ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 4, Fall 1988

ON SOCLES OF ABELIAN P-GROUPS IN L

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0. Introduction. All groups in this paper are (separable) abelian p-groups. Our notations are standard as in [**5**]. For set theoretic notations we refer to [**2**] or [**8**].

One of the most celebrated results in the theory of p-groups is Ulm's Theorem: Each *countable* p-group A is uniquely determined by its socle $A[p] = \{x \in A | px = 0\}$, viewed as a valuated Z/pZ-vectorspace with values induced by the height-function of A.

Since each countable, separable p-group is Σ-cyclic (i.e., a direct sum of cyclics), Ulm's Theorem doesn't provide much information in the case of separable p-groups. The Σ-cyclic and the torion-complete p-groups are the only ones known to be determined by their socles in the class of all separable p-groups. If we only want to deal with separable p-groups of *cardinality* $\aleph_1$, a result due to Hill and Megibben [**6**] reads as follows:

Assume $2^{\aleph_0} < 2^{\aleph_1}$. If A is neither Σ-cyclic nor torion-complete and A has a *countable* basic subgroup, then there exists a group A' such that A and A' are not isomorphic but the socles $A[p]$ and $A'[p]$ are isometric, i.e., there exists a height-preserving isomorphism $\sigma : A[p] \to A'[p]$.

Assuming that a consequence of Gödel's axiom of constructibility holds, namely, $\diamondsuit$(E) for each stationary subset E of $\aleph_1$, we will show that one may drop the countability condition in the Hill-Megibben theorem:

THEOREM ($V = L$). *Let A be a separable, abelian p-group of cardinality $\aleph_1$. If A is neither Σ-cyclic nor torion-complete, then there exists a p-group A' such that $A \cong A'$ but $A[p]$ and $A'[p]$ are isometric.*

In our second chapter, we study (weakly) ω_1-separable p-groups A of cardinality $\aleph_1$, cf. [**9**]. Such a group has an ω_1-filtration $A = \bigcup_{\nu<\omega_1} A_\nu$ into *pure*, countable subgroups A_ν such that $A_{\nu+1}$ is a summand of

A_μ for all $\nu < \mu < \omega_1$. (ω_1 denotes the first uncountable cardinal). If $A' = \cup_{\nu<\omega_1} A'_\nu$ is another such group, A and A' are called filtration-equivalent if for suitable filtrations of A and A' we have isomorphisms $f_\nu : A_\nu \to A'_\nu$ for all $\nu < \omega_1$ such that $f_\nu(A_\mu) = A_\mu$ for all $\mu \leq \nu$ and we call f_ν a level-preserving isomorphism on A_μ.

If one only wants to construct a particular example rather than giving a characterisation as in our theorem above, it suffices to have $\diamondsuit(E)$ available for a particular stationary set E. In our next result, a weak diamond (cf. [**1**]) suffices and we only have to *assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Then there exist ω_1-separable p-groups which are filtration-equivalent, have isometric socles and isomorphic basic subgroups without being themselves isomorphic* (Theorem 2.2)

In analogy to [**3**, Def. 1.3] we say a ω_1-separable p-group A has *type* $Z(p^\infty)$ if in a suitable filtration $A = \cup_{\nu<\omega_1} A_\nu$, where $A_{\nu+1}$ always is a summand of $A_\mu, \mu > \nu$, we have $A_{\lambda+1}/A_\lambda = Z(p^\infty) \oplus C_\lambda$ for some Σ-cyclic C_λ, or A_λ is a summand of $A_{\lambda+1}$ for all $\lambda < \omega_1$. Similar to [**3**, Thm. 1.4] we obtain the

THEOREM. *Let $\xi_p\infty$ be the class of all ω_1-separable p-groups of type $Z(p^\infty)$ and of cardinality $\aleph_1$. If $A, A' \in \xi_p\infty$ have isometric socles, they are filtration-equivalent.*

Unfortunately our proof is slightly more complicated than Eklof's for the torsion free case, since we cannot use the uniqueness of division by p.

Again extending a result of [**3**] to p-groups we obtain the

THEOREM. ($MA + \neg CH$). *Let $A, A' \in \xi_p\infty$. Then $A \cong A'$ if and only if $A[p]$ and $A'[p]$ are isometric.*

Here we use the fact (cf. [**9**]) that assuming $MA + \neg CH$ (=Martin's axiom and the denial oft he continuum hypothesis) ω_1-separable p-groups are isomorphic if and only if they are filtration-equivalent.

Therefore it is undecidable in ZFC if groups in $\xi_p\infty$ are determined by their socles.

Finally we will construct ω_1-separable p-groups (in ZFC) which are *not quotient equivalent* (cf. [**7**] or [**2**, **3**]) *but have the same basic subgroup, the same* Γ*-invariant and isometric socles* (Theorem 2.7).

1. Constructing abelian p-groups supported by the same socle. In the following, each group will be an abelian p-group without elements of infinite height. We omit the proof of the well-known

LEMMA 1.1. *Let G and H be pure and dense subgroups of a torsion complete p-group $\overline{B}$. Then G and H are isomorphic iff there exists $\varphi \in Aut(\overline{B})$ such that $\varphi(G) = H$.*

LEMMA 1.2. *Let G be a p-group, $n \geq 0$ and $H' \subseteq G[p^{n+1}]$ such that*

(a) $G[p^{n+1}] = H' + p^m G[p^{n+1}]$ *for all $m \in N$.*

(b) $p^{m+1}G \cap H'[p^n] \subset p(H' \cap p^m G)$ *for all $m \in N$.*

Then there exists a pure and dense subgroup H of G such that $H[p^{n+1}] = H'$.

REMARK. If $n = 0$, one doesn't need (b) and Lemma 1.2 is well known in this case.

PROOF. Let H be a subgroup of G maximal with respect to $H[p^{n+1}] = H'$.

CLAIM 1. H is pure in G:

Since obviously $pG \cap H = pH$ we may assume

$$(1) \qquad p^m G \cap H = p^m H.$$

Let $p^{m+1}e \in p^{m+1}G \cap H$. If $p^m e \in H$, we use (1) and obtain a $\gamma \in H$ such that $p^m e = p^m \gamma$ and $p^{m+1}e = p^{m+1}\gamma \in p^{m+1}H$. Hence we may assume

$$(2) \qquad p^m e \notin H.$$

Because of the maximality of H there is $b \in G[p^{n+1}]$ and $\gamma \in H$ such that

$$b = p^m e + \gamma \tag{3}$$

We apply (a) and get $p^m e' \in p^m G[p^{n+1}]$ and $\gamma' \in H' = H[p^{n+1}]$ such that $b = p^m e' + \gamma'$. Hence $p^m e + \gamma - \gamma' = p^m e'$ and $\gamma - \gamma' = -p^m(e - e') \in p^m G \cap H = p^m H$.

Since $p^{m+1}e \in H$, we obtain $p^{m+1}e + p(\gamma - \gamma') = p^{m+1}e' \in H$ and $p^m e' \in G[p^{n+1}]$ implies $p^{m+1}e' \in Gp^n \cap H = H'[p^n]$ and $p^{m+1}e' \in H'[p^n] \cap p^{m+1}G \subseteq p(H' \cap p^m G)$ because of (b). Hence there exists $\gamma'' \in H' \cap p^m G$ such that $p^{m+1}e' = p\gamma''$ and therefore

$$p^{m+1}e = p(\gamma' - \gamma) + p\gamma'' = p(\gamma' - \gamma + \gamma'')) \text{ and } \gamma' - \gamma, \gamma'' \in p^m H. \tag{4}$$

Finally we get $p^{m+1}e \in p^{m+1}H$ and H is pure in G.

CLAIM 2. H is dense in G, i.e., $G = H + p^m G$ for all $m \in \mathbf{N}$: We will prove by induction that $G[p^\ell] \subseteq H + p^m G$ for all ℓ and $m \in \mathbf{N}$.

Because of (a) we have $G[p^\ell] \subseteq H + p^m G$ for all $m \in \mathbf{N}$ and all $\ell \leq n + 1$. Suppose $\ell \geq n + 2$ and $G[p^s] \subseteq H + p^m G$ for all $s < \ell$ and all $m \in \mathbf{N}$.

Let $b \in G[p^\ell] - G[p^{\ell-1}]$ and $m \in \mathbf{N}$. Then we have $0 \neq \mathrm{pb} \in G[p^{\ell-1}]$ and we obtain $p^{m+1}e \in p^{m+1}\overline{G}$ and $\gamma \in H$ such that $\mathrm{pb} = p^{m+1}e + \gamma$. Since H is pure in G, we find $\gamma' \in H$ with $p\gamma' = \gamma$. Therefore $pb = p^{m+1}e + p\gamma'$ and $b = p^m e + \gamma' + a$ for some $a \in G[p]$. By our assumption we have $a = \gamma'' + p^m e'$ for some $\gamma'' \in H, e' \in G$. Hence $b = \gamma' + \gamma'' + p^m(e + e') \in H + p^m G$ and $G[p^\ell] \subseteq H + p^m G$ for all $\ell, m \in \mathbf{N}$ and H is dense in G.

