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LOCAL ESTIMATES THAT LEAD TO WEIGHTED ESTIMATES FOR OSCILLATING KERNELS IN TWO DIMENSIONS

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ABSTRACT. We prove here in 2 dimensions for the weights $w(x) = (1 + |x|^2)^\alpha$ that

$$||K_{a,b+iy} * f||_{q,w} \leq c(1 + |y|^2)||f||_{q,w}$$

for $2 \leq q \leq \frac{a}{1-b}$ and $0 \leq \alpha \leq bq + a - 2$.

Also we obtain estimates independent of R and v for the expression

$$|\int_R K_{a,b}(t)e^{-it\cdot v}dt|$$

for various rectangles R and all points v in 2 dimensions.

0. Introduction. Throughout we shall suppose that $R = [\alpha, \beta] \times [\gamma, \tau]$, a rectangle with sides parallel to the coordinate axis. Furthermore we set

$$||h|| = \sup_{t\in R} |h(t)|, t = (t_1, t_2) \text{ and } dt = dt_1 dt_2.$$

We wish to analyze the weighted mapping properties of the kernels

$$K_{a,b}(t) = \frac{e^{i|t|^a}}{(1 + |t|^n)^b} \text{ with } b \geq 1 - \frac{a}{2} \text{ and } a > 1. \tag{0.1}$$

Here n in (0.1) is determined by the dimension of t. For the most part this paper is concerned with $n = 2$ dimensions.

We generalize Lemma 4.5 of [**1**] to $n = 2$ dimensions, namely,

THEOREM 1. *Let* $w(x) = (1+|x|^2)^\alpha, 1 < a, 1 - \frac{a}{2} \leq b \leq 1,$ *and* $n = 2$. *Then for* $2 \leq q \leq \frac{a}{1-b}$ *and* $0 \leq \alpha \leq bq + a - 2$,

$$||K_{a,b+iy} * f||_{q,w} \leq c(1 + |?y|^2)||f||_{q,w}.$$

Furthermore, we generalize Lemma 3.1 and Theorem 3.2 of [1] to two dimensions. These results are used to prove in a forthcoming paper with W.B. Jurkat, necessary and sufficient conditions on $v(x)$, which are constant on annuli, so that for $q \geq 2$,

$$\int (1+|x|^2)^{bq+a-2} |K_{a,b} * f|^q \leq c \int v(x)|f|^q dx.$$

The proof of Theorem 1 is based upon showing that for $a > 1$

$$S(R) = |\int_R K_{a,1-\frac{a}{2}}(t) e^{-it\cdot v} dt| \leq B \tag{0.2}$$

for all rectangles parallel to the coordinate axis. The constant B is independent of the point $v = (v_1, v_2)$ and the rectangle R.

In case $a \geq 2$, (0.2) follows immediately from [5]; at the time we also did the cases $1 < a < 2$, but were reluctant to include it in that paper. In all dimensions and for all $a > 0, a \neq 1$, (0,2) was shown by P. Sjölin in [8] in case R is R^n. It is not clear to us how to generalize this result to include all rectangles. Let me add that if we replace rectangles R by spheres then in that case (0.2) becomes unbounded when $n > 2$. To see that consider the case where $K_{2,0}(t) = e^{i|t|^2}$,change to polar coordinates, and consider v near $\overline{0}$ for $n > 2$.

Although the main features of showing (0.2) can be generalized to n-dimensions ($n > 2$), we would prefer to generalize a simpler proof.

And so this paper will be organized as follows. In the first three sections we obtain those results needed to do Theorem 1. In the last section we do the cases $1 < a < 2$ for (0.2).

1. Notation, preliminary estimates, and further discussions. We note that $S(R)$ is defined in (0.2). Although the next result applies in very general situations, we shall state it in such a way that it applies in our case.

PROPOSITION 1.1. If

(i) $R^2 = \cup_m R_m$ with $|R_m \cap R_k| = 0$ for $m \neq k$, and

(ii) $S(R'_m) \leq S(R_m)$ for all $R'_m \subseteq R_m$ and all m, then for all rectangles R

$$S(R) \leq \sum_m (R_m).$$

PROOF. $R = \cup_m (R_M \cap R)$ by (i) and now by (ii) we get the result.

