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GROUPS OF ISOMETRIES ON OPERATOR ALGEBRAS II

STEEN PEDERSEN

ABSTRACT. We show that, to each C_0-group ρ of isometries on a C^*-algebra A, there corresponds a C_O-group α of automorphisms on A, and a unitary cocycle u satisfying, $\rho(t)a = u(t)\alpha(t)a$, $t \in \mathbf{R}$, $a \in A$. It is shown, that the generator of ρ is of the form, $a \to i(Ha - aK)$, where H and K are (unbounded) self-adjoint operators.

Introduction. We study the polar decomposition of a C_O-group ρ of isometries on a C^*-algebra. It is used to obtain information about the infinitesimal generator of ρ, and the implementability of ρ. The case, where the algebra contains a unit, is considered in [**9**].

It is known [**5**], [**7**], that a linear isometry, mapping a C^*-algebra onto itself, can be decomposed into a Jordan-automorphism, followed by multiplication by a unitary. The unitary may be chosen in the multiplier algebra of A. This decomposition is called the *polar decomposition.*

We prove in §1 that, if ρ is a C_0-group of isometries on a factor $\mathcal{M}$, and $\rho(t)a = u(t)\alpha(t)a$, a in $\mathcal{M}$, is the polar decomposition of each $\rho(t)$, then α is a C_O^*-group of automorphisms on $\mathcal{M}$, and u is a σ-weakly continuous unitary α-cocycle $(u(s+t) = u(s)\alpha(t)u(t))$ in$\mathcal{M}$. The corresponding result for a C_O-group of isometries on a C^*-algebra is proved in §2. In §3, we give necessary and sufficient conditions for u to be a representation of the additive group of real numbers. We prove, in §4, that it is possible to choose a representation of A such that $\rho(t)a = U(t)aV(t)$, for a pair of unitary C_O-groups U and V. We study the infinitesimal generator of a group of this form, see also [**9**, §4]. In the final section, we consider the case where A is a C^*-algebra of compact operators.

Notation. Let X be a Banach space. A group on X is a homomor-

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phism from the additive group of real numbers $\mathbf{R}$ into the multiplicative group of invertible elements in $B(X)$ = the ring of bounded linear operators on X. Let X^* be the Banach dual of X, if $x \in X$ and $\varphi \in X^*$. Then we write $\langle x, \varphi\rangle$ for the value of φ at the point x. A group ρ on X is a C_O-group, if $\rho(t)x$ is a continuous function of t, for each x in X. The generator δ is defined by

$$\delta(x) = \lim_{t \to O} (\rho(t)x - x)/t;$$

the domain $\mathcal{D}(\delta)$ of δ is the set of x in X for which the limit exists. We say that ρ is a C_O^*-group on X^*, if there exists a C_O-group ρ_* on X such that $\rho(t)$ is the adjoint of $\rho_*(t)$ for each t. The generator of ρ is then the adjoint of the generator of ρ_*.

If $\mathcal{H}$ is a Hilbert space and f is a vector in $\mathcal{H}$, then we denote by $[f]$ the projection onto the one-dimensional subspace of $\mathcal{H}$, spanned by f.

Let A be a C^*-algebra, a linear bijection α from A onto A is said to be a Jordan-automorphism (resp., automorphism) if $\alpha(1) = 1, \alpha(a^*) = \alpha(a)^*$ for a in A and $\alpha(a^2) = \alpha(a)^2$ for all self-adjoint a in A (resp., $\alpha(a^*) = \alpha(a)^*$, and $\alpha(ab) = \alpha(a)\alpha(b)$, for all a and b in A). We refer to [**8**], [**11**] for the theory of C^*- and von Neumann algebras.

1. von Neumann algebras.

