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SUDDEN SYMMETRY IN SIMULTANEOUS APPROXIMATION

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ABSTRACT. In the theory of simultaneous approximation to a set of n formal power series and using n rational functions with a common denominator along the lines of simultaneous Padé approximation of type II use the set $f(z), f(wz), f(w^2z), \cdots, f(w^{n-1}z)$ where w is a primitive n-th root of unity and f belongs to one of three specific classes of hypergeometric functions. In the case that the approximants are calculated with the aid of the same number of coefficients from each of the series, the invariance of the n-tuple of functions under rotation over $2\pi/n$ is transferred to the denominator polynomial, which therefore turns out to be a polynomial in z^n.

1. Introduction. Let n be an arbitrary natural number, $n \geq 2$, and consider an n-tuple of formal power series over $\mathbf{C}$ given by

$$f_j(z) = \sum_{m\geq 0} c_{j,m} z^m, \quad c_{j,0} \neq 0 \quad (j = 1, 2, \cdots, n).$$

For any $(n+1)$-tuple of non-negative integers $(r_0, r_1, \cdots, r_n), s = r_0 + r_1 + \cdots + r_n$, we pose the approximation problem of finding polynomials $P_0(z), P_1(z), \cdots, P_n(z)$ over $\mathbf{C}$ satisfying

$$(1) \qquad \begin{cases} \deg P_j(z) \leq s - r_j & (j = 0, 1, \cdots, n) \\ P_0(z) f_j(z) - P_j(z) = O(z^{s+1}) \text{ as } z \to 0 & (j = 1, 2, \cdots, n) \end{cases}$$

It is well known that there exist several classes of functions such that this approximation problem – the polynomials usually are called the *type* II or *German* polynomials for the functions $1, f_1, \cdots, f_n$ - has a unique solution $(P_1(z)/P_0(z), P_2(z)/P_0(z), \cdots, P_n(z)/P_0(z))$ if only the condition $P_0(0) = 1$ is added; cf. [**3**], [**5**], [**6**], [**8**] and for $n = 1$ cf. [**9**].

As the inverted denominators of the approximants, i.e., $z^{s-r_0}P_0(1/z)$, can be identified as orthogonal polynomials in the setting of indefinite

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innerproduct spaces (a matter outside the scope of this paper) the location of the zeros of these polynomials is important. Symmetry in certain plots for the zeros in the case of approximation of the pair e^z, e^{-z} and the explicit form for the denominators at the points (p, k, k) being *even* polynomials, started the investigation of the following situation. Consider a *nice* function $f(z)$ and let w be a primitive nth root of unity ($w^n = 1, w^j \neq 1$ for $j = 1, 2, \cdots, n-1$) and define the n-tuple of functions by

$$f_j(z) = f(w^{j-1}z) \quad \text{for} \quad j = 1, 2, \cdots, n.$$

Try to solve the approximation problem for the choice $r_0 = p, r_j = k$, $j = 1, 2, \cdots, n$ where p and k are non-negative integers, subject to the normalization $P_0(0) = 1$ (so the same amount of information is used from each of the functions, i.e., the coefficients of $z^{p+(n-1)k+1}, \cdots, z^{p+nk}$). For the exponential function system $\exp(w^{j-1}z)$,

$j = 1, 2, \cdots, n$, the explicit formulae (cf. [**5**], [**6**], [**7**]) show that the denominator polynomials $P_0(z)$ are polynomials in z^n. From the explicit formulae for the system of hypergeometric functions ${}_1F_1(1; c; w^{j-1}z)$ $, j = 1, 2, \cdots, n$, in [**1**] it follows that the same happens in this case (of course $c \notin (0, -1, -2, \cdots)$; for $c = 1$ we recover the previous result).

That this property is not restricted to a set of ${}_1F_1$'s will follow from the results in §2, the proofs will be given in §3.

2. Main results. First a determinantal condition will be given which ensures the symmetry property, and after that three classes of functions will be given that satisfy the condition.

