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# NONSCATTERED ZERO-DIMENSION REMAINDERS

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ABSTRACT. If a completely regular Hausdorff space $X$ is locally compact, then the maximal compactification $\phi X$ of $X$ having zero-dimensional remainder has the property that $\phi X - X$ is non-scattered if and only if every compact metric space is a remainder of $X$. In this paper characterization of when $\phi X - X$ is nonscattered are presented for the case when $X$ is not locally compact. The results are related to conditions on $R(X)$, the set of points in $X$ which do not possess compact neighborhoods, and they apply in case $X$ is rimcompact so that $\phi X$ is the Freudenthal compactification of $X$.

**1. Introduction.** Substantial attention (for example, see [**2, 3, 6, 7, 13**], etc.) has been devoted to the question of existence and properties of the compactification $\phi X$ of a non-locally compact, completely regular Hausdorff space $X$,, where the remainder $\phi X - X$ is zero-dimensional and $\phi X$ is maximum with respect to this property. In case $X$ is locally compact, $\phi X$ always exists and it follows from [**5**] that $\phi X - X$ is non-scattered if and only if all compact metric spaces are remainders of $X$. The results of [**5**] are extended by Ünlü in [**12**]. The purpose of this paper is to characterize when $\phi X - X$ is non-scattered in case $X$ is not locally compact. If $R(X)$, the set of points in $X$ which do not possess compact neighborhoods (in $X$), is compact, we show that $\phi X - X$ is non-scattered if and only if each compact metric space $M$ is an open subset of the remainder of some compactification $\alpha_M X$, where $\alpha_M X \leq \phi X$ in the lattice of compactifications of $X$. If $R(X)$ is locally compact, then $\phi X - X$ is non-scattered whenever each compact metric space $M$ is a subset of some $\alpha_M X - X$, where $\alpha_M X \leq \phi X$. In both cases $\phi X - X$ is non-scattered whenever each compact metric space $M$ is a subset of some $\alpha_M X - X$, where $\alpha_M X \leq \phi X$. In both cases $\phi X - X$ is non-scattered if and only if $\phi X - X$ contains a compact, non-scattered subset. Also, conditions internal to $X$ are provided which characterize these properties in each case.

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Examples show that, without restrictions on $R(X), \phi X - X$ can be non-scattered without containing a compact, non-scattered subset and that $R(X)$ can have a compact, zero-dimensional, non-scattered remainder when $\phi X - X$ is scattered.

## 2. Preliminaries.

Herein all spaces $X$ are completely regular Hausdorff spaces and compactifications are always Hausdorff. If $\alpha X$ and $\gamma X$ are compactifications of $X$, we say that $\gamma X \leq \alpha X$ if and only if there is a continuous mapping $f$ of $\alpha X$ onto $\gamma X$, where the restriction of $f$ to $X$ is the identity homeomorphism. The mapping $f$ will be called the natural mapping. Recall that $\mathrm{Cl}_{\alpha}(\alpha X - X) = (\alpha X - X) \cup R(X)$ [**6**], for any compactification $\alpha X$. If the remainder $\alpha X - X$ is zero-dimensional, we say that $\alpha X$ is a zero-dimension compactification of $X$. Scattered subsets of any space are defined and studied in [**11**]. It is well known that if $S$ is compact and totally disconnected, then the following are equivalent:

(i) $S$ is non-scattered.

(ii) The Cantor set $C$ is a continuous image of $S$.

(iii) Each compact metric space is a continuous image of $S$.

PROPOSITION 2.1. *If $X$ is locally compact and non-scattered, then $X$ contains a compact, non-scattered subset.*

PROOF. It follows from the definition of a non-scattered set that $X$ must contain a closed, non-scattered subset $S$ with no isolated points. Let $p \in S$. Choose $N$ to be an open neighborhood of $p$ in $X$ having compact closure. Since $S$ has no isolated points, neither does $N \cap S$. But then the closure of $N \cap S$ in $X$ is compact and has no isolated points, thus the proof is complete.

The next result follows from a theorem of Magill [**9**] and 1.2 of [**7**].

PROPOSITION 2.2. *Let $X$ be locally compact and suppose $\alpha X$ is a compactification of $X$. If $f$ is a continuous mapping of $\alpha X - X$ onto a Hausdorff space $K$, then there is a compactification of $\gamma X$ of $X$ with*

*$\gamma X - X$ homeomorphic to $K$ and $\gamma X \leq \alpha X$.*

Following Magill [**8**] we say that a family $\{G_1, \ldots, G_n; K_n\}$ of subsets of $X$ is an $n$-star of $X$ provided:

(i) Each $G_i$ is open.