Note that if R_1, R_2 are rectangles with sides parallel to the coordinate axis then so is $R_1 \cap R_2$. And now by Proposition 1.1, if (0.2) holds for a partition of R^2 into rectangles with sides parallel to the coordinate axis and $\sum_m S(R_m) \leq B$, then (0.2) holds for all rectangles with sides parallel to the coordinate axis. Notice also in this special case that (ii) need only hold for rectangles R'_m with sides parallel to the coordinate axis.

Note our rectangles R are denoted as $R = [\alpha, \beta] \times [\gamma, \tau]$. We say our rectangles R satisfy (1.1) if,

$$(1.1) \qquad \begin{cases} (1.1') & 0 < \frac{|\alpha|}{2} \leq |\beta| \leq 2|\alpha| \text{ and } |\gamma| + |\tau| \leq 2|\beta| \text{ or} \\ (1.1'') & 0 < \frac{|\tau|}{2} \leq |\tau| \leq 2|\gamma| \text{ and } |\alpha| + |\beta| \leq 2|\tau|. \end{cases}$$

In fact, if R satisfies (1.1), then it was shown in (1′) on p. 249 of [**5**] that

$$(0.2') \qquad |\int_R K_{a,1-\frac{a}{2}+iy}(t)e^{-it\cdot v}dt| \leq B(1+|y|^2) \sup_{R' \subseteq R} S(R'),$$

where R' is a rectangle with sides parallel to the coordinate axis.

Next suppose that $R^2 = \cup_m R_m$, a partition where R_m satisfies (1.1) for $m = 1, 2, \ldots$. Then in order to show (0.2) (and hence (0.2′)) it suffices to prove that

$$(1.2) \qquad \sum_m S(R_m) \leq B.$$

The idea we use in order to show (0.2) is to rewrite it as

$$S(R) = |\int_R e^{i(\phi(t)-t\cdot v)}(\frac{\partial \phi}{\partial t_i} - v_i) \cdot \frac{f_i(t)}{(1+|t|^2)^{1-\frac{a}{2}}} dt|$$

where $f_i(t) = (v_i - \frac{\partial\phi}{\partial t_i})^{-1}, i = 1,2$ with $\phi(t) = |t|^a$ and $v = (v_1, v_2)$; then we use either Lemma B, Lemma B' or Lemma C in order to separate $\frac{f_i(t)}{(1+|t|^2)^{1-\frac{a}{2}}}$ from the rest of the integrand. Next we integrate out the t_i-variable and hence we are left with a 1-dimensional problem. Our main concern is where $\frac{\partial\phi}{\partial t_i} - v_i = 0$ and in particular where our critical point ρ lies, i.e., that point where $\frac{\partial\phi}{\partial t_i}(\rho) = v_i$ for $i = 1$ and 2. And so we cut up the plane into rectangles R, respecting these critical curves, and we have the option of taking $i = 1$ or $i = 2$.

The variable point $v = (v_1, v_2)$ and we set $f_i(t) = (v_i - \frac{\partial\phi}{\partial t_i})^{-1}, i = 1,2$ where $\phi(t) = |t|^a, a > 0, a \neq 1$.

By $|t| \sim u$ we mean that $c_1 u \leq |t| \leq c_2 u$ for two constants c_1, c_2.

Let $\rho = (\rho_1, \rho_2)$ denote the critical point where $\frac{[\partial\phi}{\partial t_i](\rho)=v_i, i=1,2}$. Also set $\delta = |\rho|^{1-\frac{a}{2}}$ and let $\mu_0 = \frac{3\cdot(2-a)}{(a-1)} + 1$ when $1 < a < 2$.

We let $B_1, B_2, \ldots, c_1, c_2, \ldots,$ stand for positive constants and use the letters B, c generically.

