ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 3, Summer 1988

ERGODIC SEQUENCES AND A SUBSPACE OF B(G)

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ABSTRACT. J. Blum and B. Eisenberg studied conditions on a sequence $\{\mu_n\}$ of probability measures on a locally compact abelian group G which ensured that, for any strongly continuous unitary representation π of G on a Hilbert space H and for any $\xi \in H$, $\{\int_G \pi(x)\xi d\mu_n(x)\}$ converges to a G-invariant member of H. In this paper their result is (essentially) generalized to non-abelian G. The generalization involves $\mathbf{B}_I(G)$, the closure of the linear span of the coefficients of the irreducible representations of G; thus $\mathbf{B}_I(G)$ contains $\mathbf{AP}(G)$ always, and equals $\mathbf{A}(G)$ if G is compact or abelian. The relationships of $\mathbf{B}_I(G)$ to $\mathbf{AP}(G)$ and to $\mathbf{C}_0(G)$ are investigated and $\mathbf{B}_I(G)$ is identified for some non-abelian groups, in particular, for the Heisenberg group, for which $\mathbf{B}_I(G)$ is not an algebra.

1. Introduction. Let G be a locally compact abelian group. By *representation* of G, we shall mean a strongly (equivalently, weakly) continuous unitary representation π of G on a Hilbert space H (as in [**7**; §13.1]) The fixed point set of π is

$$H_f = \{\xi \in H : \pi(x)\xi = \xi \text{ for all } x \in G\}.$$

A sequence $\{\mu_n\}$ of probability measures on G is called a *a strong operator ergodic* (s.o. ergodic) sequence or a generalized summing sequence if, for every representation π of G on a Hilbert space H and for every $\xi \in H, \{\pi(\mu_n)\xi\}$ converges in norm to a member of H_f. It is readily seen (via [**10**, §23], for example) that $\{\mu_n\}$ is s.o. ergodic if and only if, for every representation π of G on H, $\pi(\mu_n) \to P$ in the strong operator topology, where P is the orthogonal projection onto H_f.

Blum and Eisenberg [**1**] proved the following interesting.

1980 *Mathematics Subject Classification*: Primary 22D10, 22D40, 43A07, 43A35.
Key words and phrases: Locally compact group, weak or strong operator ergodic sequence, Fourier-Stieltjes algebra, invariant mean.
Research supported in part by NSERC grant A7857. The second author is also indebted to the Royal Society and the Carnegie Trust for the Universities of Scotland for financial assistance.
Received by the editors on April 18, 1986.

THEOREM. *The following are equivalent:*

(i) $\{\mu_n\}$ *is s.o. ergodic.*

(ii) $\hat{\mu}_n(\gamma) \to 0$ *for all* $\gamma \in \hat{G}\backslash\{1\}$.

(iii) $\{\mu_n\}$ *converges weakly to Haar measure on the Bohr compactification of* G.

A natural question to ask is: what happens if G is not assumed to be abelian? To answer this question we consider *weak operator ergodic* (w.o. ergodic) sequences $\{\mu_n\}$ of probability measures on G, for which $\{\pi(\mu_n)\}$ converges in the weak operator topology to P for every representation π of G. In the next section we consider the possibility that w.o. and s.o. ergodicity are equivalent notions, and determine this is the case for groups G if every $\pi \in \hat{G}$ is finite dimensional. However, many groups almost trivially have w.o. ergodic sequences that are not s.o. ergodic. Also, in §2 we present our generalization of the Blum-Eisenberg theorem, (i) and (ii) of which generalize in a natural way. (iii) is more intriguing. One might expect to be dealing with the space $\mathbf{AP}(G)$ of almost periodic functions here, but in fact we must deal with the closed subspace $\mathbf{B}_I(G)$ spanned by the coefficient functions of *all* the irreducible representations of G. (iii) then translates to: $\{\mu_n\}$ converges weakly to the invariant mean on $\mathbf{B}_I(G)$.

Thus $\mathbf{B}_I(G)$ appears as an object worthy of study in its own right. An interesting feature about $\mathbf{B}_I(G)$ is that, because only irreducible representations are involved, we can actually calculate it for some groups whose duals are known. This is in contrast to the cognate space $\mathbf{B}(G)$, the Fourier-Stieltjes algebra, which can be much bigger. However, the algebraic structure of $\mathbf{B}(G)$ contains information of how tensor products of representations "decompose", and this is lacking for $\mathbf{B}_I(G)$. Nonetheless, all of $\hat{G}$ is, in a sense, contained in $\mathbf{B}_I(G)$, so that $\mathbf{B}_I(G)$, though not an algebra in general, should give substantial information about duality for G. In §3 we calculate $\mathbf{B}_I(G)$ for some groups.