(ii) $G_i \cap G_j = \phi$, for all $i \neq j$.

(iii) $K_n = X - \cup\{G_i : i = 1, \ldots, n\}$ is compact.

(iv) $K_n \cup G_i$ is non-compact, for each $i = 1, \ldots, n$.

We reserve the notation $\phi X$ for a maximum zero-dimensional compactification of $X$ (when it exists, cf. [**1, 2, 7**], for example). The following results, due to McCartney [**7**], will be utilized in what follows.

PROPOSITION 2.3. *If $\phi X$ exists and $\{G_1, \ldots, G_n; K_n\}$ is any $n$-star of $X$, then $(\phi X - X) \cap \mathrm{Cl}_\phi G_i \cap \mathrm{Cl}_\phi G_j = \phi$, for all $i \neq j$.*

PROPOSITION 2.4. *If $\phi X$ exists and $G$ is an open subset of $X$ with compact boundary, then $\mathrm{Cl}_\phi G$ is a maximum zero-dimensional compactification of $\mathrm{Cl}_X G$.*

**3. Characterization of non-scattered remainders.** During the remainder of this paper we study spaces $X$ for which $\phi X$ exists and, for these spaces, present characterizations of when $\phi X - X$ is non-scattered. If $A$ is a subspace of $X$, we denote the boundary of $A$ in $X$ by $\mathrm{Fr}_X A$.

THEOREM 3.1. *Suppose $\phi X$ exists. Then the following are equivalent.*

(1) *$(\phi X - X) - \mathrm{Cl}_\phi R(X)$ is non-scattered.*

(2) *$(\phi X - X) - \mathrm{Cl}_\phi R(X)$ contains a compact non-scattered subspace.*

(3) *Each compact metric space $M$ is an open subspace of some remainder $\alpha_M X - X$ of $X$ with $\alpha_M \leq \phi X$.*

(4) *Each compact metric space $M$ is an open subspace of some remainder $\alpha_M X - X$ of $X$.*

(5) *There is a 2-star* $\{U, V; F\}$ *of* $X$ *with* $R(X) \subseteq V$ *and the Cantor set is a remainder of* $X - V$.

PROOF. (1) implies (2). This is immediate from 2.1.

(2) implies (3). Let $K$ denote a compact non-scattered subset of $(\phi X - X) - \mathrm{Cl}_{\phi} R(X)$. Since $\phi X - X$ is zero-dimensional, each point of $K$ possesses a compact $\mathrm{Cl}_{\phi}(\phi X - X)$-open neighborhood which misses $\mathrm{Cl}_{\phi} R(X)$. Thus there exists a partition $\{A, B\}$ of $\mathrm{Cl}_{\phi}(\phi X - X)$ into compact $\mathrm{Cl}_{\phi}(\phi X - X)$-open sets with $K \subseteq A$ and $\mathrm{Cl}_{\phi} R(X) \subseteq B$. Let $M$ be any compact metric space. Since $A$ is compact, zero-dimensional and non-scattered there exists a continuous surjection $p$ from $A$ onto $M$. Let $Y = \phi X - A$. Then $Y$ is locally compact and $\phi X = \alpha Y$ is a compactification of $Y$, so by Proposition 2.2, there exists a compactification $\gamma\gamma$ of $\gamma$ such that $\gamma Y \leq \alpha Y$ and $\gamma Y - Y \simeq M$. Note that $\gamma Y$ is a compactification $\gamma X$ of $X$ and $\gamma X \leq \phi X$. Finally, if $f$ is the natural map from $\alpha Y$ to $\gamma Y, f(A) = \gamma Y - Y$ and $f(B) = B$ is compact, so $M \simeq \gamma Y - Y = (\gamma X - X) - B$ is open in $\gamma X - X$.

(3) implies (4). This is obvious.