THEOREM 1. *Let n be a natural number, $n \geq 2$, and let $f(z) = \sum_{m \geq 0} c_m z^m$ be a formal power series with*

$$(2) \qquad \begin{vmatrix} c_q & c_{q+n} & q+2n & \cdots & c_{q+(k-1)n} \\ c_{q+1} & c_{q+n+1} & c_{q+2n+1} & \cdots & c_{q+(k-1)n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{q+k-1} & c_{q+n+k+1} & c_{q+2n+k-1} & \cdots & c_{q+(k-1)n+k-1} \end{vmatrix} \neq 0$$

for all $q \geq 0$, $k \geq 1$. *Then the entry* $(P_1(z)/P_0(z), \cdots, P_n(z)/P_0(z))$ *from the Padé-n-table for the functions* $f(z), f(wz), f(w^2z), \cdots, f(w^{n-1}z)$ *at points* $(p, k, \cdots, k) \in (\mathbf{N} \cup \{0\})^{n+1}$ *with* $p \geq k-1$, *where* w *is a primitive nth root of unity* ($w^n = 1, w^j \neq 1$ *for* $j = 1, 2, \cdots, n-1$), *following from*

(3)
$$\begin{cases} P_j(z) \in \mathbf{C}[z], & j = 0, 1, \cdots, n, \\ \deg P_0(z) \leq nk; \deg P_j(z) \leq p + (n-1)k, & j = 1, 2, \cdots, n, \\ P_0(z) f(w^{j-1}z) - P_j(z) = O(z^{nk+p+1}) \text{ for } z \to 0, & j = 1, 2, \cdots, n, \end{cases}$$

has denominator polynomial satisfying

$$P_0(z) = \sum_{0 \leq m \leq k} d_m (z^n)^m \text{ with } d_0 = 1, d_k \neq 0. \tag{4}$$

THEOREM 2. *If* f *is the power series expansion of a function from one of the classes given below, the conditions* (2) *are satisfied. The classes are*

A. $f(z) = {}_1F_1(1; c; z)$; $c \notin \{0, -1, -2, \cdots\}$.

B. $f(z) = {}_2F_1(a, 1; c; z)$; $a, c, a-c, \notin \{0, -1, -2, \cdots\}$.

C. $f(z) = {}_2F_0(a, 1; z)$; $a \notin \{0, -1, -2, \cdots\}$.

REMARK. Of course, if it is known that the approximation problem (3) has a unique solution up to a multiplicative constant, the result (4) follows from the simple observation that replacing z by wz leads to a permutation of the n-tuple of functions. The conditions (3) then show that the polynomial $P_0(z)$ has to be invariant under rotation over $2\pi/n$, thus it must be a polynomial in z^n. If we do not know about uniqueness, the situation is not that simple.

3. Proof of the main result.

For the proof we use the reformulation of (3) in terms of a system of linear equations for the unknown coefficients of the polynomials P_j (cf. [2]) and some results on the explicit calculations of determinants given in the following lemmas.

LEMMA 1. *Let the elements of the* $N \times N$ *determinant* $|e_{r,s}|$ *satisfy* *the relations*

$$e_{r,s+1} = (y_r - x_s)e_{r,s}, \qquad 1 \le r \le N,\ 1 \le s \le N-1, \tag{4}$$

where $x_1, x_2, \cdots, x_{N-1}, y_1, y_2, \cdots, y_N$ *are complex numbers. Then*

$$|e_{r,s}| = \prod_{1\le r\le N} e_{r,1} \cdot \prod_{1\le s<r\le N} (y_r - y_s). \tag{5}$$

PROOF. See [**4**]. □

LEMMA 2. *Let* $|e_{r,s}|$ *be as in Lemma* 1, *satisfying* (4) *and moreover let* $p_s(y)$ *be a polynomial of degree* $\le N-s$, $1 \le s \le N; x_N$ *arbitrary. Then*

$$|e_{r,s}p_s(y_r)| = |e_{r,s}|p_1(x_1)p_2(x_2)\cdots p_N(x_N). \tag{6}$$

PROOF. See [**4**]. □

PROOF OF THEOREM 1. Put $P_0(z) = \sum_{0\le m\le nk} b_m z^m, P_j(z) = \sum_{0\le m\le (n-1)k+p} a_{j,m}z^m$, $j = 1, 2, \cdots, n$, and $f(w^{j-1}z) = f_j(z) = \sum_{m\ge 0} c_{j,m}z^m (j = 1, 2, \cdots, n)$.