2. Ergodic sequences and the space $\mathbf{B}_\mathbf{I}(\mathbf{G})$. It is natural to ask when the two notions of ergodicity for sequences $\{\mu_n\}$ are equivalent. A partial answer is given by the following proposition. Note that the separability condition is not required if G is abelian (and in fact the following proof is a non-abelian version of that given by Blum and

Eisenberg for the implication (ii) $\Rightarrow$ (i) of their theorem). It is also not required if G is compact, since then every representation decomposes into a direct sum of irreducible representations, and an easy argument gives the result.

PROPOSITION 1. *Let G be a separable, locally compact group such that every irreducible representation is finite dimensional. Then every w.o. ergodic sequence is s.o. ergodic.*

PROOF. Let $\{\mu_n\}$ be a w.o. ergodic sequence for G and let π be a representation of G on H^π. Since G is separable, H^π is a direct sum of closed, separable, $\pi(G)$-invariant subspaces; hence we can suppose that H^π is separable. A well-known theorem [**15**; p. 127] then asserts that π can be decomposed into a direct integral

$$\pi = \int_X \pi_x \, d\nu(x)$$

with each $\pi_x \in \hat{G}$. So, $H^\pi = \int_X H^{\pi_x}\, d\nu(x)$ and $\pi(\mu_n) = \int_X \pi_x(\mu_n) d\nu(x)$. Since $\{\mu_n\}$ is w.o. ergodic, $\pi(\mu_n) \to P^\pi$ in the weak operator topology, where P^π is the orthogonal projection of H^π onto H^π_f. Similarly, for each $x \in X, \pi_x(\mu_n) \to P^{\pi_x}$ in the weak operator topology, and hence in the strong operator topology, since π_x is finite dimensional. Now let $\xi \in H_\pi, \xi = \{\xi_x\} \in L_2(\nu, \{H^{\pi_x}\})$. We claim $||(\pi(\mu_n) - P^\pi)\xi||_2 \to 0$. For, since $P^\pi = \{P^\pi_x\}$ (in terms of integral decomposition), we have

$$||(\pi(\mu_n)\xi - P^\pi\xi)(x)||^2 = ||\pi_x(\mu_n)\xi_x - P^{\pi_x}\xi_x||^2 \leq 2||\xi_x||^2,$$

and hence by the Lebesgue dominated convergence theorem,

$$||(\pi(\mu_n) - P^\pi)\xi||^2 = \int_X ||\pi_x(\mu_n)\xi_x - P^{\pi_x}\xi_x||^2 \, d\nu(x) \to 0.$$

Thus $\pi(\mu_n) \to P^\pi$ in the strong operator topology as required. □

We now need the space $\mathbf{B}_I(G)$, the closure in $\mathbf{C}(G)$ of the linear span of the set of coefficient functions of the irreducible representations of G. (Such a coefficient function is of the form $F^\pi_{\xi,\eta}(x) = (\pi(x)\xi, \eta)$ for a $\pi \in \hat{G}$ and $\xi, \eta \in H^\pi$.) Since $\mathbf{B}(G)$ is closed in $\mathbf{C}(G)$ if and only if G is finite [**9**], we do not have $\mathbf{B}_I(G) \subset \mathbf{B}(G)$ in general. (Of course, it is true that $\mathbf{B}_I(G) = (\mathbf{B}_I(G) \cap \mathbf{B}(G))^-$.) Note that $\mathbf{B}_I(G) \supset \mathbf{AP}(G)$, the space of almost periodic functions on G and $\mathbf{B}_I(G) = \mathbf{AP}(G)$ if G is compact or abelian. Also, $\mathbf{B}_I(G)$ is always a translation invariant subspace of $\mathbf{W}(G)$, the space of weakly almost periodic functions on G. There exists a unique invariant mean m on $\mathbf{W}(G)$ and, as well as $\mathbf{W}(G) \supset \mathbf{B}(G)^- \supset \mathbf{AP}(G)$, we have $\mathbf{W}(G) = \mathbf{AP}(G) \oplus \mathbf{W}_0(G)$, where $\mathbf{W}_0(G) = \{f \in \mathbf{W}(G) : m(|f|) = 0\}$. Since $\mathbf{AP}(G) \subset \mathbf{B}_I(G)$, we also have $\mathbf{B}_I(G) = \mathbf{AP}(G) \oplus \mathbf{B}_I^0(G)$, where $\mathbf{B}_I^0(G) = \mathbf{B}_I(G) \cap \mathbf{W}_0(G)$; $\mathbf{B}_I^0(G)$ is the closure in $\mathbf{C}(G)$ of the linear span of the coefficient functions of the infinite dimensional representations in $\hat{G}$. The restriction of m to $\mathbf{B}_I(G)$ gives an invariant mean on $\mathbf{B}_I(G)$ which is unique. (See [**4**] for all these matters.)