(4) implies (5). Since the Cantor set $\mathcal{C}$ is a compact metric space, there exists an $\alpha X$ with $\mathcal{C}$ an open subset of $\alpha X - X$. But then $\alpha X - (X \cup \mathcal{C})$ is closed in $\alpha X - X$, so $B = \mathrm{Cl}_{\alpha}(\alpha X - (X \cup \mathcal{C})) = (\alpha X - (X \cup \mathcal{C})) \cup R(X)$. Thus $\mathcal{C}$ and $B$ partition $\mathrm{Cl}_{\alpha}(\alpha X - X)$ into compact $\mathrm{Cl}_{\alpha}(\alpha X - X)$-open sets. Choose $\hat{U}$ and $\hat{V}$ to be disjoint $\alpha X$ open sets with $\mathcal{C} \subseteq \hat{U}$ and $B \subseteq \hat{V}$, and let $U = \hat{U} \cap X$, $V = \hat{V} \cap X$. Then $\mathrm{Cl}_{\alpha}(X - V) \subseteq \alpha X - \hat{V}, \mathrm{Cl}_{\alpha}(V) \cap \mathcal{C} = \phi$ so that $\mathrm{Cl}_{\alpha}(X - V) - (X - V) = \mathcal{C}$. Thus $\{U, V; X - (U \cap V)\}$ is the desired 2-star of $X$.

(5) implies (1). Let $\{U, V; F\}$ be a 2-star of $X$ with the properties in (5). Then Proposition 2.3 shows that $A = \mathrm{Cl}_{\phi}(U \cup F) \cap (\phi X - X)$ and $\mathrm{Cl}_{\phi}(V \cup F) \cap (\phi X - X)$ partition $\phi X - X$ into $(\phi X - X)$-open and closed sets. Since $U \cup F$ is closed in $X$ and misses $R(X)$ it is now also evident that $A \cap \mathrm{Cl}_{\phi} R(X) = \phi$. Also, $\mathrm{Fr}_{X}(U \cup F) \subseteq F$, and so $\mathrm{Fr}_{X}(U \cup F)$is compact. It now follows from Proposition 2.4 that $\mathrm{Cl}_{\phi}(U \cup F)$ is the maximal zero-dimensional compactification of the locally compact $U \cup F$. Since the Cantor set is a remainder of $U \cup F$ it follows that $A$ can be mapped continuously onto the Cantorset. Thus $A$ is non-scattered and hence $(\phi X - X) - \mathrm{Cl}_{\phi} R(X)$ is non-scattered, as desired. □

REMARK 3.2. In light of Theorem 2.1 in [**5**], we observe that the statement $X-V$ has the Cantor set as a remainder can be characterized internally in $X-V$, so that (5)is really a characterization internal to $X$.

In case $R(X)$ is compact, Theorem 3.1 immediately yields the following result.

COROLLARY 3.3. *Suppose $\phi X$ exists and $R(X)$ is compact. Then the following are equivalent.*

(1) *$\phi X - X$ is non-scattered.*

(2) *$\phi X - X$ contains a compact non-scattered subspace.*

(3) *Each compact metric space $M$ is an open subspace of some remainder $\alpha_M X - X$ of $X$, with $\alpha_M X \leq \phi X$.*

(4) *Each compact metric space $M$ is an open subspace of some remainder $\alpha_M X - X$ of $X$.*

(5) *There is a 2-star $\{U, V; F\}$ in $X$ with $R(X) \subseteq V$ and the Cantor set is a remainder of $X_V$.*

In the case of the above corollary, if $\phi X - X$ is non-scattered, then the Cantor set of a remainder of some locally compact closed subspace of X which has empty intersection with $R(X)$. We now consider the case when $R(X)$ is locally compact.

THEOREM 3.4. *Suppose $\phi X$ exists and $R(X)$ is locally compact. Then the following are equivalent.*

(1) *$\phi X - X$ is non-scattered.*

(2) *$\phi X - X$ contains a compact non-scattered subset.*

(3) *Each compact metric space $M$ is contained in some remainder $\alpha_M X - X$ of $X$ with $\alpha_M X \leq \phi X$.*

PROOF. If $(\phi X - X) - \mathrm{Cl}_\phi R(X)$ is non-scattered, then the proof follows immediately from Theorem 3.1. We thus assume $(\phi X - X) - \mathrm{Cl}_{\phi} R(X)$ is scattered and now prove the result for this case.

(1) implies (2). Assume $\phi X - X$ is non-scattered. Then, since $(\phi X - X - \mathrm{Cl}\, R(X)$ is scattered, $\mathrm{Cl}_{\phi} R(X) - R(X)$ is nonscattered and hence is the desired compact subspace.

