that $f \geq 0$ since $f \sim |f|$. In fact, we shall suppose that $f \geq 1$ since $f \sim \max\{f, 1\}$. In addition we may assume that f grows as slowly as we please, for if ϕ is any function increasing to ∞ on $[0, +\infty)$, then $f \sim f_\phi$, where

$$f_\phi(z) = \min\{f(z), \phi(|z|)\}.$$

□

We shall choose a suitable ϕ later.

Set $D_n = \emptyset$, for $n = 0, 1, 2$, and, for $n > 2$, we may by condition-Ω inductively choose a sequence $r_n \nearrow \infty$ such that, for the discs $D_n = D(0, r_n)$, we have

$$D_{n+1} \supset \tilde{\overline{D}}_n.$$

For $n = 0, 1, \ldots$, set

$$E_n = \overline{\Omega(F \cup \overline{D}_n) \cup \Omega(F \cup \overline{D}_{n+2})}$$

and

$$F_\infty = \left[F \cup \cup_{n=0}^{\infty} E_n\right].$$

If F satisfies condition-Ω, then the following lemmas hold.

LEMMA 2. *F_∞ is closed and $\overline{\mathbf{C}}\backslash F_\infty$ is connected and locally connected.*

LEMMA 3. *For some compact K,*

$$F\backslash K \subset \partial F_\infty.$$

PROOF OF LEMMA 2. First of all F_∞ is closed since it is the union of a locally-finite family of closed sets.

Let $z \in \mathbf{C}\backslash F_\infty$ and choose $n(\geq 2)$ such that $z \notin D_n$, $z \in D_{n+1}$.

Case 1. $z \notin \overline{\Omega(F \cup \overline{D}_n)}$. Then z can be connected to ∞ by an arc γ_z,

$$\gamma_z \subset \mathbf{C}\backslash(F \cup \overline{D}_n).$$

We may assume γ_z is polygonal. Such an arc must leave any $E_m, m > n$, eventually. Hence, γ_z can be modified to an arc in $\mathbf{C}\backslash F_\infty$. Let us see more closely how such a modification can be done. If the arc γ_z meets some E_m, it must necessarily enter and leave E_m by crossing the circle ∂D_{m+2}. Let γ_n be a component of $\gamma_z \cup E_m$. Then γ_m is a polygonal segment with first and last points a_m and b_m on ∂D_{m+2}. Suppose $a_m \neq b_m$ and let α_m be the arc of ∂D_{m+2} which together with γ_m bounds a Jordan domain disjoint from D_{m+2}. Let $\tilde{a}_m$ be a point of $\gamma_z \cap D_{m+2}$ which is close to a_m and precedes a_m along γ_z. Let $\tilde{b}_m$ be a point of $\gamma_z \cap D_{m+2}$ which is close to b_m and succeeds b_m along γ_z. Replace the arc of γ_z from $\tilde{a}_m$ to $\tilde{b}_m$ by an arc in D_{m+2} from $\tilde{a}_m$ to $\tilde{b}_m$ near α_m.

We may construct $\tilde{\alpha}_m$ such that $\tilde{\alpha}_m \cap F = \emptyset$. Indeed, if this intersection were not empty, then the component of $\Omega(F \cup D_m)$ which meets γ_m would contain a point p in $\tilde{\alpha}_m \cap F$. For such a p, we have

$$p \in \partial F\backslash\overline{D}_{n+1} \cap (F \widehat{\cup} D_n)$$

which contradicts the way the D_n's were chosen. Thus $\tilde{\alpha}_m \cap F = \emptyset$, and we may modify γ_z by replacing γ_m by $\tilde{\alpha}_m$.

If $a_m = b_m$ the above procedure is, if anything, even simpler.

Since γ_z is polygonal, we see that the family of all such modifications is locally finite, and if we perform these modifications, we end up with a path $\tilde{\gamma}_z$ from z to ∞ outside of F_∞.

Case 2. $z \in \overline{\Omega(F \cup \overline{D}_n)}$. Then $z \notin \overline{\Omega(F \cup \overline{D}_{n-2})}$ and again we construct an appropriate path $\tilde{\gamma}_z$ as in Case 1.

Since any $z \in \mathbf{C} \backslash F_\infty$ can be connected to ∞ by a path outside of F_∞, it follows that $\mathbf{C} \cup \{\infty\} \backslash F$ is connected and locally connected. □

PROOF OF LEMMA 3. This follows from the fact that, for each n, $F \cap E_n^\circ = \emptyset$, and if $m > n$, then

$$E_n \backslash D_{m+2} \subset E_m.$$

Now choose ϕ corresponding to F_∞^0 by Lemma 1. We may assume $\log f \leq \phi$ by Lemma 3 and since, as we said earlier, we may suppose f grows as slowly as we please.

By Lemma 2 it is easy to see that $\mathbf{C} \backslash \overline{F_\infty^0}$ is also connected, and so each point of ∂F_∞^0 is regular for the Dirichlet problem. Thus, by Lemma 1, with $U = F_\infty^0$, and extending f to ∂U,

$$H = H_{\log f}^U$$

is continuous on $\overline{F_\infty^0}$ and harmonic on F_∞^0. If we set $H = \log f$ on the rest of F_∞, then H is continuous on F_∞ and harmonic on its interior.