2. A local L^∞-estimate and weighted L^2-estimates.

In this section, (at the slight risk of confusion) let

$$S(R) = |\int_R K_{a,b+iy}(t) e^{-ty\cdot v} dt|. \tag{2.1}$$

Actually, we should attach a, b, y and v to $S(R)$ but find it more convenient not to. In this section and the last section it will be clear what the symbol denotes; in order to avoid confusion, early on it will be made clear whether (0.2) or (2.1) is being used.

Before the main result of this section is stated we need some notation. Let $I_u = [-u, u] \times [-u, u]$ a rectangle in R^2, and set $R(u_1, u_2) = I_{u_2} - I_{u_1}$ with $u_2 > u_1$, which denotes an "annulus". Let $R(u) = R(u, 2u)$.

Next we state a generalization of Lemma 3.1 of **[1]** to two dimensions.

LEMMA 2.1. *Let $a > 1, u \geq 1, n = 2$ and $d = \max(4 \cdot 8^{\frac{a}{2}-1}, 8)$. Set $Q = R(\frac{a}{5}u^{a-1}, adu^{a-1})$ and $J = R^2 - Q$. Then there is a positive constant $c(a,b)$ so that*

$$S(R(u)0 \leq c(1 + |y|^2) u^{2(1-b-(\frac{a}{2}))} \{\chi_Q(v) + (u^2)^{\frac{-a}{4}} \chi_J(v)\}. \tag{2.2}$$

REMARK 2.1. Note that $S(R(u))$ is bounded trivially in case $u \leq 1$, since the integrand is bounded by 1.

We shall begin with an n-dimensional result due to Sjölin, Theorem 1(b) of [**8**]. Here n could be any integer greater than or equal to 1. In case $n = 1$, this can also be found in [**3**].

THEOREM A. *Let* $1 - (\frac{a}{2}) \leq b \leq 1, a > 0$ *and* $a \neq 1$. *Then*

$$\text{If } b < 1, ||K_{a,b+iy} * f||_q \leq c(1 + |y|^n)||f||_q \quad \text{for } \frac{a}{a+b-1} \leq q \leq \frac{a}{1-b}, \tag{2.3}$$

and

$$\text{if } b = 1, ||K_{a,y} * f||_q \leq c(1 + |y|^n)||f||_q \quad \text{for } 1 \leq q < \infty. \tag{2.4}$$

(Here, we set $K_{a,y} \equiv K_{a,1+iy}$ and in case $q = 1$ we mean the real Hardy space H^1 of R^n, see [**2**].)

The next result can be found in [**5**], as well as Lemma 1 of [**4**].

LEMMA B. *Let* $f, \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \frac{\partial^2 f}{\partial t_2 \partial t_1}$ *and* g *be continuous on a rectangle* R. *Furthermore, if* $f, \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}$, *do not change sign in* R *and* f *is real-valued, then for some* $R' \subseteq R$ *and vertex* P *of* R

$$|\int_R fg| \leq c(|f(P)| + \int_R |\frac{\partial^2 f}{\partial t_2 \partial t_1}|)|\int_{R'} g|.$$

Using the supremum notation $|| \cdot ||$ introduced in §0 over R, we get

LEMMA 2.2. *Let* $a \neq 1$ *and* R *satisfy* (1.1) *for* (2.5) *and* (1.1′) *for* (2.6). *Then*

$$S(R) \leq c(1 + y^2)||(1 + t^2)^{-1}||^{b+\frac{a}{2}-1} \text{ if } b \geq 1 - \frac{a}{2} \text{ and} \tag{2.5}$$

$$S(R) \leq c(1 + y^2)||(1 + t^2)^{-b}|| \; ||(\frac{\partial^2 \phi}{\partial t_2^2})^{-1}||^{\frac{1}{2}}. \tag{2.6}$$

$$\cdot [||f_1|| + \int_R |\frac{\partial^2 f_1}{\partial t_2 \partial t_1}|]$$
$$\text{for } |v_1| \leq \frac{a}{5} \inf_R |t|^{a-1} \text{ or } |v_1| \geq 2a||t||^{a-1}.$$

REMARK 2.2. We obtain a corresponding result to (2.6) for $f_2(t)$. In that case, we replace $\frac{\partial^2 \phi}{\partial t_2^2}$ by $\frac{\partial^2 \phi}{\partial t_1^2}$, f_1 by f_2, v_1 by v_2 and (1.1′) by (1.1″).