(2) implies (3). Let $M$ be a compact metric space. Since $\mathrm{Cl}_{\phi} R(X) - R(X)$ is compact, non-scattered and zero-dimensional, there exists a continuous surjection $p$ from $\mathrm{Cl}_{\phi} R(X) - R(X)$ onto $M$. Now $Y = \phi X - (\mathrm{Cl}_{\phi} R(X) - R(X))$ is locally compact, and since $\phi X$ is also a compactification of $Y$, we apply Proposition 2.2 to obtain a compactification $\gamma Y$ of $Y$ such that $\gamma Y - Y \simeq M$ and $\gamma Y \leq \phi X$. It is now evident that $\gamma Y = \gamma X$ is the desired compactification of $X$.

(3) implies (1). Let $\alpha X$ be a compactification of $X$ such that $\alpha X - X$ contains a copy of the Cantor set $\mathcal{C}$ and $\alpha X \leq \phi X$. If $p$ is the natural mapping from $\phi X$ onto $\alpha X$, $p{-}1(\mathcal{C})$ is a compact non-scattered subspace of $\phi X - X$. $\square$

We observe that Theorem 3.4 does not contain a condition analagous to condition (4) of Theorem 3.1 and Corollary 3.3. The following example demonstrates that $\phi X - X$ can be scattered even when $R(X)$ is compact and each compact metric space $M$ is contained in some remainder $\alpha_M X - X$ of $X$.

EXAMPLE 3.5. Let $I$ denote the closed unit interval and $N$ denote the positive integers. Choose spaces $Y$ and $Z$ such that $\beta Y - Y = I$ and $\beta Z - Z = N$. Let $X$ be the free union of $Y$ and $Z$. Then $\beta X$ is the free union of $\beta Y$ and $\beta Z, R(X)$ is compact and $\phi X - X$ is countable. However, since the Cantor set is contained in $I$ it follows from Proposition 2.2 that every compact metric space is contained in some remainder $\alpha_M Y - Y$of $Y$, and hence is contained in some remainder of $X$.

In case $X$ is almost rimcompact and $R(X)$ is locally compact, we now obtain an internal characterization of when $\phi X - X$ is non-scattered. Now $\phi X - X$ is non-scattered if and only if $\phi X - \mathrm{Cl}_{\phi} R(X)$ is non-scattered or $Cl_{\phi} R(X) - R(X)$ is non-scattered. The first possibility is resolved by Theorem 3.1, while the second will be discussed in Theorem 3.9. However, before stating that result we present an example which demonstrates that it is possible for $R(X)$ to be locally compact, $(\phi X - X) - \mathrm{Cl}_\phi R(X)$ to be scattered, and $\mathrm{Cl}_\phi R(X) - R(X)$ to be non-scattered.

EXAMPLE 3.6. Choose $Y$ so that $\beta Y - Y = \beta N$, and take a copy $\beta N_1$ of $\beta N$ with $\beta N_1 \subseteq \beta N - N$ (see 6.10 A of [**4**]). Let $X = Y \cup [\beta N - N - (\beta N_1 - N_1)]$. Now $\beta X = \beta Y = \phi X$ and $\beta X - X = N \cup (\beta N_1 - N_1)$. Thus $\mathrm{Cl}_\phi(\beta X - X) = \beta N$ so that $R(X) = \beta N \cap X = (\beta N - N) - (\beta N_1 - N_1)$is locally compact. Also, $N_1 \subseteq R(X)$ implies $\beta N_1 - N_1 \subseteq \mathrm{Cl}_\phi R(X)$, so that $\mathrm{Cl}_\phi R(X) - R(X)$ is non-scattered and $(\phi X - X) - \mathrm{Cl}\, R(X) = N$ is scattered.

Recall that $X$ is almost rimcompact if and only if $\phi X - X$ has a base of neighborhoods (in $\phi X$) whose boundaries are contained in $X$. (See [**2**]). Since rimcompact spaces are almost rimcompact, the following results apply when $\phi X$ is the Freudenthal compactification of $X$. It is easily seen that the space $X$ in Example 3.6 is rimcompact.

LEMMA 3.7. *Suppose $X$ is almost rimcompact and $A$ and $B$ are disjoint, non-empty compact subsets of $\phi X - X$. Then there are disjoint open sets $U$ and $V$ in $\phi X$ such that $A \subseteq U, B \subseteq V$ and $\{U \cap X, V \cap X; X - (U \cup V)\}$ is a 2-star of $X$.*