By the harmonic analogue to Arakelyan's Theorem due to Gauthier, Hengartner, and Labréche [**5**], there is a function h, harmonic on $\mathbf{C}$ such that

$$|h(z) - H(z)| < 1, \quad z \in F_\infty.$$

Let $\tilde{h}$ be a harmonic conjugate to h. Then the entire function

$$g = e^{h + i\tilde{h}}$$

is equivalent to f and the sufficiency is proved. □

REMARK. One could consider this problem more generally by replacing $\mathbf{C}$ by a domain D. If F is a (relatively) closed subset of D, we call two functions f and g on F equivalent if they are bounded on the same sequences of F. We call F a Rubel set (relative to D) if each $f \in A(F)$ is equivalent to a function $g \in H(D)$. It seems that our proof of necessity goes through for general domains if we replace the Stray Maximum Principle by its more general form [**7**]. However, our proof of sufficiency works only for simply connected domains since we have used the existence of a harmonic conjugate.

Returning to the case $D = \mathbf{C}$, let us now conisder the following condition from [**6**].

DEFINITION. (CONDITION-G). For each compact set K, there exists a compact set $\tilde{K}$ such that, if V is a bounded connected open set with $\partial V \subset F \cup K$, then either

$$V \subset F \cup \tilde{K} \tag{1}$$

or

$$(V/\tilde{K}) \cap F = \emptyset. \tag{2}$$

Let us call Condition-$\tilde{G}$ the condition obtained from Condition-G by deleting (1). It is easy to see that if $F^0 = \emptyset$ and Condition-$\tilde{G}$ fails then Condition-Ω fails. Thus, from our theorem, it follows that Condition-$\tilde{G}$, and a fortiori Condition-G, are necessary in order for F, with empty interior, to be a Rubel set.

However, if $F^0 \neq \emptyset$, then Condition-G, and a fortiori Condition-$\tilde{G}$, are no longer necessary as the following example shows.

EXAMPLE. Let F_n be the (empty) rectangle with vertices

$$1 + i2^{-n},\ 1 + i(2^{-n} - 2^{-(n+2)},\ n + i2^{-n},\ n + i(2^{-n} - 2^{-(n+2)}),$$

$n = 1, 2, \ldots$, and let F_{-n} be the reflection of F_n in the imaginary axis. Then

$$F = \cup_{n=1}^{\infty}(F_n \cup \hat{F}_{-n}) \cup \{x + iy : y = 0\}$$

is a Rubel set but fails to satisfy Condition-G.

To see that F is a Rubel set, suppose $f \in A(F)$. Let $F_r = F \cap \{x \geq 1\}$ and $F_l = F \cap \{x \leq -1\}$. Then $F_r^0 = \emptyset$ and F_r satisfies Condition-Ω. Hence there is an entire function g_r such that $g_r \sim f$ on F_r. Now

$F_l \cup \hat{F}_r$ is an Arakelian set and there exists an entire function g such that

$$|g - f| < 1 \text{ on } F_l \text{ and } |g - g_r| < 1 \text{ on } \hat{F}_r.$$

Then $g \sim f$, so F is a Rubel set.

We have shown that Condition-G is not necessary in order for F to be a Rubel set, except in the case $F^0 = \emptyset$.

In [**6**] it was conjectured that Condition-G is sufficient in order that F be a Rubel set. We present a counterexample F to this conjecture. Moreover, $F^0 = \emptyset$.

With $z = x + iy$, we set

$$\begin{aligned} E &= \{z : y = x^{-1}|\sin x^{-1}|,\ 0 \leq x \leq 1\}, \\ E_n &= \{(2^{-n}x + 2^{-n}) + i(y + n) : x + iy \in E\}, \\ l_n &= \{z : x = 2^{-n},\ 0 \leq y + \infty\}, \\ z_n &= (2^{-n} + 2^{-(n+1)}) + i(n-1), \end{aligned}$$

for $n = 1, 2, \ldots$, and

$$F = \{\cup_{n=1}^{\infty}[E_n \cup l_n \cup \{z_n\}]\}^{-}.$$

Then $F^0 = \emptyset$, and F satisfies Condition-G but not Condition-Ω. Hence, by our theorem, F is not a Rubel set, and so, Condition-G is not sufficient in order that F be a Rubel set, even if $F^0 = \emptyset$.

Added in proof. After the present paper was submitted, we learned that A.H. Nersesyan has also announced a characterization of Rubel sets [Harmonic approximation and a solution to a problem of L.A. Rubel, Dokl. Akad. Nauk. Arm. SSR 84, 104-106 (1987)].

REFERENCES

1. J.M. Anderson, K.F. Barth and D.A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. **9** (1977), 129-162.

2. N.U. Arakelyan, *Uniform approximation on closed sets by entire functions* (Russian), Akad. Nauk SSR, Izvestia, ser. mat. **28** (1964), 1187-1206.

3. ——— and P.M. Gauthier, *On tangential approximation by holomorphic functions*, Akad. Nauk Arm. SSR, Izvestia, Mat. **17** (1982), 421-441.

4. D. Gaier, *Vorlesungen über Approximation im Komplexen*, Birkhäuser Verlag, 1980.

5. P.M. Gauthier and W. Hengartner, *Approximation uniforme qualitative sur des ensembles non-bornés*, Presses de l'Université de Montréal, 1982.

6. M. Goldstein, *An example of an Arakelian glove which is a weak Arakelian set*, Ill. J. Math. **27** (1983), 138-144.

7. A. Stray, *Decomposition of approximable functions*, Ann. Math. **120** (1984), 225-235.

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