Let $(\mathcal{M}, \mathcal{H})$ be a von Neumann algebra and let ρ be a C_O^*-group of isometries on $\mathcal{M}$. If u, and α, are determined by

$$u(t) = \rho(t)1, \text{ and } \alpha(t)a = u(t)^*\rho(t)a,$$

for t in $\mathbf{R}$, and a in $\mathcal{M}$. Then (u, α) is the polar decomposition of ρ in the sense of [**9**], i.e., $\rho(t)a = u(t)\alpha(t)a$, $t \in \mathbf{R}, a \in \mathcal{M}$, and each $\alpha(t)$ is a Jordan-automorphism on $\mathcal{M}$ [**5**, Theorem 7]. It is easy to see that u is then an α-cocycle, i.e.,

$$u(s+t) = u(s)\alpha(s)u(t)$$

for all s and t in $\mathbf{R}$. Our first result concerns the continuity properties of u and α.

PROPOSITION 1.1. *Let (u, α) be the polar decomposition of a C_O^*-group. Then* a) *u is strongly continuous;* b) *α is pointwise σ-weakly continuous.*

PROOF. Fix φ in $\mathcal{M}_*$, and choose ξ_n, η_n in $\mathcal{H}$, such that $\sum |\xi_n|^2 < \infty, \sum |\eta_n|^2 < \infty$ and $\varphi(a) = \sum(a\xi_n, \eta_n)$ for all a in $\mathcal{M}$.

(a). By assumption, $\langle u(t), \varphi\rangle = \langle \rho(t)1, \varphi\rangle$ is a continuous function of t. In particular, u is σ-weakly continuous, and therefore,

$$(u(t) - u(s)^*(u(t) - u(s)) = 2 - u(t)^*u(s) - u(s)^*u(t)$$

converges weakly to zero as t tends to s. Hence u is strongly continuous.

(b). If a is in $\mathcal{M}$, then

$$\langle \alpha(t)a - \alpha(s)a, \varphi\rangle = \sum(\rho(t)a - \rho(s)a)\xi_n, u(s)\eta_n) + \sum(\rho(t)a\xi_n, (u(t) - u(s))\eta_n).$$

The first sum converges to zero (as t tends to s), by assumption, and the second sum tends to zero by the dominated convergence theorem and (a).

THEOREM 1.2. *Let $(\mathcal{M}, \mathcal{H})$ be a von Neumann algebra and let ρ be a C_O^*-group of isometries on $\mathcal{M}$ with polar decomposition (u, α). If $\mathcal{M}$ is either abelian or a factor, then α is a C_O^*-group of automorphisms on $\mathcal{M}$.*

PROOF. If each $\alpha(t)$ is an automorphism, then a short computation shows that the group property of ρ and the cocycle property of u implies that α has the group property. Hence, we must prove that each $\alpha(t)$ is an automorphism. Only the case where $\mathcal{M}$ is a factor requires a proof, since a Jordan automorphism of a commutative algebra is an automorphism. In this case, each $\alpha(t)$ is either an automorphism or an anti-automorphism by [**5**, Theorem 10]. Hence, the Theorem follows from the continuity of α and the connectedness of $\mathbf{R}$.

PROPOSITION 1.3. *Let $(\mathcal{M}, \mathcal{H})$ be a von Neumann algebra; let α be a C_O^*-group of automorphisms on $\mathcal{M}$, and finally let u be a σ-weakly continuous unitary α-cocycle in $\mathcal{M}$. If $\rho(t)a = u(t)\alpha(t)a$ for $a \in \mathcal{M}, t \in \mathbf{R}$; then ρ is a C_O^*-group of isometries on $\mathcal{M}$.*

PROOF. It is easy to see that ρ satisfies the algebraic conditions. We will prove pointwise σ-weak continuity. Let $a \in \mathcal{M}$, and let φ be a

σ-weakly continuous state on $\mathcal{M}$. The Cauchy-Schwartz inequality for positive functionals yields the estimate,

$$\begin{aligned}|\langle \rho(t)a - a, \varphi\rangle|^2/2 \\ &\leq |\langle u(t)(\alpha(t)a - a), \varphi\rangle|^2 + |\langle (u(t) - 1)a, \varphi\rangle|^2 \\ &\leq \langle \alpha(t)(a^*a) - \alpha(t)(a^*)a - a^*\alpha(t)(a) + a^*a, \varphi\rangle \\ &\quad + |\langle (u(t) - 1)a, \varphi\rangle|^2,\end{aligned}$$

which, in turn, implies the desired continuity of ρ.