PROOF. Since $X$ is almost rimcompact, each point of $A$ is contained in a $\phi X$-open neighborhood $N_a$ with boundary in $X$ such that $N_a \cap B = \phi$. Hence $A$ can be contained in an open set $U$ which misses $B$ and whose boundary is in $X$. Let $V = \phi X - \mathrm{Cl}_\phi U$. Evidently, $B \subseteq V$, and $U \cap X$ and $V \cap X$ are disjoint. Also, $\phi X - (U \cup V) = \mathrm{Fr}_\phi U$ is compact. Since $A \subseteq \mathrm{Cl}_\phi(U \cap X)$ and $B \subseteq \mathrm{Cl}_\phi(V \cap X), (U \cap X) \cup \mathrm{Fr}_\phi U$ and $(V \cap X) \cup \mathrm{Fr}_\phi U$ are noncompact. Thus $\{(U \cap X, V \cap X; X - (U \cup V)\}$ is a 2-star of $X$. □

DEFINITION 3.8. If $A \subseteq X$ and $\{G_1, \ldots, G_n; F_n\}$ is an $n$-star of $X$, then we say $\{G_1, \ldots, G_n; F_n\}$ hereditarily determines an $n$-star of $A$ if $\{G_1 \cap A, \ldots, G_n \cap A; F_n \cap A\}$ is an $n$-star of $A$.

THEOREM 3.9. *Suppose $X$ is almost rimcompact and $R(X)$ is locally compact. Then* $\mathrm{Cl}_{\phi}R(X) - R(X)$ *is non-scattered if and only if there exists a sequence* $\{\mathcal{G}_n : n_e N\}$ *such that, for each positive integer $n$:*

(i) $\mathcal{G}_n = \{G_1^n, \dots, G_{2^n}^n; F_{2^n}\}$ *is a* $2^n$*-star of* $X$.

(ii) $\mathcal{G}_n$ *hereditarily determines a* $2^n$*-star of* $R(X)$.

(iii) $G_{2i-1}^{n+1} \cup G_{2i}^{n+1} \subseteq G_i^n, i = 1, \dots, 2^n$.

PROOF. If $\mathrm{Cl}_{\phi}R(X) - R(X)$ is compact, zero dimensional and non-scattered, it can be mapped continuously onto the Cantor set. For each $n \in N$, let $\mathcal{A}_n = \{A_i^n : i = 1, \dots, 2^n\}$ be a partition of $\mathrm{Cl}_{\phi}R(X) - R(X)$ into compact sets satisfying $A_{2i-1}^{n+1} \cup A_{2i}^{n+1} = A_i^n, i = 1, \dots, 2^n$.

By Lemma 3.7, there is a 2-star $\mathcal{G}_1 = \{G_1^1, G_2^1; F_2\}$ of $X$ such that $A_i^1 \subseteq \mathrm{Cl}_{\phi}G_i^1$, $i = 1, 2$. We note that $R(X) \cap F_2$ is compact and $G_i^1 \cap R(X) \neq \phi$, $i = 1, 2$, so that $\mathcal{G}_1$ hereditarily determines a 2-star of $R(X)$.

For $n = 2$, let $\{U, V; X - (U \cup C)\}$ be 2-star of $X$ such that $A_1^2 \cup A_3^2 \subseteq \mathrm{Cl}_{\phi}U$ and $A_2^2 \cup A_4^2 \subseteq \mathrm{Cl}_{\phi}V$. Now set $G_1^2 = G_1^1 \cap U, G_2^2 = G_1^1 \cap V, G_3^2 = G_2^1 \cap U$ and $G_4^2 = G_2^1 \cap V$. Then $F_4 = X - \cup\{G_i^2 : i = 1, 2, 3, 4\} = F_2 \cup (X - (U \cup V))$ is compact. Clearly $\mathcal{G}_2 = \{G_1^2, G_2^2, G_3^2, G_4^2; F_4\}$ is a 4-star of $X$ which hereditarily determines a 4-star of $R(X)$ and satisfies (iii) of Theorem 3.9. The proof of necessity can now be completed by induction.

Conversely, assume the $\mathcal{G}_n$'s exist. Then, for each positive integer $n$, $\{\mathrm{Cl}_{\phi}G_i^n \cap (\phi X - X) : i = 1, \dots, 2^n\}$ is a partition of $\phi X - X$ into $(\phi X - X)$-open sets. Thus, for each $n$, $\{\mathrm{Cl}_{\phi}G_i^n \cap (\mathrm{Cl}_{\phi}R(X) - R(X)) : i = 1, \dots, 2^n\}$ is a partition of $\mathrm{Cl}_{\phi}R(X) - R(X)$ into $\mathrm{Cl}_{\phi}R(X) - R(X)$-open sets. Also, for each $n \in N, G_{2i-1}^{n+1} \cup G_{2i}^{n+1} \subseteq G_i^n, i = 1, \dots 2^n$; hence

$$\begin{aligned}&(\mathrm{Cl}_{\phi}G_{2i-1}^{n+1} \cup \mathrm{Cl}_{\phi}G_{2i}^{n+1}) \cap (\mathrm{Cl}_{\phi}R(X) - R(X))\\ &= \mathrm{Cl}_{\phi}G_i^n \cap (\mathrm{Cl}\, R(X) - R(X)).\end{aligned}$$