LEMMA 1.3. *Let $\overline{B}$ be a torsion complete p-group and G a pure and dense subgroup of $\overline{B}$. Let $z \in \overline{B}[p] - G[p]$ and $\eta \in \mathbf{N}$. Moreover let H' be a subgroup of $\overline{B}[p^{n+1}]$ such that*

(1) $G[p^{n+1}] \subseteq H' + \langle z \rangle$

(2) $H'[p^n] = G[p^n]$

(3) *z is an element of the p-adic closure of $H' \cap G[p]$ in $\overline{B}$, i.e. there exists a sequence $\{z_m\}_{m\in\mathbf{N}}$ such that $z_m \in H' \cap G[p]$ and $z - z_m \in p^m\overline{B}$.*

Then H' satisfies the conditions (a) and (b) of Lemma 2.

PROOF. We first show (a): Let $b \in \overline{B}[p^{n+1}], 0(b) = p^s$. Since G is dense in $\overline{B}$, we find $g \in G$, $b' \in \overline{B}$ such that $b = g + p^m b'$, and $0 = p^s b = p^s g + p^{m+s} b'$ and $p^{m+s} b' = -p^s g \in p^{m+2}\overline{B} \cap G = p^{m+2} G$. Therefore there exists $g' \in G$ with $p^{m+s} b' = p^{m+s} g'$ and $b = (g + p^m g') + (p^m b' - p^m g') \in G + p^m \overline{B}[p^{n+1}]$. Since $G[p^{n+1}] \subseteq H' + < z\rangle$ we get $g \in H', \ell \in N$ such that $g + p^m g' = q + \ell z$ and $\ell z - \ell z_m \in p^m \overline{B}$. Therefore $b = (q + \ell z_m) + (\ell z - \ell z_m + p^m b' - p^m q') \in H' + p^m \overline{B}[p^{n+1}]$ and $\overline{B}[p^{n+1}] \subseteq H' + p^m \overline{B}[p^{n+1}]$ is shown.

To prove (b), let $p^{m+1} b \in H'[p^\eta] = G[p^\eta]$. Since G is pure in $\overline{B}$, we get $g \in G$ such that $p^{m+1} b = p^{m+1} g$ and since $p^m g \in G[p^{u+1}]$, we obtain $q \in H'$ and $\ell \in N$ such that $p^m g = q + \ell z$ and again $p^g - (\ell z - \ell z_m) = q + \ell a_m \in H'$ and $h_{\overline{B}}(q + \ell z_m) = h_{\overline{B}}(p^m g - (\ell z - \ell(z_m)) \geq \min \{h_{\overline{B}}(p^m g), h_{\overline{B}}(\ell z - \ell z_m)\} \geq m$. Therefore the p-height of $q + \ell z_m$ in $\overline{B}$, $h_{\overline{B}}(q + \ell z_m) \geq m$ and $p(q + \ell z_m) = p(p^m q - (\ell_z - \ell z_m)) = p^{m+1} g = p^{m+1} b$ and $p^{m+1} b \in p(H' \cap p^m \overline{B})$.

All the set theoretical notations we will use in this paper may be found in P. Eklof's remarkable paper [**3**] on ω_1-separable torsion free groups.

LEMMA 1.4. *Let G be a p-group of regular, uncountable cardinality κ. Then there exists a pure κ-filtration $G = \cup_{\alpha<\kappa} G_\alpha$ of G. (A κ-filtration is called pure, if all the G_α's are pure in G).*

We'll omit the routine proof.

DEFINITION 1.5. Let $G = \cup_{\alpha<\kappa} G_\alpha$ be a pure κ-filtration of the separable p-group G and $m \in \mathbf{N}$. G_α is called *not p^m-closed* if there is a $y \in G$ such that $0(y) = p^m$, $\langle y\rangle \cap G_\alpha = 0$ and $y \in G_\alpha[p^m] + p^k G$ for each $k \in \mathbf{N}$. Let $\gamma(G) = \{\alpha < \kappa | \lim(\alpha)$ and G_α is not p^m-closed$\}$, $P(\kappa)/\sim$ the Boolean algebra $P(\kappa)$ modulo the ideal of non-stationary subsets of κ and $\Gamma^m_\kappa(G) = \gamma(G)/\sim$. Then $\Gamma^m_\kappa(G)$ is an invariant of G, cf. [**3**] or [**7**].

REMARK 1.6. Let $\overline{B}$ be a torsion complete p-group of regular cardinality κ and G a pure subgroup of $\overline{B}$ and $|G| = \kappa$. Then there exist pure κ-filtrations $G = \cup_{\alpha<\kappa} G_\alpha$ and $\overline{B} = \cup_{\alpha<\kappa}\overline{B}_\alpha$ of G and $\overline{B}$ such that $G_\alpha \subseteq \overline{B}_\alpha$ for each $\alpha < \kappa$.

DEFINITION 1.7. Let G be a pure subgroup of the separable p-group $A, |G| = \kappa$ regular, $G = \cup_{\alpha<\kappa} G_\alpha$ a pure κ-filtration of G and $G'_\alpha = G_\alpha[p^{n+1}]$. We may assume $|G_\alpha| \geq \aleph_0$ for all $\alpha < \kappa$. A subgroup H of $A[p^{n+1}]$ is called (n, z, α)-admissible in G if $z \in A[p] - G$ and

(1) $H \cap \langle z \rangle = 0$

(2) $H[p^n] \subseteq G[p^n]$

(3) $H \subseteq G[p^{n+1}] + \langle z \rangle$

(4) $|H| \langle \kappa$

(5) $G'_\alpha \subseteq H + G[p^n] + \langle z \rangle$.

LEMMA 1.8. *Same notation as in* 1.8. *If H is a (n, z, α)-admissible subgroup, then there exists an $(n, z, \alpha + 1)$-admissible subgroup H' such that $H + G_{\alpha+1}[p^n] \subseteq H'$.*

PROOF. Let $\mathcal{M}$ be the set of all subgroups $\tilde{H}$ of $H + G'_{\alpha+1}$ such that:

(a) $\tilde{H} \cap \langle z \rangle = 0$

(b) $\tilde{H}[p^n] \subseteq G[p^n]$

(c) $H + (G_{\alpha+1}[p^n]) \subseteq \tilde{H}$.

We will show that $H + G_{\alpha+1}[p^n] \in \mathcal{M}$.

We may assume $n \geq 1$. In order to show (a), let $h + g = kz \in (H + G_{\alpha+1}[p^n]) \cap \langle z \rangle$ where $h \in H, g \in G_{\alpha+1}[p^n]$ and $k \in \mathbf{Z}$. Since $h \in H[p^n] \subseteq G[p^n], h + g = kz \in G \cap \langle z \rangle = 0$ and (a) holds. To prove (b), let $a \in (H + G_{\alpha+a}[p^n])[p^n], a = h + g$ with $h \in H$ and $g \in G_{\alpha+1}[p^n]$. Again, $0(h) \leq p^n$ and $a = h + g \in G[p^n]$. This shows (b) and we have $\mathcal{M} \neq \emptyset$. Since the $\mathcal{M}$ is inductive, we may apply Zorn's Lemma to obtain a maximal element H' in $\mathcal{M}$.

Conditions (a), (b) and (c) imply (1), (2) of 1.8 and moreover $H + G_{\alpha+1}[p^n] \subseteq H'$. Since $H \subseteq G[p^{n+1}] + \langle z \rangle, 0(z) = p$ and

$G'_{\alpha+1} \subseteq G[p^{n+1}]$ we obtain $H' \subseteq H + G'_{\alpha+1} \subseteq G[p^{n+1}] + \langle z \rangle$. Hence 1.8(3) holds for H' as well and $H' \subseteq A[p^{n+1}]$. Now $|H| \langle \kappa, |G'_{\alpha+1}| \langle \kappa$ and $|H'| \leq |H + G'_{\alpha+1}| \leq |H| + |G'_{\alpha+1}| \langle \kappa$. This implies 1.8(4). We have to show $G'_{\alpha+1} \subseteq H' + G[p^n] + \langle z \rangle$. Let $g \in G'_{\alpha+1}$. If $0(g) \leq p^n$ (c) implies $g \in H'$. Hence we may assume $0(g) = p^{n+1}$ and $g \notin H'$. Then we have $H + G_{\alpha+1}[p^n] \subseteq H' + \langle g \rangle$ and the maximality of H' implies $(H' + \langle g \rangle \cap \langle z \rangle \neq 0$ or $(H' + \langle g \rangle)[p^n] \subseteq G[p^n]$. We'll consider two cases:

CASE 1. $(H' + \langle g \rangle) \cap \langle z \rangle \neq 0$.