REMARKS 1.4. a) The main results of this paper remain true if $\mathbf{R}$ is replaced by an arbitrary connected topological group.

b) Theorem 1.2 is a partial converse to Proposition 1.3. But it cannot be extended to a full converse. Specifically, there exists a von Neumann algebra, which admits a C_O^*-group α of Jordan-automorphisms such that $\alpha(t)$ is not an automorphism for some t, cf., [**2**, p. 158].

c) Our result should be compared with the known fact that *a* C_O-group of isometries on a von Neumann algebra is automatically norm continuous, [**9**, Corollary 1.7].

EXAMPLE 1.5. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M} = B(\mathcal{H})$, and $\rho(t)a = U(t)a$, where U is a given unitary C_O-group on $\mathcal{H}$. The polar decomposition $(u, \alpha)$of ρ is then given by $u(t) = U(t)$ and $\alpha(t)a = a$.

2. C^*-algebras. We study groups of isometries on general C^*-algebras. The case where the C^*-algebra contains a unit was considered earlier in [**9**].

Let A be a C^*-algebra, and let ρ be a surjective isometry on A. By [**7**, Theorem 1] there exists a Jordan-automorphism α on A, and a unitary u in the multiplier algebra $M(A)$ of A, such that

$$\rho(a) = u\alpha(a)$$

for all a in A. By [**7**, Lemma 3], the pair (u, α) is uniquely determined by the above conditions. The pair (u, α) is called the *polar decomposition* of ρ. By uniqueness, this polar decomposition coincides with the one defined in [**9**], if A is assumed to have a unit. If ρ is a C_O-group of isometries on A, then the polar decomposition (u, α) is determined by the following condition: For each $t \in \mathbf{R}$, the pair $(u(t), \alpha(t))$ is the polar decomposition of $\rho(t)$. Our first result states that, in this case, α is a C_O-group of automorphisms on A, and u is a α''-cocycle when α'' is defined by $\alpha''(t) = \alpha(t)''$. Note that $\alpha''(t)$ is the σ-weakly continuous extension of $\alpha(t)$ to the universal enveloping von Neumann algebra A'' of A.

THEOREM 2.1. *Let A be a C^*-algebra and let $M(A)$ be the multiplier algebra of A.*

There is a canonical bijection between the set of C_O-groups ρ of isometries on A, and the set of pairs (u, α), where α is a C_O-group of automorphisms on A, and u is a $M(A)$ valued unitary α''-cocycle, such that the mapping

$$t \to \langle u(t), \varphi \rangle, \tag{1}$$

is continuous for each φ in $A^* = (A'')_* \subset M(A)^*$. The bijection is given by

$$\rho(t)a = u(t)\alpha(t)a, \tag{2}$$

for a in A and t in $\mathbf{R}$. That is, (u, α) is the polar decomposition of ρ.

PROOF. Let the pair (u, α) be specified as above. Then it follows from Proposition 1.3 that ρ is a weakly continuous group of isometries on A. Hence, ρ is a C_O-group by general semi-group theory [**3**, Corollary 3.1.8].

Conversely, assume that ρ is given, and let (u, α) be the polar decomposition of ρ. The listed properties of u follow directly from the discussions in §1. Moreover, an easy calculation shows that

$$\begin{aligned}\alpha(t)a^2 - \alpha(s)a^2 =& (\rho(t)a - \rho(s)a)^*(\rho(t)a + \rho(s)a) \\ &- (\rho(t)a - \rho(s)a)^*\rho(s)a + (\rho(s)a)^*(\rho(t)a - \rho(s)a)\end{aligned}$$

for $a = a^*$. Hence $||\alpha(t)a^2 - \alpha(s)a^2|| \leq 4||a||\,||\rho(t)a - \rho(s)a||$. It is well known that every element $a \in A$ decomposes as a linear combination of squares, $a = \sum c_j a_j^2$ $(j = 1, 2, 3, 4)$, where each c_j is a complex number of modulus one, and $a_j = a_j^* \in A, ||a_j||^2 \leq ||a||$. It follows that

$$(3) \qquad ||\alpha(t)a - \alpha(s)a|| \leq 4||a||^{1/2} \sum_{j=1}^{4} ||\rho(t)a_j - \rho(s)a_j||.$$

That is, α is strongly continuous. An argument in [9] shows that strong continuity of α implies that each $\alpha(t)$ is an automorphism. See also [**3**, proof of Corollary 3.2.12]. Therefore, each $\alpha(t)'' = \alpha''(t)$ is an automorphism. Now, the cocycle property of u, and a simple computation, shows that α'' is a group. Then, of course, α is a group as well.