For a measure μ on G, the function $\hat{\mu}$ on $\mathbf{C}(G)$ (or any subspace of $\mathbf{C}(G)$ is defined by

$$\hat{\mu}(f) = \int f d\mu.$$

The next result generalizes the Blum-Eisenberg result to the non-abelian (separable) case.

THEOREM 2. *Let G be a separable, locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures on G are equivalent*:

(i) *$\{\mu_n\}$ is a w.o. ergodic sequence .*

(ii) *$\pi(\mu_n) \to 0$ in the weak operator topology for every $\pi \in \hat{G} \backslash \{1\}$.*

(iii) *$\hat{\mu}_n \to m_1$ the weak* topology, where m_1 is the unique invariant mean on $\mathbf{B}_I(G)$.*

PROOF. (i) implies (ii). Let $\pi \in \hat{G}$. Then $\pi(\mu_n) \to P^\pi$ (in our earlier notation) and $P^\pi \in (\pi(G))^c$, the commutant of $\pi(G)$. Since π is irreducible, $P^\pi \in \mathbf{C}I$; since P^π is a projection, $P^\pi = 0$ or I. If $P^\pi = I, H_f^\pi = H^\pi$, and irreducibility tells us that H^π is one-dimensional and that π is the trivial representation 1. So, if $\pi \neq 1$, then $\pi(\mu_n) \to 0$ in the weak operator topology.

(ii) implies (iii). Let $\pi \in \hat{G}$ and let $\xi, \eta \in H^\pi$. Then

$$\mu_n(F^\pi_{\xi,\eta}) = \int F^\pi_{\xi,\eta}(x)d\mu_n(x) = \int(\pi(x)\xi,\eta)d\mu_n(x)$$
$$= (\pi(\mu_n)\xi,\eta) \to (P^\pi\xi,\eta).$$

Since the left translate $L_yF^\pi_{\xi,\eta}$ satisfies

$$L_yF^\pi_{\xi,\eta}(x) = F^\pi_{\xi,\eta}(yx) = (\pi(yx)\xi,\eta) = F^\pi_{\xi,\pi(y^{-1})\eta}(x),$$

it follows that

$$\hat{\mu}_n(L_yF^\pi_{\xi,\eta}) \to (P^\pi\xi,\pi(y^{-1})\eta) = (P^\pi\xi,\eta),$$

i.e., $\hat{\mu}_n \to m_1$.

(iii) implies (i). Let $\pi \in \hat{G}$. By (iii), $\{(\pi(\mu_n)\xi,\eta)\}$ converges for all $\xi,\eta \in H^\pi$, and hence $\pi(\mu_n) \to T$ in the weak operator topology for some $T \in B(H^\pi)$. Clearly $(T\xi,\eta) = m_1(F^\pi_{\xi,\eta})$, and, since m_1 is invariant, we have $\pi(y)T = T = T\pi(y)$, and hence $T = P^\pi$, and is 0 or I. Now argue using disintegration theory (as in the proof of Proposition 1) to obtain that $\pi(\mu_n) \to P^\pi$ in the weak operator topology for all representations π. This gives (i).

We now proceed to some simple properties of $\mathbf{B}_I(G)$, relating this space to $\mathbf{AP}(G)$ and $\mathbf{C}_0(G)$. The next result shows that, under certain circumstances $\mathbf{B}_I(G)$ is an $\mathbf{AP}(G)$-module.

PROPOSITION 3. *Suppose that every finite dimensional $\pi \in \hat{G}$ is one dimensional. Then $\mathbf{AP}(G)\mathbf{B}_I(G) \subset \mathbf{B}_I(G)$.*

PROOF. Let $\pi \in \hat{G}$ and let α be a character of G. Then $\pi \otimes \alpha \in \hat{G}$ (where $\pi \otimes \alpha(x) = \alpha(x)\pi(x) \in B(H^\pi)$). Thus $\alpha F^\pi_{\xi,\eta} \in \mathbf{B}_I(G)$ for all $\xi,\eta \in H$, and hence $\alpha\mathbf{B}_I(G) \subset \mathbf{B}_I(G)$. Since $\mathbf{AP}(G)$ is the closure of the span of the characters of G, the desired result follows. □

COROLLARY. *If G is connected and solvable or is minimally almost periodic, then $\mathbf{AP}(G)\mathbf{B}_I(G) \subset \mathbf{B}_I(G)$.*

PROOF. It is well known that every connected, solvable group satisfies the hypothesis of Proposition 3 [**13**; 29.42]. Also, a minimally almost periodic group satisfies $\mathbf{C}1 = \mathbf{AP}(G)$ by definition.