Then there exists $h \in H'$ and $k \in \mathbf{Z}$ such that $z = h + kg$. Since $G_{\alpha+1}[p^n] \subseteq H'$ and $H' \cap \langle z \rangle = 0, p$ doesn't divide k and there exists $k' \in \mathbf{Z}$ with $k'kg = g$. This implies $g = k'(z - h) \in H' + \langle z \rangle \subseteq H'G[p^n] + \langle z \rangle$.

CASE 2. $(H' + \langle g \rangle)[p^n] \subseteq G[p^n]$.

Here we have $n \geq 1$ and $h \in H', k \in \mathbf{Z}$ with $h + kg \notin G[p^n]$ and $0(h + kg) \leq p^n$.

Since $H' \subseteq H + G_{\alpha+1} \subseteq G[p^{n+1}] + \langle z \rangle$, there exists $g \in G[p^{n+1}]$ and $\ell \in \mathbf{Z}$ such that $h + kg = \tilde{g} + \ell z$.

Now $0 = p^n(h + kg) = p^n(\tilde{g} + \ell z), 0(z) = p$ and $n \geq 1$ imply $p^n \tilde{g} = 0$ and hence $\tilde{g} \in G[p^n]$.

We obtain

$$kg = -h + \tilde{g} + \ell z \in H' + G[p^n] + \langle z \rangle.$$

If p does not divide k, we are finished. Suppose p divides k. Then $kg \in G[p^n]$ and $0(h + kg) \leq p^n$ implies $h \in H'[p^n]$, a contradiction to our choice of $h + kg$.

This shows $G'_{\alpha+1} \subseteq H' + G[p^n] + \langle z \rangle$ and H' is $(n, z, \alpha + 1)$-admissible.

In the next lemma, we use diamonds $\diamondsuit_\kappa(E)$, claiming the existence of Jensen functions on the stationary subset of the regular cardinal κ, cf. ([**2**], [**8**]).

LEMMA 1.9. *Assume $\diamondsuit_\kappa(\Gamma^{n+1}_\kappa(G))$ holds for some fixed $n \in \mathbf{N}$ and let G be a separable abelian p-group of cardinality κ. Moreover let $G \neq \overline{G}$ be the torsion completion of $G, \Gamma^{n+1}_\kappa(G) \neq 0, z \in \overline{G}[p] - G[p]$ and $|\overline{G}| = \kappa$. Then there exists $H \subseteq \overline{G}[p^{n+1}]$ such that*

(a) $G[p^{n+1}] + \langle z \rangle = H + \langle z \rangle$

(b) $G[p^n] = H[p^n]$

(c) For all $\varphi \in \mathrm{Aut}(\overline{G})$, $\varphi(G[p^{n+1}]) \neq H$.

(d) *z is an element of the p-adic closure of $H \cap G[p]$ in $\overline{G}$.*

PROOF. For each $k \in \mathbf{N}$ take $z_k \in G[p]$ such that $z - z_k \in p^k G$ and a pure κ-filtration $G = \cup_{\alpha\langle\kappa} G_\alpha$ with $z_k \in G_\omega$ for all $k \in \mathbf{N}$ and a κ-filtration $\overline{G} = \cup_{\alpha\langle\kappa} \overline{G}_\alpha$ such that $G_\alpha \subseteq \overline{G}_\alpha$, cf. (6). Let $E = \{\alpha < \kappa |$ lim (α) and G_α not p^{n+1}-closed$\}$ and $\{f_\alpha : \overline{G}_\alpha \to \overline{G}_\alpha | \alpha \in E\}$ be a collection of Jensen functions to witness $\diamondsuit_\kappa(E)$.

By induction we will define a subgroup $H = \cup_{\alpha<\kappa} H_\alpha$ of $\overline{G}[p^{n+1}]$ such that

(0) $H_\alpha = G_\alpha[p^{n+1}]$ for $\alpha < \omega$,

(1) $H_\beta \subseteq H_\alpha$ for all $\beta \leq \alpha$,

(2) $H_\alpha = \cup_{\beta<\alpha} H_\beta$ if α is a limit ordinal,

(3) $z \notin H_\alpha$,

(4) $H_\alpha[p^n] \subseteq G[p^n]$,

(5) $H_\alpha \subseteq G[p^{n+1}] + \langle z \rangle$,

(6) $|H_\alpha| < \kappa$,

(7) If $\alpha = \beta + 1, \beta \in E, f_\beta(G_\beta[p^{n+1}]) = H_\beta : f_\beta = \phi \restriction \overline{G}_\beta$ for some $\phi \in \mathrm{Aut}(\overline{A})$ such that $z \notin \phi(G), \phi(G[p^n]) = (G[p^n])$ and $\phi(G[p^{n+1}]) \subseteq G[p^{n+1}] + \langle z \rangle$, let H_α be defined by $H_\alpha = H_\beta + \langle \phi(Y_\beta) + z \rangle$ where

(7a) $y_\beta \in \hat{G}_\beta$, the closure of G_β in G

(7b) $0(y_\beta) = p^{n+1}$

(7c) $\langle y_\beta > \cap G_\beta = 0$.

(8) If $\alpha = \beta + 1$ and β doesn't fit into (7) we have $G_\alpha[p^n] \subseteq H_\alpha$ and H_α is a (n, z, α)-admissible subgroup.

For $n < \omega$ let $H_n = G_n[p^{n+1}]$ and the H_n's satisfy the conditions (0) - (8). Let $\delta \geq \omega$ and $\delta < \kappa$. Suppose we have constructed H_α for all $\alpha < \delta$ satisfying (0) - (8).

If δ is a limit ordinal let $H_\delta = \cup_{\beta<\delta} H_\beta$. Since κ is regular and $|H_\beta| < \kappa$ we have $|H_\delta| < \kappa$. Conditions (2) -(5) for H_δ are obvious. Now assume $\delta = \beta + 1$.

CASE 1. β satisfies (7). Since $\beta \in E$, we find y_β satisfying (7a) - (7c). Let $H_{\beta+1} = H_\beta + \langle \phi(Y_\beta) + z\rangle$ and assume $z \in H_{\beta+1}$. Then $z = h + \ell(\phi(Y_\beta) + z)$ for some $h \in H_\beta, \ell \in \mathbf{Z}$ and $\ell \equiv 0 \bmod p$. Hence $z(1-\ell) = h + \ell\phi(y_\beta)$ and $\phi^{-1}(z(1-\ell)) = \phi^{-1}(h) + \ell y_\beta \in G$ because $y_\beta \in G$ and $\phi(G_\beta[p^{n+1}]) = H_\beta$. Therefore $z(1-\ell) \in \phi(G)$ which implies $\ell \equiv 1 \bmod p$. Now $z = h+\ell(\phi(y_\beta)+z) = h+\ell\phi(y_\beta)+z$ and $\ell\phi(y_\beta) \in H_\beta$. Hence $\ell y_\beta \in \phi^{-1}(H_\beta) = G_\beta[p^{n+1}]$ and by (7c) we get $\ell y_\beta = 0$. Now $z = h \in H_\beta$, a contradiction to (3). Therefore $z \notin H_{\beta+1}$ and we have to show that $H_{\beta+1}$ satisfies (4): $H_{\beta+1}[p^n] \subseteq G[p^n]$. So let $h \in H_\beta, \ell \in \mathbf{Z}$ such that $p^n x = 0$ where $x = h+\ell(\phi(y_\beta)+z) \in H_{\beta+1}$. This implies $0 = p^n x = p^n h + p^n \ell\phi(y_\beta)$ and $p^n \ell y_\beta \in \phi^{-1}(H_\beta) = G_\beta[p^{n+1}]$. Therefore $p^n \ell y_\beta = 0$ and $\ell \equiv 0 \bmod p$ and hence $x = h + \ell\phi(y_\beta) \in G[p^n]$ because $0 = p^n x = p^n h, H_\beta[p^n] \subseteq G[p^n]$ and $p\phi(y_\beta) \in \phi(G[p^n]) = G[p^n]$. This shows (4). Condition (5) is obvious because $\phi(G[p^{n+1}]) \subseteq G[p^{n+1}]+\langle z\rangle$ and $H_\beta \subseteq G[p^{n+1}]+\langle z\rangle$. By the definition of $H_{\beta+1}$ we have $H_\beta \subseteq H_{\beta+1}$ and $|H_{\beta+1}| < \kappa$.