COROLLARY 2.2. *Theorem* 2.1 *is also true if we replace the continuity condition* (1) *on* u *by the condition that* $t \to u(t)a$ *is continuous for each* a *in* A.

PROOF. If ρ is given, then the stated continuity of u follows from (2) and the strong continuity of α.

COROLLARY 2.3. *Let* α *be a* C_O*-group of automorphisms on a* C^**-algebra* A. *Let* u *be a* $M(A)$ *valued unitary* α''*-cocycle. The following two conditions are equivalent:*

(i) *The mapping* $t \to \langle u(t), \varphi \rangle$ *is continuous for each* φ *in* A^*;

(ii) *The mapping* $t \to u(t)a$ *is continuous for each* a *in* A. *If it is further assumed that* A *has a unit, then* (i) *and* (ii) *are equivalent to* (iii) *below*:

(iii) *The mapping* $t \to u(t)$ *is continuous in the norm of* A.

We give an example below of a C^*-algebra A (necessarily without unit) and a C_O-group ρ of isometries on A such that the unitary part of the polar decomposition of ρ is not norm continuous (relative to the norm of $M(A)$).

EXAMPLE 2.4. Let A be the C^*-algebra of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$, and let U be a C_O-group (with unbounded generator) of unitaries on $\mathcal{H}$. If we determine a C_O-group ρ of isometries on A by $\rho(t)a = U(t)a$, then the polar decomposition (u, α) of ρ is given by $u(t) = U(t)$ and $\alpha(t)a = a$ for $t \in \mathbf{R}$ and $a \in A$.

3. The unitary part. Let (u, α) be the polar decomposition of a continuous group ρ of isometries on an operator algebra. We show that the unitary part u is a group if the pair $(u(t), \alpha(t))$ satisfies a certain algebraic relation for each t.

THEOREM 3.1. *Let $\mathcal{M}$ be a von Neumann algebra, let α be a C_O^*-group of automorphisms on $\mathcal{M}$, and let u be a $\mathcal{M}$-valued σ-weakly continuous unitary α-cocycle. The conditions* (1) *and* (2) *below are equivalent:*

(1) $u(s + t) = u(s)u(t)$ *for all s and t in* $\mathbf{R}$.

(2) $u(t)\alpha(t)a = \alpha(t)(u(t)a)$ *for all a in $\mathcal{M}$, and t in* $\mathbf{R}$.

PROOF. Condition (1) is equivalent to the following:

$$u(t) = \alpha(s)(u(t)); \; s, t, \in \mathbf{R}, \tag{3}$$

since u is an α-cocycle. Moreover, (3) implies (2), since each $\alpha(t)$ is an endomorphism. Finally, we will argue that condition (2) implies (3). Since ρ and α are groups, (2) and a computation shows that $u(nt) = u(t)^n$ for t in $\mathbf{R}$ and $n = 1, 2, 3, \ldots$; therefore

$$u(s)u(t) = u(s + t) \tag{4}$$

if s and t are rational numbers with the same sign. By continuity (4) holds whenever s and t are real numbers and sign $(s) =$ sign (t). Hence $u(t) = \alpha(s)(u(t))$, if $s, t \geq 0$, by the cocycle property of u. Applying $\alpha(-s)$ to both sides of the last equality yields

$$u(t) = \alpha(s)(u(t)); \; s \in \mathbf{R}, \; t \geq 0.$$

Similarly we have $u(t) = \alpha(s)(u(t))$ for $s \in \mathbf{R}$ and $t \leq 0$. Condition (3) follows from this.