The solution of (3) can then be found in two steps:

1. Put $b_0 = 1$ and try to solve

$$\begin{pmatrix} c_{j,(n-1)k+p} & c_{j,(n-1)k+p+1} & \cdots & c_{j,p+1-k} \\ c_{j,(n-1)k+p+1} & c_{j,(n-1)k+p} & \cdots & c_{j,p+2-k} \\ \vdots & \vdots & & \vdots \\ c_{j,nk+p-1} & c_{j,nk+p-2} & \cdots & c_{j,p} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{nk} \end{pmatrix}$$
$$= \begin{pmatrix} -c_{j,(n-1)k+p+1} \\ -c_{j,(n-1)k+p+2} \\ \vdots \\ -c_{j,nk+p} \end{pmatrix} \quad (j = 1, 2, \cdots, n)$$

(n systems of k linear equations each for the nk unknowns $b_m, 1 \le m \le nk$)

2. If the b's are known, we can calculate the $a_{j,m}$ from

$$a_{j,m} = c_{j,m} + c_{j,m-1}b_1 + c_{j,m-2}b_2 + \cdots + c_{j,m-nk}b_{nk},$$
$$0 \le m \le (n-1)k + p; \quad j = 1, 2, \cdots, n,$$

(here the convention $c_{j,m} = 0$ for $m < 0$ is used).

In the sequel we frequently need

$$\sum_{1 \le j \le n} w^{(j-1)t} = \begin{cases} 0 \text{ if } t \not\equiv 0 \bmod n \\ n \text{ if } t \equiv 0 \bmod n. \end{cases} \tag{7}$$

We start with the equations for the b's: n blocks of k equations each. Row # i of k-block # j is, after division by the coefficient of c_{p+i-k} being $w^{(p+i-k)(j-1)}$, just

$$(w^{(r-p+k)(j-1)}c_{r+i}\ w^{(r-p+k-1)(j-1)}c_{r+i-1} \cdots w^{j-1}c_{p+i-k+1} | -c_{r+i+1})$$
$$1 \le i \le k; \ 1 \le j \le n, \ r = (n-1)k + p - 1$$

Insert r and write the equation for sake of simplicity as

$$[w^{(nk-s)(j-1)}c_{p+i-k+nk-s}, 1 \le s \le nk; -c_{p+i-k+nk}].$$

Add this row # i of k-block # j for $j = 1, 2, \cdots, n-1$ to row # i of k-block #n and use (7). The last mentioned row will then contain a lot of zeros and, after division by n, just the $c_{r+i+1-s}$ for those s that satisfy $nk - s \equiv 0 \bmod n$; thus for $s = nv$ with $v = 1, 2, \cdots, k$ it takes the form

(8)
$$(0 \cdots 0c_{p+i-k+n(k-1)}0 \cdots 0c_{p+i-k+n(k-2)} \cdots 0 \cdots 0c_{p+i-k} | -c_{p+i-k+nk})$$

in which each c before the vertical bar is preceded by a string of $n-1$ zeros.