CASE 2. $\beta = \beta' + 1$ and β' is not a limit ordinal. Then $\beta' \notin E$ and H_β is a (n, z, β)-admissible subgroup. We may apply Lemma 1.8 to get a $(n, z, \beta+1)$-admissible subgroup $H_{\beta+1}$ with $H_\beta + G_{\beta+1}[p^n] \subseteq H_{\beta+1}$. This shows that $H_{\beta+1}$ satisfies the conditions (0) - (8).

CASE 3. $\beta = \beta' + 1$ and β' is a limit ordinal. Since $H_{\alpha+2}$ is a $(n, z, \alpha + 2)$-admissible subgroup for all $\alpha < \beta'$ we have $G_{\beta'}[p^{n+1}] \subseteq H_{\beta'} + G[p^n] + \langle z\rangle$. Therefore $G_{\beta'}[p^{n+1}] \subseteq H_\beta + G[p^n] + \langle z\rangle$ and with (3) - (6) for H_β we have H_β is (n, z, β')-admissible. Now apply Lemma 1.9 two times to obtain a $(n, z, \beta + 1)$-admissible subgroup $H_{\beta+1}$ such that $H_\beta + G_{\beta+1}[p^n] \subseteq H_{\beta+1}$.

This completes our construction. We will show that H satisfies (a), (b), (c) and (d). Condition (b) is obvious because of (4) and (8) and $H + \langle z\rangle \subseteq G[p^{n+1}] + \langle z\rangle$ follows from (5). Since $H_{\alpha+2}$ is a $(n, z, \alpha+2)$-admissible subgroup for all $\alpha < \kappa$ and (b) we have $G[p^{n+1}] \subseteq H + \langle z\rangle$. This shows (a) and since Z is in the closure of $G_\omega[p^{n+1}] = H_\omega$ we obtain (d).

To prove (c), let $\varphi \in \mathrm{Aut}(\overline{G})$ such that $\varphi(G[p^{n+1}]) = H$. Since $C = \{\alpha < \kappa | \varphi(G_\alpha[p^{n+1}]) = H_\alpha\}$ is a cub and $S = \{\alpha | \varphi \restriction \overline{G}_\alpha = f_\alpha\}$ is stationary we have a limit ordinal $\beta \in C \cap S.\varphi(G[p^n]) = G[p^n]$ is obvious because of (b) and $\varphi(G[p^{n+1}]) \subseteq G[p^{n+1}] + \langle z \rangle$ follows from (a). Since $z \notin H \supseteq \varphi(G[p])$ we have $z \notin \varphi(G)$. Therefore φ satisfies condition (7). But $\varphi(Y_\beta) \in \varphi(G[p^{p+1}]) = H$ and if $\phi \in \mathrm{Aut}(\overline{G})$ is the map used in the definition on $H_{\beta+1}$ we have $\phi \restriction \overline{G}_\beta = f_\beta = \varphi \restriction \overline{G}_\beta$ and $\phi(Y_\beta) = \varphi(Y_\beta)$ because of the continuity of automorphisms. But $Y_\beta \in G[p^{n+1}]$ implies $\phi(Y_\beta) = \varphi(Y_\beta) \in H$ and $\phi(Y_\beta) + z \in H$ by construction, so we obtain the contradiction $z \in H$. This proves (c).

We are now able to prove our main result.

THEOREM 1.10. $(V = L)$. *Let G be a separable p-group of regular cardinality κ, $\overline{G}$ its torsion-completion and $\Gamma_\kappa^{n+1}(G) \neq 0$ and $G \neq \overline{G}$. Then there exists a subgroup $H \subseteq \overline{G}$ such that*

(i) *H is pure and dense in $\overline{G}$*

(ii) $G[p^n] = H[p^n]$

(iii) $H \cong G$.

PROOF. We apply Lemma 1.9 and get $\tilde{H} \subseteq \overline{G}[p^{n+1}]$ satisfying (1.9a), (1.9b) (1.9c) and (1.9d). Now apply Lemma (1.2) and (1.3) to obtain a pure and dense subgroup H of $\overline{G}$ such that $\tilde{H} = H[p^{n+1}]$ and $G[p^n] = \tilde{H}[p^n] = H[p^n]$.

Assume $G \cong H$. Then there exists a $\varphi \in \mathrm{Aut}(\overline{G})$ such that $\varphi(G) = H$ and hence $\varphi(G[p^{n+1}]) = H[p^{n+1}] = \tilde{H}$ contradicting (1.9c).

THEOREM 1.11. $(V = L)$. *Let $n \in \mathbf{N}$ and G a separable p-group of cardinality $\aleph_1$ such that G is neither Σ-cyclic nor torsion-complete. Then there exists a separable p-group H such that H is not isomorphic to G but there exists a height-preserving isomorphism $\varphi : H[p] \to G[p]$.*

For the proof of (1.11) we need

LEMMA 1.12. *Let $\tilde{H}$ be a separable p-group, C' a subgroup of $\tilde{H}$ and S a dense subsocle of $\tilde{H}$* (cf. [**5**]). *If $C'[p] \subseteq S$ then there exists a pure subgroup H of $\tilde{H}$ such that $C' \leq H, H[p] = S$ and $\tilde{H}/H$ is divisible.*

Let

$$\mathcal{M}_{C'} = \{H \leq \tilde{H} | H[p] = S \text{ and } C' \leq H\} \text{ and}$$
$$\mathcal{M} = \{H \leq \tilde{H} | H[p] = S\}. \text{ Obviously } C' + S \in \mathcal{M}_{C'} \subseteq \mathcal{M}$$

because $C'[p] \subseteq S$. Since $\mathcal{M}_{C'}$ is inductive, we may apply Zorn's Lemma to obtain a maximal element H in $\mathcal{M}_{C'}$. Then H is a maximal element in $\mathcal{M}$. By [**5**, 66.3] we have that H is pure and dense in $\tilde{H}$. This shows (1.12).

PROOF OF (1.11). Let $\overline{G}$ be the torsion-completion of G. Since G is not torsion-complete we have $G \neq \overline{G}$. If $\Gamma^2_{\aleph_1}(G) \neq 0$ then (1.10) implies that there is a pure and dense subgroup H such that $H \cong G$ and $H[p] = G[p]$. It is obvious that the identity map on the socles of H and G is an isometry because H and G are pure subgroups of $\overline{G}$. This shows (1.11) in the case $\Gamma^2_{\aleph_1}(G) \neq 0$.

Now assume $\Gamma^2_{\aleph_1}(G) = 0$. Since G is not Σ-cyclic we have $\Gamma^1_{\aleph_1}(G) \neq 0$, cf. [**4**].

We will show that there is a subgroup H of $\overline{G}$ such that (a) H is pure in $\overline{G}$, (b) $H[p] = G[p]$ and (c) $\Gamma^{n+1}_{\aleph_1}(H) \neq 0$ for all $n \in \mathbf{N}$. Let $G[p] = U_{\alpha<\omega_1} S_\alpha$ be an ω_1-filtration of $G[p]$ (with $S_0 = 0$) and for $\alpha < \omega_1$ let $\overline{S}_\alpha$ be the closure of S_α in G. Then $S_\alpha \leq \overline{S}_\alpha \leq G[p], \overline{S}_\beta \leq \overline{S}_\alpha$ for all $\beta < \alpha$ and $G[p] = \cup_{\alpha<\omega_1} \overline{S}_\alpha$. By induction we will define for each $\alpha < \omega_1$ subgroups H_α and $\tilde{H}_\alpha$ such that

(1) $H_\beta \leq H_\alpha$ for all $\beta \leq \alpha$,

($\tilde{1}$) $\tilde{H}_\beta \leq \tilde{H}_\alpha$ for all $\beta \leq \alpha$,

(2) $H_\alpha = \cup_{\beta<\alpha} H_\beta$ if α is a limit ordinal

($\tilde{2}$) H_α is a pure subgroup of $\tilde{H}_\alpha$,

(3) $H_\alpha[p] = S_\alpha$,

($\tilde{3}$) $\tilde{H}_\alpha[p] = \overline{S}_\alpha$,

(4) H_α is pure in $\overline{G}$,

($\tilde{4}$) $\tilde{H}_\alpha$ is pure in $\overline{G}$,

(5) $\tilde{H}_\alpha/H_\alpha$ is divisible.

Let $H_0 = 0$ and $\tilde{H}_0 = 0$. Since G is separable and $S_0 = 0$ we have $\overline{S}_0 = 0$. Hence H_0 and $\tilde{H}_0$ satisfy the conditions (1)-(5) and ($\tilde{1}$)-($\tilde{4}$). Let $\delta > 0$ and $\delta < \omega_1$. Suppose we have constructed H_α and $\tilde{H}_\alpha$ for all $\alpha > 0$ satisfying (1)-(5) and ($\tilde{1}$)-($\tilde{4}$).