COROLLARY 3.2. *Let $A = \mathcal{M}$ be a C^*-algebra and let ρ be a C_0-group of isometries on A. The polar decomposition (u, α) of ρ satisfies the conclusion of Theorem* 3.1.

PROOF. The polar decomposition (u, α'') of ρ'' satisfies the assumptions in Theorem 3.1, and the corollary follows.

COROLLARY 3.3. *Let (u, α) be as in Theorem* 3.1 (*or as in Corollary* 3.2). *For each t in $\mathbf{R}$, define an operator $L(u(t))$ on $\mathcal{M}$ by the assignments $L(u(t))a = u(t)a$ for all a in $\mathcal{M}$. If $\rho(t) = L(u(t))\alpha(t)$, then the following five conditions are equivalent*:

(i) $\rho(t)\alpha(t) = \alpha(t)\rho(t)$ *for all t in $\mathbf{R}$*;

(ii) $L(u(t))\alpha(t) = \alpha(t)L(u(t))$ *for all t in $\mathbf{R}$*;

(iii) $u(s+t) = u(s)u(t)$ *for all s and t in $\mathbf{R}$*;

(iv) $L(u(s))\alpha(t) = \alpha(t)L(u(s))$ *for all s and t in $\mathbf{R}$*;

(v) $\rho(s)\alpha(t) = \alpha(t)\rho(s)$ *for all s and t in $\mathbf{R}$*.

4. Implemented groups. The generator of a C_O-group of isometries on a C^*-algebra is shown to be of the form $a \to i(Ha - aK)$, where H and K are (unbounded) self-adjoint operators. The only restriction, which we may impose on H and K in general, is that the mapping $a \to \exp(itH)a\exp(-itK)$ leaves the algebra invariant for each t.

LEMMA 4.1. *Let A be a C^*-algebra. If a is in $M(A)$, then $||a|| = \sup||ab||$, where the supremum is over all b in A with norm less than or equal to one.*

PROOF. Let $||a||_O = \sup\{||ab|| \mid b \in A, ||b|| \leq 1\}$. We will show that $||\cdot||_O$ is a C^*-norm on $M(A)$. Define a linear map L from $M(A)$ into the Banach algebra of all bounded linear maps on A by $L(a)b = ab$, $b \in A$. Clearly, $||a||_O = ||L(a)||$. Since the kernel, $\ker L$, of L is an ideal in $M(A)$ and the intersection $A \cap \ker L = \{0\}$, the known thickness of A in $M(A)$ [**11**, p. 169] implies that L is injective, i.e., $||a||_O$ is a Banach algebra norm on $M(A)$. Let $a \in M(A)$ and $\varepsilon > 0$ be given and choose

$b \in A$ such that $||b|| \leq 1$, and $||ab|| \geq (1-\varepsilon)||a||_O$. Then

$$||a^*a||_O \geq ||(ab)^*ab|| \geq (1-\varepsilon)^2||a||_O^2.$$

It follows that $||a||_O^2 \leq ||a^*a||_O$, which in turn implies that $||a||_O$ is a C^*-norm on $M(A)$.

THEOREM 4.2. *Let ρ be a norm continuous group of isometries on a C^*-algebra A. There exist norm continuous unitary groups U and V in A'' such that $\rho(t)a = U(t)aV(-t)$ for a in A and t in $\mathbf{R}$.*

PROOF. By eq. (3) of §3, $||\alpha(t) - \alpha(s)|| \leq 16||\rho(t) - \rho(s)||$. There exists, by a result in [**8**, Theorem 8.5.2], a norm continuous unitary group V in A'', such that $\alpha(t)a = V(t)aV(-t)$. If we take the supremum over all a in A with $||a|| \leq 1$, then

$$\begin{aligned} ||u(t) - u(s)|| &= \sup ||(u(t) - u(s))\alpha(s+t)a|| \\ &\leq \sup ||(\rho(t) - \rho(s))\alpha(s)a|| + \sup ||(\alpha(s) - \alpha(t))a|| \\ &\leq 17||\rho(t) - \rho(s)||. \end{aligned}$$

Now let $U(t) = u(t)V(-t)$, and the Theorem follows.