As the procedure can be used for each i with $1 \le i \le k$, we find that k-block # n has turned into a system of k linear equations in the k unknowns $b_{nv}(1 \le v \le k)$ of the form

$$\begin{pmatrix} c_{p+1-k+n(k-1)} & c_{p+1-k+n(k-2)} & \cdots & c_{p+1-k} \\ c_{p+2-k+n(k-1)} & c_{p+2-k+n(k-2)} & \cdots & c_{p+2-k} \\ \vdots & \vdots & & \vdots \\ c_{p+n-k+n(k-1)} & c_{p+n-k+n(k-2)} & \cdots & c_{p+n-k} \end{pmatrix} \begin{pmatrix} b_n \\ b_{2n} \\ \vdots \\ b_{kn} \end{pmatrix} = \begin{pmatrix} -c_{p+1-k+nk} \\ -c_{p+2-k+nk} \\ \vdots \\ -c_{p+n-k+nk} \end{pmatrix}. \tag{9}$$

The condition (2) with $q = p + 1 - k$ ensures that (9) has a unique solution in which $b_{kn} \neq 0$ while Cramer's rule shows that $(-1)^k b_{kn}$ is the quotient of two determinants of the form (2): the numerator with $q = p+1-k+n$ and the denominator with $q = p+1-k$. Having found the b's with index divisible by n, we turn to the problem of finding the others.

Subtract row # i of k-block # n (i.e.(8)) from row # i of k-block # j for $j = 1, 2, \cdots, n-1$. Then

(i) All right hand sides become zero: the equations are homogeneous

(ii) The coefficients in row # i of k-block # j become

$$\begin{cases} w^{(nk-s)(j-1)} - 0 & \text{for } s \not\equiv 0 \bmod n \\ w^{(nk-s)(j-1)} - 1 = 0 & \text{for } s \equiv 0 \bmod n \ (\text{as } w^n = 1). \end{cases}$$

Thus we have $(n-1)k$ homogeneous linear equations for the $(n-1)k$ unknowns $b_m, 1 \leq m \leq nk-1$, m not divisible by n. The final stage of the proof consists of showing that this system has a nonzero determinant, leading to $b_m = 0$ for m not divisible by n.

Let s reduced modulo n be denoted by $\underline{s}$ (is, $s = nv + \underline{s}$ with $0 \leq \underline{s} \leq n-1$), then row # i, $1 \leq i \leq k$, of k-block # j, $1 \leq j \leq n-1$, has the following form if the zero coefficients in the columns corresponding to the b_{nv}, $1 \leq v \leq k$, have been omitted:

$$(10) \qquad \begin{aligned} (w^{-\underline{s}(j-1)} c_{r+i+1-s}; 1 \leq s \leq n-1, n+1 \leq s \leq 2n-1, \cdots, \\ (k-1)n+1 \leq s \leq kn-1 | 0). \end{aligned}$$

Instead of viewing the system as $n-1$ blocks of k equations each, we view it as k blocks of $n-1$ equations each by putting rows # i from k-block # j, $1 \leq j \leq n-1$, together as an $(n-1)$ block # i. Consider now $(n-1)$-block # i for a certain fixed i and introduce the new unknowns $\underline{b}_m (1 \leq m \leq nk-1, m$ not divisible by $n)$ by $\underline{b}_m = c_{r+i+1-m} b_m$. The structure of the matrix of the $(n-1)$-block becomes rather simple now: k copies next to one another of the $(n-1) \times (n-1)$ matrix A given by

$$A = (w^{-(j-1)t})_{1 \leq j,t \leq n-1}.$$

As $\det A = V(w^{-1}, w^{-2}, \cdots, w^{-(n-1)})$ is the $(n-1) \times (n-1)$ Vandermonde determinant on $w^{-1}, w^{-2}, \cdots, w^{-(n-1)}$ and these numbers are pairwise distinct, we find $\det A \neq 0$. Thus there is a method of reduction of the system of equations to change A into the $(n-1) \times (n-1)$ unit matrix I manipulating on rows. As the rows of the $(n-1)$-block # i consist of k copies of A, we have at the final stage of the reduction k copies of I next to one another. The proof is now obvious: rewrite the system with the original b's as unknowns and reassemble the equations in the form of $n-1$ k-blocks. The k-block # j, $1 \leq j \leq n-1$, then has as row # i $(1 \leq i \leq k)$