Let $C = \cup_{\alpha<\delta}\tilde{H}_\delta$. Then C is a pure subgroup of $\overline{G}$ because of ($\tilde{1}$) and ($\tilde{4}$) for all $\alpha < \delta$. Since $\overline{S}_\alpha$ is contained in $\overline{S}_\delta$ for all $\alpha < \delta$ we have $C[p] = (\cup_{\alpha<\delta}\tilde{H}_\alpha)[p] = \cup_{\alpha<\delta}\overline{S}_\alpha \leq \overline{S}_\delta$. By [**5**, 74 (e) and 74.1] we get a pure subgroup $\tilde{H}_\delta$ of $\overline{G}$ such that $C \leq \tilde{H}_\delta$ and $\tilde{H}_\delta[p] = \overline{S}_\delta$. Hence $\tilde{H}_\delta$ satisfies the conditions ($\tilde{1}$), ($\tilde{3}$) and ($\tilde{4}$). Now we have to construct H_δ.

Let $C' = \cup_{\alpha<\delta}H_\alpha$. Then $C'[p] = (\cup_{\alpha<\delta}H_\alpha[p] = \cup_{\alpha<\delta}S_\alpha \leq S_\sigma \leq \overline{H}_\delta[p]$ and $C = \cup_{\alpha<\delta}H_\alpha \leq \cup_{\alpha<\delta}\tilde{H}_\alpha \leq \tilde{H}_\delta$. Condition (1) and (4) for all $\alpha < \delta$ imply that C' is a pure subgroup of $\overline{G}$. Therefore C' is a pure subgroup of $\tilde{H}_\delta$. Next we show that $H_\delta[p] = S_\delta + p^k\tilde{H}_\delta[p]$ for all $k \in \mathbf{N}$. Let $k \in \mathbf{N}$. Since G and $\tilde{H}_\delta$ are pure subgroups of $\overline{G}, \overline{S}_\delta \leq G$ and $\overline{S}_\delta = \tilde{H}_\delta[p] \leq \tilde{H}_\delta$, we have

$$\begin{aligned} p^kG \cap \overline{S}_\delta &= p^k\overline{G} \cap G \cap \overline{S}_\delta = p^k\overline{G} \cap \overline{S}_\delta = p^k\overline{G} \cap \tilde{H}_\delta \cap \overline{S}_\delta = \\ &= p^k\tilde{H}_\delta \cap \overline{S}_\delta = p^k\tilde{H}_\delta \cap \tilde{H}_\delta[p] = p^k\tilde{H}_\delta[p]. \end{aligned}$$

Therefore, since $\overline{S}_\delta$ is the closure of S_δ in G, $\tilde{H}_\delta[p] = \overline{S}_\delta = S_\delta + (p^kG \cap \overline{S}_\delta) = S_\delta + p^k\tilde{H}_\delta[p]$. Now we may apply (1.12) to get a pure subgroup H_δ of $\tilde{H}_\delta$ such that $C' \leq H_{\delta'} H_\delta[p] = S_\delta$ and $\tilde{H}_\sigma/H_\sigma$ is divisible. Since H_δ is pure in $\tilde{H}_\delta$ and $\tilde{H}_\delta$ is pure in $\overline{G}$, we have H_δ is pure in $\overline{G}$. Hence H_σ and $\tilde{H}_\sigma$ satisfy the conditions (1)-(5) and ($\tilde{1}$)-($\tilde{4}$) and our construction works.

Let $H = \cup_{\alpha<\omega_1}\tilde{H}_\alpha$. Condition (1), (4) and (3) imply that H is a pure subgroup of G and

$$H[p] = (\cup_{\alpha<\omega_1}H_\alpha)[p] = \cup_{\alpha<\omega_1}\overline{S}_\alpha = G[p].$$

This shows (a) and (b).

From (1), (4), (3) and ($\tilde{2}$) we conclude that $\cup_{\alpha<\omega_1}H_\alpha$ is a pure subgroup of $\overline{G}, (\cup_{\alpha,\omega_1}H_\alpha)[p] = G[p]$ and $\cup_{\alpha<\omega_1}H_\alpha \leq H$. Therefore $H[p] = (\cup_{\alpha<\omega_1}H_\alpha)[p]$ and $\cup_{\alpha<\omega_1}H_\alpha$ is a pure subgroup in H. By [**5**, 26 (j), p. 115] we have $H = \cup_{\alpha<\omega_1}H_\alpha$. Since $|S_\alpha| < \omega_1$ and $H_\alpha[p] = S_\alpha$

we infer that $|H_\alpha| < \omega_1$. Now in view of (1) and (2) we have that $H = \cup_{\alpha<\omega_1} H_\alpha$ is a ω_1-filtration of H. Let $E = \{\alpha < \omega_1 | S_\alpha$ is not closed in $g\}$ and $G = \cup_{\alpha<\omega_1} G_\alpha$ a pure ω_1-filtration of G. Then

$$\begin{aligned} E/\sim &= \{\alpha < \omega_1 | G_\alpha[p] \text{ is not closed in } G\}/\sim \\ &= \{\alpha < \omega_1 | G_\alpha \text{ is not } p-\text{closed in } G\}/\sim = \Gamma^1_{\aleph_1}(G) \neq 0 \end{aligned}$$

Hence E is a stationary subset of ω_1. Let $\alpha \in E$ and $n \in \mathbf{N}$. Then $S_\alpha \neq \overline{S}_\alpha$ and therefore $\tilde{H}_\alpha/H_\alpha \neq 0$. Since $\tilde{H}_\alpha/H_\alpha$ is divisible and H_α is pure in $\tilde{H}_\alpha$ we find a $y_n \in \tilde{H}_\alpha$ such that $0(y_n) = p^{n+1}$, $\langle y_n\rangle \cap H_\alpha = 0$ and $y_n \in H_\alpha[p^{n+1}] + p^k\tilde{H}_\alpha \subseteq H_\alpha[p^{n+1}] + p^k H$ for all $k \in \mathbf{N}$. This shows that $\alpha \in E_{n+1} = \{\alpha < \omega_1 | H_\alpha$ is not p^{n+1}-closed in $H\}$. Hence $\Gamma^{n+1}_{\aleph_1}(H) \neq 0$, because E is stationary in ω_1. This shows (c). Since G and H are pure subgroups of $\overline{G}$ and $G[p] = H[p]$, the groups have isometric socles. But $G \cong H$ because $\Gamma^2_{\aleph_1}(G) = 0$ and $\Gamma^2_{\aleph_1}(H) \neq 0$. This completes the proof of (1.11).

2. ω_1-separable p-groups with equal socles. In this chapter we will construct - using weak diamonds - ω_1-separable p-groups having isometric socles. Similar constructions may be found in [**3**], [**4**].

Let B be Σ-cyclic p-group, $B = \oplus_{\alpha<\omega_1} \oplus_{n<\omega} (\alpha,n)\mathbf{Z}$ such that $0(\alpha,n) = p^{n+k}$ for all $\alpha < \omega_1$ and some fixed $k \in \mathbf{N}$. We fix a stationary subset $E \subseteq \omega_1$ such that $E \subseteq \{\alpha < \omega_1 | \lim\,(\alpha)\}$, i.e., all elements of E are limit ordinals.

For each $\lambda \in E$ fix a ladder $\{\lambda_n\}_{n\in\mathbf{N}}$, i.e., $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbf{N}$ and $\lambda = \sup\{\lambda_n | n \in \mathbf{N}\}$. Moreover we choose $\tilde{z}_{(\lambda,n)} \in \mathbf{Z}$ such that $\tilde{z}_{(\lambda,n)} \equiv 1 \bmod p^k$. For $\lambda < \omega_1$ let $B_\lambda = \oplus_{\alpha<\lambda} \oplus_{n<\omega} (\alpha,n)\mathbf{Z}$ and $\hat{B}_\lambda$ the torsion-completion of B_λ and if $\lim\,(\lambda)$, let $\tilde{B}_\lambda = \cup_{\alpha<\lambda}\hat{B}_\alpha$. For each $\lambda \in E$, define $\lambda^\circ_m = \Sigma_{n\geq m}(\lambda_n, n)p^{n-m} \in \hat{B}_\lambda - \tilde{B}_\lambda$. Define $z(\lambda,n)$ to be 1 if n is odd and $z(\lambda,2n) = \tilde{z}(\lambda,n)$ and assume $\tilde{z}(\lambda,n) \equiv 1 \bmod p^{k+1}$. Let $\lambda^1_m = \Sigma_{n\geq m}(\lambda_n,n)z(\lambda,n)p^{n-m}$. Observe that for $\varepsilon = 0,1$ we have $p\lambda^\varepsilon_{m+1} - \lambda^\varepsilon_m \in B_\lambda$ and $\lambda^\circ_\circ = \lambda^1_0$ for all $\lambda \in E$. Set $G_\alpha = \langle B_\alpha, \lambda^\varepsilon_m | m < \omega, \varepsilon \in \{0,1\}, \lambda < \alpha\rangle$. Then $G = \cup_{\alpha<\omega_1} G_\alpha$ is an ω_1-filtration of the pure subgroup G of $\hat{B} := \hat{B}_{\omega_1}$.