PROOF. We first show (2.5). Note that

$$S(R) = |\int_R (K_{a,1-\frac{a}{2}+iy}(t)e^{-it\cdot v})(1+|t|^2)^{-b-\frac{a}{2}+1}dt|.$$

Applying the second-mean-value theorem in each of the variables we obtain,

$$S(R) \leq c||(1+|t|^2)^{-b-\frac{a}{2}+1}||\,|\int_{R'} K_{a,1-\frac{a}{2}+iy}(t)e^{-it\cdot v}dt|$$

where $R' \subseteq R$. By (0.2) (or (0.2′)) the result follows. To see (2.6), note that we can suppose that R lies in the first quadrant. For the v_1 in the lemma and with $g = K_{a,b+iy}(t)(v_1 - \frac{\partial \phi}{\partial t_1})e^{-it\cdot v}$, $f = (v_1 - \frac{\partial \phi}{\partial t_1})^{-1} = f_1$ observe that the hypothesis of Lemma B is satisfied.

Now by Lemma B for these v's

$$S(R) \leq c(||f_1|| + \int_R |\frac{\partial^2 f_1}{\partial t_2 \partial t_1}|dt)|\int_{R'} g(t)dt|.$$

By two applications of the second-mean-value theorem $|\int_{R'} g(t)dt| \leq c||(1+|t|^2)^{-b}||\,|\int_{R''} e^{i\phi(t)}e^{-iv\cdot t}(v_1 - \frac{\partial \phi}{\partial t_1})dt|$ where $R'' \subseteq R'$. Next, noting that we have an exact differential in the t_1-variable, and then applying Van der Corput's lemma in the t_2-variable, we get our result.

And now we are in a position to prove Lemma 2.1.

PROOF OF LEMMA 2.1. Because of our earlier remarks it suffices to estimate $S(R)$ where $R = [u, 2u] \times [0, 2u]$.

Case 1. $\frac{a}{5}u^{a-1} \leq |v| \leq adu^{a-1}$:

In case $b \geq 1 - \frac{a}{2}$, use (2.5) of Lemma 2.2. Now for all other choices of b, since for $t \in R, |t| \sim u$, the proof of (2.5) still applies.

Case 2. $|v| \leq \frac{a}{5}u^{a-1}$:

Here with $f(t) = f_1(t)$ (2.6) can be applied. Since for $t \in R, |\frac{\partial^2 \phi}{\partial t_2^2}| \geq cu^{a-2}$ and $|\frac{\partial^2 f_1}{\partial t_1 \partial t_2}| \leq c\frac{t_2}{t_1^{a+2}}$ the result holds for this range of v.

Case 3. $|v| \geq a \cdot d \cdot u^{a-1}$:

This implies either $|v_1| \geq \frac{a \cdot d}{2}u^{a-1}$ or $|v_2| \geq \frac{a \cdot d}{2}u^{a-1}$. Suppose $|v_2| \geq \frac{a \cdot d}{2}u^{a-1}$ (the other case is similar). Then for $t \in R$,

$$|\frac{\partial^2 \phi}{\partial t_1^2}| \geq cu^{a-2} \text{ and } |\frac{\partial^2 f_2}{\partial t_2 \partial t_1}| \leq c\{\frac{t_1 \cdot |t|^{a-4}}{v_2^2} + \frac{t_1 t_2 |t|^{2a-6}}{v_2^3}\}$$

Now using the counterpart to (2.6) explained in Remark 2.2 we obtain our result for this range of v.

This completes the proof of Lemma 2.1.

Set $K_{a,b+iy}(t;u) = K_{a,b+iy}(t)\chi(t \notin I_u)$ (and for $b = 1$ drop the b as in (2.4)). It follows by (2.2) that $||K_{a,b+iy}(\cdot;u) * f||_2 \leq c(1 + |y|^2)(1 + u^2)^{1-b-(a/2)}||f||_2$ for $b \geq 1 - \frac{a}{2}$. In particular, each of these kernels $K_{a,b+iy}(\cdot;;u)$ maps L^2into L^2. In fact, we shall show the following result $||K_{a,b+iy}(\cdot;u) * f||_q \leq c(1+|y|^2)||f||_q$ for $\frac{a}{a+b-1} \leq q \leq \frac{a}{1-b}$ where c is independent of u and y.