Finally, each $\mathcal{G}_n$ hereditarily determines a $2^n$-star of $R(X)$ so that each $\mathrm{Cl}_{\phi}G_i^n \cap (\mathrm{Cl}_{\phi}R(X) - R(X)$ so that each $\mathrm{Cl}_{\phi}G_i^n \cap (\mathrm{Cl}_{\phi}R(X) - R(X)) \neq \emptyset$. Thus $\mathrm{Cl}_{\phi}R(X) - R(X)$ has a dyadic decomposition, and so can be mapped continuously onto the Cantor set. (See 8.4.4 of **[11]**.) □

**4. Examples and further results.** The following example shows that, for arbitrary $R(X)$, a non-scattered remainder $\phi X - X$ of a rim-compact space $X$ need not contain a compact non-scattered subset. Thus, without restrictions on $R(X)$, we cannot obtain results analogous to Theorems 3.3 and 3.4.

EXAMPLE 4.1. Let $W^*$ be the compact first uncountable ordinal space and $\omega_1$ be the first uncountable ordinal. Take $Y = W^* \times I$, and let $X = Y - \{(\omega_1, r) : r \in I$ and $r$ is irrational). Then $\beta X = \phi X = Y$ and $R(X) = \{(\omega_1, r) : r \in I$ and $r$ is irrational$\}$. Let $\alpha$ be any ordinal satisfying $\alpha < \omega_1$ and let a be an irrational in $I$. Now set $G_1 = X - ((W^* \times [0, a/22]] \cup ([0, \alpha] \times I))$ and $G_2 = X - (W^* \times [a/2, 1]) \cup ([0, \alpha] \times I))$. Then $\{G_1, G_2; X - (G_1 \cup G_2)\}$ is a 2-star of $X$ which hereditarily determines a 2-star of $R(X)$. Similarly, using the points $a/4, a/2, 3a/4$ of $I$, a 4-star of $X$ can be constructed which satisfies the conditions of Theorem 3.9. Now proceeding inductively, a sequence of $2^n$-stars of $X$ can be constructed so that the conditions of Theorem 3.9 hold, $\phi X - X$is non-scattered yet $\phi X - X$ contains no compact non-scattered subset.

By removing the set $(\{\omega_1\} \times \mathcal{C}) \cup \{(\omega_1, r) : r \in I$ and $r$ is rational$\}$ from $Y$, we obtain a space $X$ which satisfies the $2^n$-star condition of Theorem 3.9 and where $\phi X - X$ does contain a compact non-scattered subset.

The following result affords a characterization of when $\phi X - X$ contains a compact non-scattered subset. The proof is similar to that of Theorem 3.4 and is therefore omitted.

THEOREM 4.2. *Assume $\phi X$ exists. The following are then equivalent:*

1) *$\phi X - X$ contains a compact, non-scattered subset.*

2) *Each compact metric space $M$ is a subset of $\alpha_M X - X$, for some compactification $\alpha_M X$ of $X$, where $\alpha_M X \leq \phi X$.*

The next example shows that $\mathcal{C}$ can be a remainder of $R(X)$ when $\phi X - X$ is scattered and $R(X)$ is locally compact. Clearly, $\mathrm{Cl}_{\phi X} R(X) \neq \phi R(X)$ in this case, and $\phi R(X) - R(X)$ non-scattered does not insure that $\phi X - X$ is non-scattered.

EXAMPLE 4.3. Let $X$ be the plane $\mathbf{R}^2$ with the sequences $\{(n, 1k) : k \in N\}$ deleted, for each $n \in N$. Then $R(X) = \{(n, 0) : n \in N\}$. And $\phi X$ ist he one-point compactification of $\mathbf{R}^2$ by 3.10 of [**7**]. Since $R(X)$ is a copy of $N$, $\mathcal{C}$ is a remainder of $R(X)$ (see [**13**, p. 143], for example) but $\phi X - X$ is scattered.

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