Recall that simple, non-compact, Lie groups with finite centers are minimally almost periodic [**19, 3**], as is the alternating group on infinitely many symbols.

It is well known and easy to prove that $\mathbf{C}_0(G) \subset \mathbf{W}_0(G)$. The next result gives a condition which ensures that $\mathbf{C}_0(G) \subset \mathbf{B}_I(G)$. We denote by $\mathbf{M}(G)$ the algebra of bounded, regular, Borel measures on G.

PROPOSITION 4. *Suppose G is not compact and let $\phi \subset \hat{G}$. Suppose that $F^{\pi}_{\xi,\eta} \in \mathbf{C}_0(G)$ for all $\pi \in \phi$ and $\xi, \eta \in H^{\pi}$ and that, for each $\mu \in \mathbf{M}(G)$, there exists a $\pi \in \phi$ with $\pi(\mu) \neq 0$. Then $\mathbf{C}_0(G) \subset \mathbf{B}_I(G)$.*

PROOF. Let $A = \overline{\mathrm{sp}}\{F^{\pi}_{\xi,\eta} : \pi \in \phi, \xi, \eta \in H^{\pi}\}$. Then A is a closed subspace of $\mathbf{C}_0(G) \cap \mathbf{B}_I(G)$. If $A \neq \mathbf{C}_0(G)$, then, by the Hahn-Banach theorem, there is a $\mu \in \mathbf{M}(G) = \mathbf{C}_0(G)^*$ such that $\hat{\mu}(A) = 0$ and $\mu \neq 0$. But for all $\pi \in \phi$ and $\xi, \eta \in H^{\pi}$

$$0 = \hat{\mu}(F^{\pi}_{\xi,\eta}) = \int (\pi(x)\xi, \eta) d\mu(x) = (\pi(\mu)\xi, \eta),$$

and hence $\pi(\mu) = 0$ for all $\pi \in \phi$. But this implies $\mu = 0$, and this contradiction shows $A = \mathbf{C}_0(G)$. □

3. The space $\mathbf{B_I(G)}$ for certain locally compact groups. It has been noted already that $\mathbf{B}_I(G) = \mathbf{AP}(G)$ if G is compact or abelian. The next proposition shows it is possible for $\mathbf{B}_I(G)$ to contain the cognate space $\mathbf{B}(G)$; we even have $\mathbf{B}_I(G) = \mathbf{B}(G)^- = \mathbf{W}(G)$.

PROPOSITION 5. *Let G be a non-compact, simple, analytic group with finite centre. Then $\mathbf{B}_I(G) = \mathbf{C}1 \oplus \mathbf{C}_0(G)$.*

PROOF. By [**19**], $\mathbf{B}(G)^- = \mathbf{W}(G) = \mathbf{C}1 \oplus \mathbf{C}_0(G)$. Let $\phi = \hat{G}/\{1\}$. By Proposition 4, it suffices to prove that, for each $\mu \in \mathbf{M}(G)$, there exists a $\pi \in \phi$ such that $\pi(\mu) \neq 0$. Let π_1 be the left regular representation of G. Since G is of type I [**20**; **I**, pp. 313, 336], we have a Plancherel measure ν on $\hat{G}$ and a disintegration

$$\pi_1 = \int_{\hat{G}} n_\pi \pi d\nu(\pi) \text{ on } \int_{\hat{G}} n_\pi H^\pi d\nu(\pi)$$

[**7**; p. 176]. Since G is not amenable, 1 is not in the support of ν [**8**; p. 252], and hence we can write

$$\pi_1 = \int_{\hat{G}\setminus\{1\}} n_\pi \pi d\nu(\pi).$$

Let $\mu \in \mathbf{M}(G)$ and suppose $\pi(\mu) = 0$ for all $\pi \in \hat{G}\setminus\{1\}$. Then, if $f, g \in L_2(G)$, $f = \{f_\pi\}$, $g = \{g_\pi\}$, where $f_\pi, g_\pi \in n_\pi H^\pi$, we have

$$(\pi_1(\mu)f, g) = \int_{\hat{G}\setminus\{1\}} ((\pi \otimes I)(\mu) f_\pi, g_\pi) d\nu(\pi) = 0.$$

But π_1 is faithful on $\mathbf{M}(G)$, and hence $\mu = 0$. The proof is complete. □

NOTES. A little care is needed in the proposition above. There are groups G for which $\pi(\mu) = 0$ for all $\pi \in \hat{G}\setminus\{1\}$, but $\mu \neq 0$, e.g., if μ is Haar measure on a compact group.

If G is as in Proposition 5 and $\{x_n\} \subset G, x_n \to \infty$, then $\{x_n\}$ is a w.o. ergodic sequence, but not a s.o. ergodic one.