In order to see (2.9) we need a concept that first appeared in [**6**], concerning regular kernels.

DEFINITION 2.3. A kernel K is called regular if it can be written as $K(t) = k(t)g(t)$ such that

i) $|g(t)| \leq c|g(x)|$ for $\frac{|x|}{2} \leq |t| \leq 2|x|$,

ii) $\int_{\{|x|>2|t|\}} |k(x-t) - k(x)|\ |g(x)|dx \leq x$ for $t \neq 0$ and

iii) K maps L^2 into L^2 (i.e., $\hat{K} \in L^\infty$).

For $0 < a, a \neq 1$, the kernels $K_{a,y}(t;u)$ are regular for each u, in fact the constants c are independent of u. To see this, set $g(t) = e^{i|t|^a}(1+|t|^2)^{\frac{a}{2}-1-iy}$ and then take $k(t) = (1+|t|^2)^{-a/2}\chi(t \notin I_u)$. We see that (i) and (ii) are easily satisfied, and (iii) follows from (2.8) with $b = 1$. We also need the following result concerning H^1-mapping properties which first appeared in [**3**]. We should add that our applications in the earlier papers were to kernels in 1-dimension but our notions such as regular kernels and their H^1-mapping properties are essentially free of dimension.

THEOREM 2.4. *Let $K = kg$ be a regular kernel. Then*

$$||K * f||_1 \leq c||f||_{H^1} \tag{2.10}$$

if, and only if there is a constant B such that

$$\int_{\{|t|>2|I|\}} |k(t)|\ |g * b(t)|dt \leq B,$$

for all $(1,2)$-atoms b supported in the n-sphere $S(0;|I|)$, centered about the origin.

*Theorem 2.4 is a trivial generalization to the n dimensions of Theorem 1 in [**3**]. Also note that in (2.10) $c \leq B + c(K)$, where $c(K)$ depends only on the constants in the definition of regular kernels and the L^∞-norm of $\hat{K}$.*

To see (2.9) for $a > 1$, because of analytic interpolation (see [**3**]), it suffices to prove that

$$\int_{\{|t|\geq 2|I|\}} |k(t)||g * b(t)|dt \leq B(1+|y|^2) \tag{2.11}$$

where $k(t) = (1+|t|^2)^{-a/2}\chi(t \notin I_u)$ and $K_{a,y}(t;u) = k(t)g(t)$.

First note that by (0.2)

$$||g * f||_2 \leq c(1+|y|^2)||f||_2.$$

Case 1. $|I| \geq 1$.

By Schwarz's inequality

$$\int_{|x|\geq 2|I|} |k(x)||g*b(x)|dx \leq (\int_{|x|\geq 2|I|} |k(x)|^2 dx)^{1/2}||g*b||_2$$
$$\leq c(1+|y|^2)|I|^{1-a}\cdot|I|^{-1} \leq c(1+|y|^2).$$

Case 2. $|I| \leq 1$.

$$\int_{|x|\geq 2|I|} |k(x)||g*b(x)|dx$$
$$= \int_{2|I|\leq|x|\leq 2|I|^{\frac{1}{2(1-a)}}} \cdots dx + \int_{2|I|^{\frac{1}{2(1-a)}}\leq|x|} \cdots dx$$
$$= U_1 + U_2.$$

Note that

$$U_1 = \int_{2|I|\leq|x|\leq 2|I|^{\frac{1}{2(1-a)}}} |k(x)||\int dt(g(x-t)-g(x))b(t)|dx$$

since $\int b = 0$. Hence

$$U_1 \leq \int dt|b(t)|\int_{2|I|\leq|x|\leq 2|I|^{\frac{1}{2(1-a)}}} |k(x)||g(x-t)-g(x)|dx \leq c.$$

Also

$$U_2 \leq (\int_{2|I|^{\frac{1}{2(1-a)}}\leq|x|} |k(x)|^2 dx)^{1/2}||g*b||_2 \leq c(1+|y|^2).$$

Hence we obtain (2.11) and as explained earlier this implies (2.9). The proof where $0 < a < 1$ is similar and will be omitted here.