We will need the following.

LEMMA 2.1. *Let A_0, A_1 be pure subgroups of $\tilde{B}_\lambda$ such that $B_\lambda \subseteq$*

$A_0 \cap A_1$ and $A_0[p^k] = A_1[p^k]$. Let $A^\varepsilon = \langle A_\varepsilon, \lambda^\varepsilon_m | m < \omega\rangle, \varepsilon = 0, 1$. Then A^0, A^1 are pure subgroups of $\hat{B}_\lambda$ such that $A^0[p^k] = A^1[p^k]$ and $A^0 \cap A^1 = \langle A_0 \cap A_1\rangle + A^0[p^k]$.

PROOF. Since $A^\varepsilon/A_\varepsilon$ is divisible and A_ε pure in $\hat{B}_\lambda$, A^ε is pure in $\hat{B}_\lambda$. If $x \in A^0[p^k], x = a_0 + \lambda^0_m r$ for some $a_0 \in A_0, m < \omega$ and $r \in \mathbf{Z}$.

Since $a_0 \in \tilde{B}_\lambda$ we obtain $(\lambda_n, n)p^{n-m}p^k = 0$ for almost all n and $p^{n-m+k}r \equiv 0 \bmod p^{n+k}$ and hence $r \equiv 0 \bmod p^m$. This implies $x \in A_0[p^k] \oplus \langle\lambda^0_0\rangle$ and a similar argument shows $A^1[p^k] = A_1[p^k] \oplus \langle\lambda^1_0\rangle$ and the above remarks show $A^0[p^k] = A^1[p^k]$. Take now any $x \in A^0 \cap A^1$. Then $x = a_0 + \lambda^0_m r = a_1 + \lambda^1_m$ where $a_\varepsilon \in A_\varepsilon, r, s \in Z$ and $m < \omega$. This implies $a_0 - a_1' = -\lambda_m r + \lambda_m s$ and again $(\lambda_n, n)(p^{n-m}z(\lambda, \eta) - p^{n-m}r) = 0$ for almost all $n < \omega$, which implies $sz(\lambda, \eta) \equiv r \bmod p^{m+k}$ for almost all $n < \omega$.

Therefore $sz(\lambda, n) \equiv sz(\lambda, n') \bmod p^{m+k}$ if $n, n' \geq n_0$. By our choice $1 \equiv z(\lambda, n) \bmod p^k$ if n is odd and $z(\lambda, n) \not\equiv z(\lambda, n+1) \bmod p^{k+1}$ which implies $s \equiv 0 \bmod p^m$ and $r \equiv 0 \bmod p^m$. Therefore $x = a_0 + \lambda^0_m \in A_0 + A^)[p^k]$ and $\lambda^0_m r = \lambda^1_m s$ also implies $a_0 = a_1 \in A_0 \cap A_1$ and $x \in (A_0 \cap A_1) + A^0[p^k] = A^0 \cap A^1$.

Recall that a stationary subset $E \subseteq \omega_1$ is *non-small*, if the weak diamond $\phi_\omega(E)$ holds. (cf. [**1**])

THEOREM 2.2. $(2^{\aleph_0} < 2^{\aleph_1})$. *Let E be a non-small subset of ω_1. There exist $2^{\aleph_1}$ many ω_1-separable p-groups $A_\alpha, \alpha < 2^{\aleph_1}$, such that*

(0) *$\Gamma(A_\alpha) = E$ for all $\alpha < 2^{\aleph_1}$,*

(1) *$A_\alpha \cong A_\beta$ if $\alpha \neq \beta$,*

(2) *$A_\alpha[p^k] \cong A_\beta[p^k]$ are isometric ,*

(3) *For all $\alpha, \beta < 2^{\aleph_1}, A_\alpha$ and A_β are filtration-equivalent.*

PROOF. Since $2^{\aleph_0} < 2^{\aleph_1}$, there exists a partition $E = \cup_{\alpha<\omega_1} E_\alpha$ into non-small subsets E_α. (c.f. [**1**]).

For each $\eta \in {}^{\omega_1}2$ we define a group $A_\eta = \cup_{\alpha<\omega_1} A_{\eta\restriction\alpha}$ such that

(j) $A_{f\eta\restriction 0} = 0$ and $A_{\eta\restriction\lambda} \subseteq \hat{B}_\lambda$, (ij) $\lim\,(\lambda) \Rightarrow A_{\eta\restriction\lambda} = \cup_{\alpha<\lambda} A_{\eta\restriction\alpha}$,

(iij) $\nu \notin E \Rightarrow A_{\eta\restriction\nu+1} = A_\eta$

(iv) If $\nu \in E, A_{\eta\restriction\nu+1} = A^{\varepsilon}_{\eta\restriction\nu'}\varepsilon = \eta(\nu+1)$

according to Lemma 2.1 (so we have $A^0_{\eta\restriction\nu} \cap A^1_{\eta\restriction\nu} = A_{\eta\restriction\nu} + A^{\varepsilon}_{\eta\restriction\nu}[p^k]$).

For each $\delta \in E$ define a partition function P_δ: If $\xi, \rho \in \omega_1$ and $h: A_\xi \to A_\rho$, let $P(\xi,\rho,h) = \begin{cases} 1 & \text{if } h \text{ lifts to } h^0 : A^0_\xi \to A^0_\rho \\ 0 & \text{otherwise} \end{cases}$.

Let ψ_α be the function provided by $\phi_{\aleph_1}(E_\alpha)$, i.e., $\{\nu \in E_\alpha | \psi_\alpha(\nu) = P_\alpha(s \restriction \nu, t \restriction \nu, g)\}$ is stationary for each $s,t < \omega_1$ and $g: A_s \to A_t$. Take $\Sigma \leq P(\aleph_1)$ such that $S = T$ if $S, T \in \Sigma$ and $S \subseteq T$ or $T \subset S$. We may choose $a\Sigma$ s.t. $|\Sigma| = 2^{\aleph_1}$.

Now define $\varphi_S \in 2^{\aleph_1}$ such that $\varphi_S(\delta) = \begin{cases} \psi_\alpha(\delta) \text{ if } \delta \in E_\alpha \text{ and } \alpha \in S \\ 0 \text{ otherwise} \end{cases}$.

Take $A_S := A_{\varphi_S} = \cup_{\alpha<\omega_1} A_{\varphi_S\restriction\alpha}$. By our construction, obviously $A_S[p^k] = A_T[p^k]$ and elements have equal heights. This implies (2). Let $S, T \in \Sigma, S \neq T$ and assume $h: A_S \to A_T$ is an isomorphism. Then the set $C = \{\nu < \aleph_1 | h(A_{\psi_S\restriction\nu}) = A_{\varphi_t\restriction\nu}\}$ is a cub.

Take any $\alpha \in T - S$ and $\lambda \in E_\alpha \cap C$. Set $\eta = \varphi_S \restriction \lambda, \varphi = \varphi_T \restriction \lambda, \theta = h \restriction A_\eta$. Since $\lambda \notin E_\beta$ for all $\beta \in S, \varphi_S(\lambda) = 0, \psi_\alpha(\lambda) = P_\alpha(\eta,\rho,\theta)$ and w.l.o.g. θ lifts to a $\tilde\theta : A^0_\eta = A_{\varphi_s\restriction(\lambda+1)} \to A_{\varphi_T\restriction(\lambda+1)} = A^{\psi\alpha(\lambda)}_{\varphi T\restriction\lambda}$ because $A_{\varphi_S\restriction(\lambda+1)}(A_{\varphi_T\restriction(\lambda+1)})$ is the p-adic closure of $A_\eta(A_\rho)$ in $A_S(A_T)$. Therefore $\psi_\alpha(\lambda) = 1$ and $\tilde\theta$ is a 1-1 map of $A^0_\eta onto A^1_{\varphi_T\restriction\lambda} \cap A^0_{\varphi_T\restriction\lambda} = (A_{\varphi_T\restriction\lambda} \cap A_{\varphi_T\restriction\lambda}) + A_{\varphi T\restriction\lambda}[p^k]$ which is impossible. Therefore $A_S \not\cong A_T$ and (1) is shown. (3) will be an immediate consequence of our next, more general, result. Observe that $A_{\varphi S\restriction(\lambda+1)}/A_{\varphi S\restriction\lambda} = \mathbf{Z}_{p^\infty}$ if $\lambda \in E$.