PROPOSITION 6. *Let H be the Heisenberg group,*

$$H = \left\{ \begin{pmatrix} 1, & x_3, & x_1 \\ 0, & 1, & x_2 \\ 0, & 0, & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbf{R}, \right\}$$

which, as a manifold, is just $\mathbf{R}^3$. *Let* $\mathbf{AP}_0(\mathbf{R}) = \{f \in \mathbf{AP}(\mathbf{R}) : m(f) = 0\}$, *where m is the invariant mean on* $\mathbf{AP}(\mathbf{R})$. *Then*

$$\mathbf{B}_I(H) = (\mathbf{AP}_0(\mathbf{R}) \overset{\vee}{\otimes} \mathbf{C}_0(\mathbf{R}^2)) \oplus (\mathbf{C} \otimes \mathbf{AP}(\mathbf{R}^2)),$$

where $\vee$ denotes the injective tensor product norm, PROOF. For the representation theory of H, see [**20**]; **I**, p. 442] or [**16**]. The irreducible finite dimensional representations are the characters $U_{a,b},\ a, b, \in \mathbf{R}$, where

$$U_{a,b}(x) = \exp\left(2\pi i(ax_2 + bx_3)\right).$$

It follows that $\mathbf{AP}(H) = \mathbf{C} \otimes \mathbf{AP}(\mathbf{R}^2)$.

To determined $\mathbf{B}_I^0(H)$, we consider the infinite dimensional representations of $\hat{H}$, which are of the form $U_a,\ a \in \mathbf{R}\backslash\{0\}$, acting on $\mathbf{L}_2(\mathbf{R})$, where

$$(U_a(x)f)(t) = \exp(2\pi i(x_1 - x_2 t)a)f(t - x_3).$$

Suppose $a \neq 0$ and let $f, g \in \mathbf{L}_2(\mathbf{R})$. Then, if $\phi = F_{f,g}^{U_a}$, we have

$$\begin{aligned}\phi(x) = (U_a(x)f, g) &= \int_{\mathbf{R}} (U_a(x)f)(t)\overline{g(t)}dt \\ &= \int_{\mathbf{R}} \exp(2\pi i(x_1 - x_2 t)a)f(t - x_3)\overline{g(t)}dt \\ &= \exp(2\pi i x_1 a)K_{f,g}^a(x_2, x_3),\end{aligned}$$

say. We claim $K_{f,g}^a \in \mathbf{C}_0(\mathbf{R}^2)$ and to prove this we may assume f and g are continuous with compact support. It then follows that $\lim_{|x_2|\to\infty} K_{f,g}^a(x_2, x_3) = 0$ for all $x_3 \in \mathbf{R}$ (Riemann-Lebesgue lemma) and $K_{f,g}^a = 0$ if x_3 is outside some bounded interval. These facts, coupled with the continuity of the translation map $s \to L_s f,\ \mathbf{R} \to \mathbf{C}_0(\mathbf{R})$, yield $F_{f,g}^a \in \mathbf{C}_0(\mathbf{R}^2)$.

We now claim that, for fixed $a \neq 0$,

$$\operatorname{sp}\ \{K_{f,g}^a : f, g \in \mathbf{L}_2(\mathbf{R})\}$$

is dense in $\mathbf{C}_0(\mathbf{R}^2)$. For, suppose not. Then there is a $\mu \in \mathbf{M}(\mathbf{R}^2), \mu \neq 0$, with

$$\int K_{f,g}^a d\mu = 0 \qquad (f, g \in \mathbf{L}_2(\mathbf{R}^2)).$$

Hence, for all f, g continuous with compact support, we have

$$\begin{aligned}0 &= \int d\mu(x_2, x_3) \int_{\mathbf{R}} \exp(-2\pi i x_2 t a)f(t - x_3)\overline{g(t)}dt \\ &= \int_{\mathbf{R}} \overline{g(t)}dt \int \exp(-2\pi i x_2 t a)f(t - x_3)d\mu(x_2, x_3),\end{aligned}$$

and since the inner integral is continuous, we must have

$$\int \exp(-2\pi i x_2 t a) f(t - x_3) d\mu(x_2, x_3) = 0 \qquad (t \in \mathbf{R}). \tag{1}$$

Since μ is a finite measure, we can find a sequence $\{f_n\}$ of functions, continuous with compact support, such that, for fixed $t, k \in \mathbf{R}$,

$$\int |f_n(t - x_3) - \exp(-2\pi i k x_3)| d\mu(x_2, x_3) \to 0.$$

It follows from (1) that

$$\int \exp(-2\pi i(t x_2 + k x_3)) d\mu(x_2, x_3) = 0$$

i.e., $\hat{\mu} = 0$ on $\mathbf{R}^2$. Hence $\mu = 0$, a contradiction.