REMARK 4.3. Alternatively, one might prove Theorem 4.2 by first extending ρ to a norm-continuous group, ρ'' say, of isometries on A'', and then apply [**9**, Theorem 4.1] to ρ''.

DEFINITION 4.4. Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Let S and T be densely defined (unbounded) linear operators on $\mathcal{H}$. Let $\mathcal{D}(\delta_{S,T})$ denote the elements a in $B(\mathcal{H})$ which satisfy conditions (1) and (2) below:

(1) The operator a maps $\mathcal{D}(T)$ = the domain of T into $\mathcal{D}(S)$.

(2) There exists b in $B(\mathcal{H})$ such that $bf = i(Sa - aT)f$ for all f in $\mathcal{D}(T)$.

Now define a linear map $\delta_{S,T}$ on $B(\mathcal{H})$, by $\delta_{S,T}(a) = b$ for a in $\mathcal{D}(\delta_{S,T})$ where b is specified as above.

THEOREM 4.5. *Let ρ be a C_O-group of isometries on a C^*-algebra A. The following hold:*

(i) *There exist a faithful representation $(\pi, \mathcal{H})$ of A and two self-adjoint operators H and K on $\mathcal{H}$, such that the infinitesimal generator of $(Ad\pi)\rho$ is $\delta_{H,K}$, restricted to elements a in $\pi(A) \cap \mathcal{D}(\delta_{H,K})$ such that $\delta_{H,K}(a)$ is in $\pi(A)$. Here $(Ad\pi)\rho$ is the C_O-group on $\pi(A)$ determined by $((Ad\pi)\rho)(t)\pi(a) = \pi(\rho(t)a)$, for a in A and t in $\mathbf{R}$.*

(ii) *If A has a unit, then there exists a bounded self-adjoint operator P on $\mathcal{H}$ and a unitary operator W on $\mathcal{H}$ such that $H = W(K+P)W^*$.*

(iii) *If ρ is assumed to be norm continuous, then we may choose H and K in A''.*

PROOF. Part (i) follows from Theorem 2.1 and [**6**, Theorem A1]. Part (ii) is a consequence of (i) and [**4**, Theorem 4.3]. (iii) is a corollary to Theorem 4.2.

REMARK 4.6. (i) Note that, if A is the C^*-algebra of all compact operators on a Hilbert space $\mathcal{H}$, and if H, K is any pair of self-adjoint operators on $\mathcal{H}$, then the formula $\rho(t)a = \exp(itH)a\exp(-itK)$, $a \in A$, $t \in \mathbf{R}$, determines a C_O-group ρ of isometries on A, by Theorem 2.1. Hence, in general, there is not a relation between the H and K in Theorem 4.5.

(ii) If we are in case (ii) of Theorem 4.5, then

$$\delta_{H,K}(a) = W\delta_{K,K}(W^*a) + iWPW^*a.$$

Part (i) of Theorem 4.5 may therefore be regarded as an extension of [**9**, Theorem 3.1].

(iii) Let u be the unitary part of the polar decomposition of a given group ρ. Assume further that u is a group. Extend ρ to a C_O^*-group ρ'' of isometries on A'', and let δ'' be the infinitesimal generator of ρ''. If the unit $1 \in \mathcal{D}(\delta'')$, then it is easy to see that $\delta''(1)$ is the infinitesimal generator of u. It follows that u is norm continuous. If A has a unit, then automatically $1 \in \mathcal{D}(\delta'')$ cf., [**9**, Theorem 3.8]. This is not true in general, however, by Example 2.4.

(iv) Let (u, α) be the polar decomposition of the C_O-group ρ from strongly continuous unitary α-cocycles v with $v(0) = 1$ and the set of all unitary C_O-groups V on $\mathcal{H}$. The correspondence is determined by $v(t) = V(t)\exp(-itK)$.

Let T be a densely defined linear operator on the Hilbert space $\mathcal{H}$. Define $\delta_T = \delta_{T,T^*}$. We studied δ_T earlier in [**9**]. Here we will show that some of the results in [**9**] have converses when the following assumption is added: $\mathcal{D}(T) \subset \mathcal{D}(T^*)$. Specifically:

PROPOSITION 4.7. *Let T be a densely defined operator on a Hilbert space $\mathcal{H}$. If $\mathcal{D}(T)$ is contained in $\mathcal{D}(T^*)$, then we have:*

(a) $\mathcal{D}(T) = \{f \in \mathcal{H} | [f] \in \mathcal{D}(\delta_T)\}$.