DEFINITION 2.3. (I) A separable abelian p-group A is weakly ω_1-separable if each countable subgroup B of A is contained in a countable ω_1-pure subgroup C of A, i.e., if $C \subseteq C' \subseteq A$ and C' is countable, then C is a summand of C'.

(II) A weakly ω_1-separable p-group A is of type $Z(p^\infty)$, if A admits a pure filtration $A = \cup_{\alpha<\omega_1} A_\alpha$ such that $A_{\alpha+1}$ is ω_1-pure in A for all $\alpha < \omega_1$ and if λ is a limit ordinal, A_λ is also ω_1-pure or $A_{\lambda+1}/A_\lambda$ is isomorphic to $\mathbf{Z}(p^\infty)$.

We will adopt parts of the proof of 1.4 Theorem in [**3**, p. 506] to show

THEOREM 2.4. *Let A, A' be weakly ω_1-separable p-groups of cardinality ω_1 and of type $Z(p^\infty)$. If there exists a height-preserving isomorphism $\sigma : A[p] \to A'[p]$, then A and A' are filtration-equivalent.*

PROOF. We first show that A and A' have pure ω_1-filtrations $A = \cup_{\nu<\omega_1} A_{\nu'} A' = \cup_{\nu<\omega_1} A'_\nu$ such that $\sigma(A_\nu[p]) = A'_\nu[p]$ and $A_{\nu+1}(A'_{\nu+1})$ are ω_1-pure in $A(A')$:

Let $A = \cup_{\nu<\omega_1} \tilde{A}_\nu, A' = \cup_{\nu<\omega_1} \tilde{A}'_\nu$ be pure ω_1-filtrations such that $\tilde{A}_{\nu+1}/\tilde{A}_\nu(\tilde{A}'_{\nu+1}/\tilde{A}'_\nu)$ are either Σ-cyclic or $\cong Z(p^\infty)$ and $\tilde{A}_{\nu+1}(\tilde{A}'_{\nu+1})$ are all ω_1-pure in $A(A')$. The set $C = \{\nu|\sigma(\tilde{A}_\nu[p]) = \tilde{A}'_\nu[p]\}$ is a cub in ω_1. Let ν_0 min C.

If $\tilde{A}_{\nu_0}$ is ω_1-pure in $A, \tilde{A}'_{\nu_0}$ is ω_1-pure as well (apply σ) and we may set $A_0 = \tilde{A}_{\nu_0}$ and $A'_0 = \tilde{A}'_{\nu_0}$. If $\tilde{A}_{\nu_0}$ is not ω_1-pure in $A, \tilde{A}_{\nu_0+1}[p]$ is the p-adic closure of $\tilde{A}_{\nu_0}[p]$ in $A[p]$ which implies $\nu_0 + 1 \in C$ and we can take $A_0 = A_{\nu_0+1}$ and $A'_0 = \tilde{A}'_{\nu_0+1}$. Suppose we have defined A_α, A'_α for all $\alpha < \beta$. If β is a limit, take $A_\beta = \cup_{\alpha<\beta} A_\alpha$. Suppose $\beta = \gamma + 1$ is a successor and $A_\gamma = \tilde{A}_{\nu_\gamma}$. Let $\nu_{\gamma+1} =$ min $\{\rho \in C|\rho > \nu_\gamma\}$ and repeat the argument used at the beginning of our induction.

Hence we may assume $A = \cup_{\alpha<\omega_1} A_\alpha$ and $A' = \cup_{\alpha<\omega_1} A'_\alpha$ are pure ω_1-filtrations, $A_{\alpha+1}(A'_{\alpha+1})$ are ω_1-pure in $A(A')$ and for $\lambda \in E \leq \omega_1$ we have $A_{\lambda+1}/A_\lambda \cong A'_{\lambda+1}/A'_\lambda \cong Z(p^\infty) \oplus C_\lambda$ where C_λ is Σ-cyclic. (Observe that a height-preserving isomorphism on the socles of Σ-cyclic p-groups is always induced by some isomorphism of the groups). For $\lambda \in E$, let $A_{\lambda+1} = A^d_\lambda \oplus B_\lambda$ where $A_\lambda \subseteq A^d_\lambda, A^d_\lambda/A_\lambda \cong Z(p^\infty)$ and $B_\lambda \cong C_\lambda$. Moreover we fix $w_\lambda \in A^d_\lambda[p](w'_\lambda \in A'^d_\lambda[p])$ such that $A_{\lambda+1}[p] = A_\lambda[p] \oplus \langle w_\lambda \rangle \oplus B_\lambda[p]$ and $\sigma(w_\lambda) = w'_\lambda$. (The element w_λ has infinite height in $A_{\lambda+1}[p]/A_\lambda[p]$).

Consider the sequence $h^\lambda = (h_{A_\lambda/A_\nu}(w_\lambda + A_\nu)|\nu < \lambda)$. This is an unbounded, increasing sequence of natural numbers. We say that the sequence h^λ has a gap at ν if $h^\lambda(\mu) < h^\lambda(\nu)$ for all $\mu < \nu$. Since

$h^\lambda(\nu)$ are finite heights, gaps don't occur at limit ordinals. Therefore we find a strictly increasing sequence of *successor* ordinals λ_n such that for $h_n = h^\lambda(\lambda_n)$ we have $h_1 < h_2 < \cdots < h_n < h_{n+1}$ and $\lambda = \sup\{\lambda_n | n < \omega\}$. Each A_{λ_n} is a summand of A_λ and if we define h'_n, λ'_n for $w'_\lambda + A'_\nu$, the existence of σ implies $h_n = h'_n$ and $\lambda_n = \lambda'_n$.

We will now study the embedding of A_λ into A^d_λ, where again $A_{\lambda+1} = A^d_\lambda \oplus B_\lambda$ and $A^d_\lambda / A_\lambda \cong Z(p^\infty)$. Let $h_n = h_{A_\lambda/A_{\lambda_n}}(w_\lambda + A_{\lambda_n})$. By induction we define elements $w_n \in A^d_\lambda$ such that

(1) $p^{h_n} w_n = w_\lambda + \tilde{a}_n, \tilde{a}_n \in A_{\lambda_n}[p]$,

(2) $p^{h_n+1} w_n = 0$,

(3) $p^h_{n+2} - h_n w_{n+1} - w_n = a_n \in A_{\lambda_n}$,

(4) $p^{h_{n+1}} a_n = 0$ and $p^{h_n} a_{n+1} = \tilde{a}_{n+1} - \tilde{a}_n$,

(5) $h_{A_\lambda/A_\nu}(a_{n+1} + A_\nu) = 0$ for all $\lambda_n \leq \nu < \lambda_{n+1}$.

We easily find $u_n \in A^d_\lambda$ such that $p^{h_n} u_n - w_\lambda = \tilde{a}_n \in A_{\lambda_n}[p]$ and $p^{h_n+2} u_n = 0$. Since $A^d_\lambda / A_\lambda \cong Z(p^\infty)$ we have $A^d_\lambda = \langle A_\lambda \cup \{u_n | n \in \mathbf{N}\}\rangle$ and $p^{h_{n+1}-h_n} u_{n+1} = u_n + b_n$ for some $b_n \in A_\lambda$.

We have to adjust the u_n's to obtain (3):

Assume we have define $w_1, \ldots, w_n$. If $\hat{A}_\lambda$ is the torsion completion of A_λ we have $A_\lambda \subseteq A^d_\lambda \subseteq \hat{A}_\lambda$ and we may choose a natural basis B of A_λ, i.e., B contains a basis B_n of A_{λ_n} and $B = \cup_n B_n$.

Now assume $b_n \in A_\lambda - A_{\lambda_{n+1}}$, i.e., there exists some $e \in B - B_{n+1}$ such that $b_n(e) \neq 0$.

We distinguish two cases

CASE 1. $w_\lambda(e) \neq 0$.

Since $h_{A^d_{\lambda}/A_{\lambda_n}}(w_\lambda) = \min\{h_{fZ}(w_\lambda(f)) | f \in B - B_n\}$, we have $h_{eZ}(\lambda(e)) \geq h_{n+1}$, and $p^{h_{n+1}} u_{n+1} = p^{h_n} u_n + p^{h_n} b_n$ implies $\tilde{a}_{n+1} - \tilde{a}_n = p^{h_n} b_n \in A_{\lambda_{n+1}}$ and therefore $p^{h_n} b_n(e) = 0$. Since $h(w_\lambda(e)) \geq h_{n+1}$ and $w_\lambda(e) \neq 0$, we get $0(e) \geq p^{h_{n+1}+1}$. This implies $h_{ez}(b_n(e)) \geq 0(e)p^{-h_n} \geq p^{h_{n+1}-h_n+1}$ and there exists an $z_e \in Z$ such that $b_n(e) = ep^{h_{n+1}-h_n} z_e$.