Thus, for $a \neq 0$, $\overline{\text{sp}}\{K^a_{f,g} : f, g \in \mathbf{L}_2(\mathbf{R}^2)\} = \mathbf{C}_0(\mathbf{R}^2)$ and it follows that $\chi_a \otimes \mathbf{C}_0(\mathbf{R}^2) \subset \mathbf{B}_I(H)$, where $\chi_a(x) = \exp(2\pi i a x)$. Since $\mathbf{AP}_0(\mathbf{R}) = \overline{\text{sp}}\{\chi_a : a \neq 0\}$, it follows from the definitions of the norms involved that

$$\overline{\text{sp}}\{F^{U_a}_{f,g} : a \neq 0, f, g \in \mathbf{L}^2(G)\} = \mathbf{AP}_0(\mathbf{R}) \overset{\vee}{\otimes} \mathbf{C}_0(\mathbf{R}^2),$$

which completes the proof. □

NOTE. The explanation of the $\mathbf{AP}_0(\mathbf{R})$ term is to be found in the bad behaviour of the dual of H at 1. 1 is the only point of $\hat{H}$ at which the topology of $\hat{G}$ is not Hausdorff [**20**; II, p. 46]. Also, a tensor product $U_a \otimes U_b$ behaves badly as $a + b \to 0$ [**14**; p. 99].

PROPOSITION 7. *Let G be the "$ax + b$" group. Then*

$$\mathbf{B}_I(G) = \mathbf{C}_0(G) \oplus (\mathbf{AP}(\mathbf{R}) \otimes \mathbf{C}.$$

PROOF. The proof is similar to that of Proposition 6, yet still needs a little work.

The finite dimensional members of $\hat{G}$ are one dimensional and of the form U_r for an $r \in \mathbf{R}$, where $U_r(a,b) = e^{ir\log a}$. There are two infinite dimensional, irreducible representations U^+ and U^-, realized on $\mathbf{L}_2([0,\infty))$ and $\mathbf{L}_2((-\infty,0])$, respectively, given by

$$U^+(a,b)f(t) = \exp(2\pi ibt)\sqrt{a}f(at),$$
$$U^-(a,b)f(t) = \exp(2\pi ibt)\sqrt{a}f(at).$$

(See [**20**; I, p. 440].) Clearly $\mathbf{AP}(G) = \mathbf{AP}(\mathbf{R}) \otimes \mathbf{C}$, where $(0,\infty)$ is identified with $\mathbf{R}$ via log.

Let $f, g \in \mathbf{L}_2(\mathbf{R})$ and let $f^+ = f|^{[0,\infty)}, f^- = f|_{(-\infty,0]}$, and similarly for g^+ and g^-. Then $\mathbf{B}_I^0(G)$ is the closure in $\mathbf{C}(G)$ of the linear span of functions K of the form

$$\begin{aligned} K(a,b) &= (U^+(a,b)f^+, g^+) + (U^-(a,b)f^-, g^-) \\ &= \int_{\mathbf{R}} \exp(2\pi ibt)\sqrt{a}f(at)\overline{g(t)}dt. \end{aligned}$$

Arguing as in the Heisenberg case, one shows that such a K is in $\mathbf{C}_0(\mathbf{R}^2$ and, if $\mu \in \mathbf{M}(G)$ satisfies $\hat{\mu}(K) = 0$ for all such K, one approximates (for fixed s and t) the function $a \to \exp(2\pi ias)$ by $a \to \sqrt{a}f(at)$ to obtain $\hat{\mu}(s,t) = 0$, hence $\mu = 0$. Thus $\mathbf{B}_I^0(G) = \mathbf{C}_0(G)$ as required. □

For the next proposition, the euclidean group of the plane $\mathbf{C} \times \mathbf{T}$ has multiplication

$$(z', w')(z, w) = (z' + w'z, w'w),$$

and provides another example as in Proposition 5 where $\mathbf{B}_I(G)$ contains the cognate space $\mathbf{B}(G)$. Again, we have $\mathbf{B}_I(G) = \mathbf{B}(G)^- = \mathbf{W}(G)$; see [**2**; Theorem 4.8].

PROPOSITION 8. *Let $G = \mathbf{C} \times \mathbf{T}$, the euclidean group of the plane. Then* $\mathbf{B}_I(G) = (\mathbf{C} \otimes \mathbf{AP}(\mathbf{T})) \oplus \mathbf{C}_0(G)$.