(b) *If δ_T is assumed norm-norm closed, it follows that T is closed.*

(c) *If S and T are both symmetric and densely defined, and further, $\delta_S \subset \delta_T$, then it follows that $S \subset T + c$ for some complex scalar c.*

PROOF. (a). The inclusion $\subset$ follows from [**9**, Proposition 4.7]. The other inclusion is immediate from the definition of the domain of δ_T and the density of $\mathcal{D}(T^*)$ in $\mathcal{H}$.

(b). Let $f_n \in \mathcal{D}(T)$ and $f, g \in \mathcal{H}$. Assume that $f_n \to f$ and $Tf_n \to g$, as $n \to \infty$. We may assume that $|f_n| = |f| = 1$, where $|f|$ denotes the norm of $f \in \mathcal{H}$. Then $||[f_n] - [f]|| \to 0$. Since

$$-i\delta_T([f_n])h = (h, f_n)Tf_n - (h, Tf_n)f_n$$

for all h in $\mathcal{H}$, we get

$$|| - i\delta_T([f_n]) - (f \otimes g - g \otimes f)|| \to 0.$$

(Recall $(f \otimes g)h := (h, f)g$ for h in $\mathcal{H}$.) Hence $[f] \in \mathcal{D}(\delta_T)$ and $-i\delta_T([f]) = f \otimes g - g \otimes f$, which in turn gives $f \in \mathcal{D}(T)$ and $Tf = g$.

(c). By (a), $\mathcal{D}(S) \subset \mathcal{D}(T)$. If f is in $\mathcal{D}(S)$, then $\delta_S([f])f = \delta_T([f])f$, and it follows that $(S - T)f = [f](S - T)f$. Hence, we may define a scalar valued function K on $\mathcal{D}(S)$ by $(S - T)f = K(f)f$ for f in $\mathcal{D}(S)$. If f and g are in $\mathcal{D}(S)$ and c is a scalar, then

$$K(cf + g)(cf + g) = cK(f)f + K(g)g,$$

so K must be a constant.

COROLLARY 4.8. *The following three conditions are equivalent:*

(i) T *is closed and symmetric.*

(ii) δ_T *is a closed derivation.*

(iii) δ_T *is a closed* $*$*-derivation.*

PROOF. Apply part (b) of Proposition 4.7 and [**9**, Theorem 4.8].

PROBLEM 4.9. Is the space of all finite rank operators in $\mathcal{D}(\delta_T)$ a *core* for δ_T? This is true if T is assumed maximal symmetric.

5. An application. Using the main theorem of [**10**] and Theorem 2.1 above, we now determine the class of C_O-groups of isometries on a C^*-algebras of compact operators.

THEOREM 5.1. *Let A be a C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$. If ρ is a C_O-group of isometries on A, then there exist unitary C_O-groups U and V on $\mathcal{H}$ such that*

$$\rho(t)a = U(t)aV(t), \quad \text{for all } t \text{ in } \mathbf{R} \text{ and } a \text{ in } A.$$

PROOF. Let (u, α) be the polar decomposition of ρ, by Theorem 2.1 and [**10**], there exists a unitary C_O-group V on $\mathcal{H}$, such that $\alpha(t)a = V(-t)aV(t)$. The existence of V can also be deduced by adapting the method of [**3**, Example 3.2.35]. If $U(t) = u(t)V(-t)$, it follows that U and V satisfy the desired conditions.

REMARK 5.2. Theorem 5.1 is related to the Theorem in [**1**].

ADDED IN PROOF. A Banach algebra version of Theorem 4.2 appeared in: A.M. Sinclair, Jordan Homomorphisms and Derivations on Semisimple Banach Algebras, Proc. Amer. Math. Soc., **24** (1970), 209-215.

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DEPARTMENT OF MATHEMATICS, IUPUI, INDIANAPOLIS IN 46223