Let $w_{n+1} = u_{n+1} = \Sigma_e e z_e$. Since the sum is finite we have again $w_{n+1} \in A^d_\lambda$.

CASE 2. $w_\lambda(e) = 0$.

Here we correct u_{n+1} and u_n such that $u_{n+1}(e) = 0 = u_n(e)$. This finally shows (3). (4) is obvious.

We have $h_{k+1} = h_{A_\lambda}(w_\lambda + \tilde{a}_{k+1}) = h_{A_\lambda/A_{\lambda_{k+1}}}(w_\lambda + A_{\lambda_{k+1}}) \geq h_{A_\lambda/A_\nu}((w_\lambda+A_\nu)+(\tilde{a}_{k+1}+A_\nu)) \geq \min\{h_{A_\lambda/A_\nu}(w_\lambda+A_\nu), h_{A_\lambda/A_\nu}(\tilde{a}_{k+1}+A_\nu)\} = \min\{h_k, h_{A_\lambda/A_\nu}(\tilde{a}_{k+1} + A_\nu)\}$ if $\lambda_n \leq \nu < \lambda_{n+1}$. Assume $h_k > h_{A_\lambda/A_\nu}(\tilde{a}_{k+1} + A_\nu)$. Then we have $h_{k+1} \leq h_{A_\lambda/A_\nu}(w_\lambda + \tilde{a}_{k+1} + A_\nu) = h_{A_\lambda/A_\nu}(\tilde{a}_{k+1} + A_\nu) < h_k$ a contradiction. Therefore $h_k \leq h_{A_\lambda/A_\nu}(\tilde{a}_{k+1} + A_\nu) = h_{A_\lambda/A_\nu}(\tilde{a}_{k+1} - \tilde{a}_k + A_\nu) = h_{A_\lambda/A_\nu}(p^{h_k}a_{k+1} + A_\nu) \leq h_k$ and $h_{A_\lambda/A_\nu}(a_{k+1} + A_\nu) = 0$.

This implies

$$(*) \qquad a_{k+1} + A_\nu \text{ generates a cyclic summand of } A_{\lambda_{n+1}}/A_\nu \text{ for all } \lambda_n \leq \nu < \lambda_{n+1}$$

It is routine to verify

If $\varphi : A_\lambda \to A'_\lambda$ is an isomorphism such that $\varphi(a_n) = a'_n$ for $n \geq n_0$, then φ lifts to an isomorphism $\varphi' : A_{\lambda+1} \to A_{\lambda+1}$.

Now let $\mu < \lambda$ (w.l.o.g. $\mu < \lambda_1$) and let $f : A_\mu \to A_\mu$ be a level preserving isomorphism such that $f \restriction A_\mu[p] = \sigma \restriction A_\mu[p]$. Using induction we may show that f extends to some level preserving $f' : A'_\mu \to A'_\mu$ where $\lambda_1 = \mu'+1$. Since $a+A_{\lambda_1}(a'+A'_{\lambda_1})$ generate cyclic summands of $A_{\lambda_1}/A_{\mu'}(A'_{\lambda_1}/A'_\mu)$ and $\sigma(p^{h_1}a) = \sigma(\tilde{a}_1) = \tilde{a}'_1 = p^{h_1}a'_1$, we can extend f to f' such that $f'(a_1) = a'_1$. Assume we already found $g_n : A_{\lambda_n} \to A'_{\lambda n}$ extending $g_{n-1}(f =: g-1)$. Again (*) implies that we may extend g_n to $g_{n+1} : A_{\lambda_{n+1}} \to A'_{\lambda_{n+1}}$ being a level preserving isomorphism.

We may now repeat Eklof's argument [**3**, p. 507] to obtain a level preserving isomorphism $f_\nu : A_\nu \to A'_\nu$ for all $\nu < \omega_1$.

COROLLARY 2.5 ($MA + \neg CH$). *Two ω_1-separable p-groups of cardinality $\aleph_1$ are isomorphic if they are both of type $Z(p^\infty)$ and have isometric socles.*

PROOF. Assumption of Martin's axiom and the denial of the continuum hypothesis makes all weakly ω_1-separable p-groups ω_1-separable, (c.f. [**9**, Thm 2.2]) and filtration-equivalence means isomorphic, c.f. [**9**, Thm 4.1] or [**2**] in the torsion free case.

Combining 2.2 and 2.5 we obtain

COROLLARY 2.6. *The question "Are ω_1-separable p-groups of type $Z(p^\infty)$ determined - up to isomorphism - by their socles" is undecidable in ZFC.*

We will conclude our paper with a construction which answers a question of M. Huber [**7**, p. 316] and also provides a proof of an assertion of Megibben's [**9**]. Consult [**7**, p. 312] for the definition of quotient-equivalence.

THEOREM 2.7 *Let E be a stationary subset of ω_1. Then there exist ω_1-separable p-groups A, A' of cardinality $\aleph_1$ such that*

(a) *$A[p]$ and $A'[p]$ are isometric.*

(b) $\Gamma(A) = \Gamma(A') = E$.

(c) *A and A' have the same basic subgroup.*

(d) *A and A' are not quotient-equivalent.*

PROOF. Let $B_\beta = \oplus\{(\alpha, n, \varepsilon)\mathbf{Z} | \alpha < \beta, n < \omega, \varepsilon \in \{0,1\}\}$ and $B = \cup_{\beta \leq \omega_1} B_\beta$ where $0(\alpha, n, \varepsilon) = p^{n+1}$. By induction on $\lambda \in E$ we define $\lambda_* < \omega_1$ such that

(I) λ_* is a limit ordinal and $\lambda_* > \lambda$.

(II) $\sup\{\mu_* | \mu < \lambda\} < \lambda_*$.

We may choose for each $\lambda \in E$ a ladder λ_n, λ_*^n of successor ordinals such that $\sup\{\lambda_n | n < \omega\} = \lambda, \sup\{\lambda_*^n | n < \omega\} = \lambda_*$ and $\lambda_*^0 > \sup\{\mu_* | \mu < \lambda\}$. For $\lambda \in E$, let $W_{\lambda,k} = \sum_{n=k}^\infty (\lambda_n, n, 0)p^{n-k}$ and $W_*^{\lambda,k} = W_{\lambda,k} + V_{\lambda,k}$ where $V_{\lambda,k} = \sum_{n=k-1}^\infty (\lambda_*^n, n, 1)p^{n-k+1}$ for $k \geq 1$ and $V_{\lambda,0} = 0$. Then we have $p^k W_{\lambda,k} = W_{\lambda,0} - \sum_{i=0}^{k-1} (\lambda_i, i, 0)p^i = p^k W_*^{\lambda,k}$ for all $\lambda \in E$ and $k \in \mathbf{N}$. Moreover, the elements $W_*^{\lambda,k}, k \geq 1$ are independent modulo $A_\lambda + 1$.

Define $A_\alpha = \langle (\nu, n, \varepsilon), W_{\lambda,k} | \nu < \alpha, n < \omega, \alpha > \lambda \in E, k < \omega, \varepsilon = 0, 1\rangle$

and $A'_\alpha = \langle (\nu, n, \varepsilon), W_*^{\lambda,k} | \nu < \alpha, n < \omega, \alpha > \lambda \in E, k < \omega, \varepsilon = 0,1 \rangle$.

For $\lambda \in E$, we have $A_{\lambda+1}/A_\lambda \cong Z(p^\infty)$ and $A'_{\lambda+1}/A'_\lambda \cong P$, the reduced Prüfer-group, i.e., $P = (\langle W_*^{\lambda,k} | k < \omega \rangle + A'_\lambda)/A'_\lambda, P[p] = \langle W_*^{\lambda,0} + A'_\lambda \rangle$ and $P/P[p] \cong \oplus_{k=1}^\infty \langle W_*^{\lambda,k} + A'_\lambda \rangle$. Observe that $A'_{\lambda+1} + B_{\lambda_*}/B_{\lambda_*}$ is divisible and the identity serves as a height-preserving isomorphism of $A_{\lambda+1}[p]$ onto $A'_{\lambda+1}[p]$. This implies that $A = \cup_{\alpha<\omega_1} A_\alpha$ and $A' = \cup_{\alpha<\omega_1} A'_\alpha$ are the desired groups. We leave the details to the reader.

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