$$(0 \cdots 0\ c_{r+i+1-j} 0 \cdots 0\ 0 \cdots 0\ c_{r+i+1-j-n} 0 \cdots 0$$

$$0 \cdots 0 c_{r+i+1-j-(k-1)n} 0 \cdots 0 | 0),$$

where the k strings of $n-1$ numbers consist of zeros with $c_{r+i+1-j-vn}$ at the j-th place, $0 \leq v \leq k-1$. Thus k-block # j actually represents a system of k homogeneous linear equations in the k unknowns $b_{j+vn}, 0 \leq v \leq k-1$, with a determinant given by (2) with $q = r+i+1-j(k-1)n$ (rows in descending order here), which leads to $b_{j+vn} = 0$, $0 \leq v \leq k-1$, for $j = 1, 2, \cdots, n-1$.

PROOF OF THEOREM 2. This is a matter of explicitly calculating the determinants (2) for the classes of functions given; the condition $p \geq k-1$ arises in a natural way while the convention $c_m = 0$ for $m < 0$ would lead to disaster otherwise. The determinant can be written as

$$(11) \quad (-1)^{k(k-1)/2} \begin{vmatrix} c_{q+k-1} & c_{q+k-2} & \cdots & c_q \\ c_{q+n+k-1} & c_{q+n+k-2} & \cdots & c_{q+n} \\ \vdots & \vdots & & \vdots \\ c_{q+(k-1)n+k-1} & c_{q+(k-1)n+k-2} & \cdots & c_{q+(k-1)n} \end{vmatrix}.$$

The elements in the $k \times k$ determinant (11) are

$$(12) \quad \begin{gathered} e_{r,s} = c_{q+(r-1)n+k-1-(s-1)} = c_{q-n+k+rn-s}, \\ 1 \leq r \leq k, \ 1 \leq s \leq k. \end{gathered}$$

In the sequel we use, for complex numbers c, the notation

$$(c)_0 = 1; \ (c)_n = c(c+1) \cdots (c+n-1) \quad \text{for } n = 1, 2, \cdots .$$

For class A formula (12) reduces to

$$e_{r,s} = 1/(c)_{q-n+k+rn-s}\ ,\ 1 \leq r \leq k,\ 1 \leq s \leq k. \tag{13}$$

This leads to $e_{r,s+1}/e_{r,s} = c + q - n + k + rn - s - 1$ and application of Lemma 1, as in [**8**], with

$$N = k; y_r = c + q - n + k + rn - 1,\ 1 \leq r \leq k, \tag{14}$$

$$x_s = s,\ 1 \leq s \leq k - 1$$

shows that the determinant (11) is different from zero, using the condition on c.

Next, for class B, we have

$$\begin{aligned} c_{q-n+k+rn-s} &= (a)_{q-n+k+rn-s}/(c)_{q-n+k+rn-s} \\ &= (a)_{q-n+rn}(a + q - n + rn)_{k-s}/(c)_{q-n+k+rn-s}. \end{aligned} \tag{15}$$

Pull out a factor $(a)_{q-n+rn}$ from row # r, $1 \leq r \leq k$; these factors are different from zero. Then apply Lemma 2 with $e_{r,s}$ as for class A, (13) and (14), and moreover

$$p_s(y) = \prod_{1 \leq j \leq k-s} (y + a - c - k + j),\ 1 \leq s \leq k - 1;\ \ p_k(y) = 1. \tag{16}$$

The conditions on a, c imply that the determinant is different from zero.

Finally, for class C, we reverse the order of the columns in (11) again, leading to

$$|(a)_{q+(r-1)n+s-1}|_{1 \leq r,s \leq k}. \tag{17}$$

Pull out a factor $(a)_{q+(r-1)n+k-1}$ from row # r, $(1 \leq r \leq k$; different from zero because of the condition on a, and we are left over with a determinant $|e_{r,s}|$ on which Lemma 1 can be applied with

$$N = k;\ y_r = a + q + (r - 1)n(1 \leq r \leq k); \tag{18}$$

$$x_s = -s + 1(1 \leq s \leq k - 1).$$

Again the condition on the parameter, here a, implies that the determinant is different from zero.

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