PROOF. The finite dimensional irreducible representations of G are the characters, $(z,w) \to w^n$, one for each $n \in \mathbf{Z}$. For each $a > 0$, there is a $U_a \in \hat{G}$ on $\mathbf{L}_2(\mathbf{T})$, where

$$U_a(z,w)f(w_1) = \exp i(z, aw_1)f(\overline{w}w_1),$$

where $(z, aw_1 = \mathrm{Re}(z\overline{aw_1})$ [**18**; p. 159]. Clearly $\mathbf{AP}(G) = \mathbf{C} \otimes \mathbf{AP}(\mathbf{T}) = \mathbf{C} \otimes \mathbf{C}(\mathbf{T})$. Since the characters on $\mathbf{T}$ form an orthonormal basis for $\mathbf{L}_2(\mathbf{T})$, $\mathbf{B}_I^0(G)$ is the closed linear span in $\mathbf{C}(G)$ of functions $K^a_{n,m}$ of the form (with notation $z = r\exp(i\theta)$, $w = \exp(i\phi), w_1 = \exp(i\psi)$)

$$K^a_{n,m}(z,w) = (U_a(z,w)f,g)(2\pi)^{-1} \int_0^{2\pi} \exp(ira \cos (\theta - \phi)) \exp(in(\psi - \phi) - im\psi)d\psi,$$

where $a > 0, n, m \in \mathbf{Z}$ and $f(\psi) = \exp(in\psi), g(\psi) = \exp(im\psi)$. Substituting $k = -n, l = n - m$ and $\gamma = \theta - \psi$, we get

$$K^a_{n,m}(z,w) = -(2\pi)^{-1}\exp i(k\phi + l\theta) \int_0^{2\pi} \exp i(ra \cos \gamma - l\gamma)d\gamma = \mathcal{K}^a_{k,l}(r,\theta,\phi),$$

say. We must show $\overline{\mathrm{sp}}\{\mathcal{K}^a_{k,l} : a > 0, k, l \in \mathbf{Z}\} =$ bf $\mathrm{B}_I^0(G) = C_0(G)$, and to do this we need some results about the Bessel functions $\{J_n : n = 0, 1, 2, \dots\}$. First, replacing θ by $\gamma + \pi/2$ and z by x in the formula for $\exp(iz \sin \theta)$ on page 22 of [**21**], we get

$$\exp(ix \cos \gamma) = J_0(x) - 2(J_2(x) \cos 2\gamma - J_4(x) \cos 4\gamma + \cdots) + 2i(J_1(x) \cos \gamma - J_3(x) \cos 3\gamma + \cdots).$$

Substituting this in the formula for $\mathcal{K}^a_{k,l}$ and integrating, we see that $\mathbf{B}_I(G)$ is the closed linear span of the functions

$$(r,\theta,\phi) \to \exp i(k\phi + l\theta)J_{|l|}(ra) \qquad (k, l \in \mathbf{Z}, a > 0).$$

To show that $\mathbf{B}_I(G) = \mathbf{C}_0(G)$, if suffices, by the Stone-Weierstrass theorem [**17**; p. 124], to prove the following lemma, for which we define J^a_n on $[0, \infty)$ for $n \geq 0$ and $a > 0$ by $J^a_n(x) = J_n(ax)$. □

LEMMA. (i) $\overline{\mathrm{sp}}\{J^a_0 : a > 0\} = \mathbf{C}_0([0,\infty))$.

(ii) For $n \geq 1$, $\overline{\mathrm{sp}}\{J^a_n : a > 0\} = \{f \in \mathbf{C}_0([0,\infty)) : f(0) = 0\} = \mathbf{C}_{00}([0,\infty))$.

PROOF. We note first that the asymptotic expansions [**21**; p. 199] imply that all the $J_n^{\prime a}s$ are in $\mathbf{C}_0([0,\infty))$. We then prove that $\overline{\mathrm{sp}}\{J_n^a : a > 0\} = \mathbf{C}_{00}([0,\infty))$ for $n \geq 1$, the proof of (i) being similar. Now, the power series expansions [**21**; p. 15] imply each $J_n^a \in \mathbf{C}_{00}([0,\infty))$ (for $n \leq 1$). We need the following equality due to Sonine [**21**; p. 394]:

$$\begin{aligned} &\int_0^\infty J_n^a(t)\exp(-p^2a^2)a^{n+1}da \\ &= t^n(2p^2)^{-n-1}\exp(-t^2/(4p^2)) \qquad (t \geq 0, p > 0). \end{aligned} \tag{2}$$

We assert that the map $a \to J_n^a$, $(0,\infty) \to \mathbf{C}_{00}([0,\infty))$. We then get that, for $0 < \varepsilon < R$, the Bochner integral

$$\int_\varepsilon^R J_n^a \exp\,(-p^2a^2)a^{n+1}da = f_{\varepsilon,\mathbf{R}}$$

is a norm limit of linear combinations of members of $\{J_n^a : a > 0\}$, and hence is in $\overline{\mathrm{sp}}\{J_n^a : a > 0\}$. Since $||J_n||_\infty \leq 1$ [**21**; p. 19, (1)], we conclude from elementary inequalities that $f_{\varepsilon,\mathbf{R}}(t)$ converges uniformly in $t \in [0,\infty)$ to the integral in (1), i.e., the function f defined by

$$f(t) = t^n(2p^2)^{-n-1}\exp(-t^2/(4p^2))$$

is in $\overline{\mathrm{sp}}\{J_n^a : a > 0\}$ for all $p > 0$. Hence g, defined by $g(t) = t^n\exp(-kt^2)$, is in $\overline{\mathrm{sp}}\{J_n^a : a > 0\}$ for all $k > 0$.

Our next claim is that $\frac{\partial g}{\partial k} \in \overline{\mathrm{sp}}\{J_n^a : a > 0\}$. To show this, fix $k > 0$ and let $\varepsilon > 0$. Then

$$\begin{aligned} g \in (t) &= (t^n\exp(-(k+\varepsilon)t^2) - t^n\exp(-kt^2))/\varepsilon \\ &= t^n\exp(-kt^2)(\exp(-\varepsilon t^2) - 1)/\varepsilon \end{aligned}$$

defines a $g_\varepsilon \in \overline{\mathrm{sp}}\{J_n^a : a > 0\}$. Further, since the Mean Value Theorem tells us that $e^{-y} - 1 + y \leq 0$ and $1 - e^{-y} - y + y^2/2 \geq 0$ and hence that $0 \leq e^{-y} - 1 + y \leq y^2/2$ for all $y \geq 0$, we get

$$\begin{aligned} g_\varepsilon(t) + t^{n+2}\exp(-kt^2) &= t^n\exp(-kt^2)(\exp(-\varepsilon t^2) - 1 + \varepsilon t^2)/\varepsilon \\ &\leq t^n\exp(-kt^2)(\varepsilon t^2)^2/(2\varepsilon) \\ &= \varepsilon t^n\exp(-kt^2)t^4/2 \end{aligned}$$

for all $t \leq 0$. Hence $||g_\epsilon - h||_\infty \to 0$, where $h(t) = t^{n+2}\exp(-kt^2)$, and $h \in \overline{\text{sp}}\{J_n^a : a > 0\}$. Continuing with this differentiating, we get $h_j^k \in \overline{\text{sp}}\{J_n^a : a > 0\}$, where $j \in \mathbf{N}$ and $k > 0$ and $h_j^k(t) = t^{n+2j}\exp(-kt^2)$.

Now suppose $\overline{\text{sp}}\{J_n^a : a > 0\} \neq \mathbf{C}_{00}([0,\infty))$. Then we can find a $\mu \in \mathbf{M}([0,\infty))$, $\mu \neq 0$, such that $\hat{\mu}(h_j^k) = 0$ for all $j \in \mathbf{N}$ and $k > 0$. Let $\nu \in \mathbf{M}([0,\infty))$ be given by $d\nu = t^n\exp(-t^2)d\mu$. Then, if $f_k^j(t) = t^{2j}\exp(-kt^2)$, we have $\hat{\nu}(f_k^j) = 0$ for all $j \geq 0, k > 0$, But $B = \text{sp}\{f_k^j : j \geq 0, k > 0\}$ is a self-adjoint subalgebra of $\mathbf{C}_0([0,\infty))$ which separates the points of $[0,\infty)$ and does not vanish at any point. It follows that $B^- = \mathbf{C}_0([0,\infty))$, and $\hat{\nu}(B) = \{0\}$, hence $\nu = 0$. Now $\frac{d\nu}{d\mu} = t^n\exp(-t)$, which implies that the support of μ is $\{0\}$. Thus $\mathbf{C}_{00}([0,\infty)) \subset \overline{\text{sp}}\{J_n^a : a > 0\} \subset \mathbf{C}_{00}([0,\infty))$ and the proof is complete.

Concluding remarks. An interesting question is whether there is always a w.o. ergodic sequence for (let's say) a separable locally compact group. Davis [**5**] has shown there is always a summing sequence for $\mathbf{AP}(G)$. According to Greenleaf [**12**], it is an open question if there is always a summing sequence for $\mathbf{B}(G)$ and $\mathbf{B}_I(G)$.)

A.T. Lau has pointed out to us that the Blum-Eisenberg result could be generalized using $\mathbf{A}(G)$ and $\mathbf{B}(G)$, and Fourier and Fourier-Stieltjes algebras, instead of $\mathbf{L}_1(G)$ and $\mathbf{M}(G)$. This approach would be "very abelian" and rather different from ours.

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