For the next lemma define

$$K^{(m)}_{a,b+iy}(t) = K_{a,b+iy}(t;2^{m+3}) - K_{a,b+iy}(t;2^{m-2}).$$

LEMMA 2.5. *Let $a > 1, 1-(\frac{a}{2}) \leq b \leq 1$, and $n = 2$. Then,*

$$\sum_{m=0}^{\infty}\int_{E_m}(1+|x|^2)^{a+2b-2}|K^{(m)}_{a,b+iy}*f(x)|^2dx \leq c(1+|y|^2)\int|f|^2x,$$

where $E_m = \{x : 2^m \le |x| \le 2^{m+1}\}$ *for* $m = 0, 1, 2, \ldots$.

PROOF. We notice that

$$I = \sum_{m=0}^{\infty} \int_{E_m} (1+|x|^2)^{a+2b-2} |K_{a,b+iy}^{(m)} * f(x)|^2 dx$$
$$\le c(1+|y|^2) \sum_{m=0}^{\infty} s^{2m(a+2b02)} \int |(K_{a,b+iy}^{(m)})\hat{}(x)|^2 |\hat{f}(x)|^2 dx.$$

By Lemma 2.1

$$I \le c(1+|y|^2) \sum_{m=0}^{\infty} s^{2m(a+2b-2)} 2^{4m(1-b-\frac{a}{2})}$$
$$(\int \chi_{Q(m)} |\hat{f}|^2 + 2^{-ma} \int \chi_{J(m)}(x) |\hat{f}(x)|^2 dx).$$

Since the sets $Q(m)$ have bounded overlaps

$$I \le c(1+|y|^2) \int |f|^2 dx$$

and hence the result.

Now we are in a position to generalize Theorem 3.2 of [**1**] to $n = 2$ dimensions.

THEOREM 2.6. *Let* $a > 1, 1 - (\frac{a}{2}) \le b \le 1, n = 2$ *and* $w(x) = (1+|x|^2)^{\alpha}$, *with* $|\alpha| \le a + 2b - 2$. *Then*

$$||K_{a,b+iy} * f||_{2,w} \le (1+|y|^2)||f||_{2,w}.$$

PROOF. The proof is like that of Theorem 3.2 in [1]. Notice that with

$$
\begin{aligned}
&E_m = \{x : 2^m \le |x| \le 2^{m+1}\},\\
&\int |K_{a,b+iy} * f|^2 w(x)dx\\
&= \int_{|x|\le 1} |K_{a,b+iy} * f|^2 w(x)dx + \sum_{m=0}^{\infty} \int_{E_m} |K_{a,b+iy} * f|^2 w(x)dx\\
&\le c\{\int_{|x|\le 1} |K_{a,b+iy} * f|^2 dx + \sum_{m=0}^{\infty} 2^{2m\alpha}\\
&\quad \int_{E_m} |\int_{|t|\le 2^{m-1}} K_{a,b+iy}(x-t)f(t)|^2 dx\\
&\quad + \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} |\int_{|t|\ge 2^{m-1}} K_{a,b+iy}(x-t)f(t)dtdt|^2 dx\}\\
&= I + II + III.
\end{aligned}
$$

For $\alpha > 0$, by Theorem A. with $q = 2$, we get that

$$
\begin{aligned}
I + III &\le c(1+|y|^2)\{\int_{|x|\le 1} |f|^2 dx + \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{|x|\ge 2^{m-1}} |f|^2 dx\}\\
&\le c(1+|y|^2)\{\int_{|x|\le 1} |f|^2 dx + \int_{|x|\ge \frac{1}{2}} |f|^2 \sum_{m=0}^{\infty} s^{sm\alpha} \chi(|x| \ge 2^{m-1})dx\}\\
&\le c(1+|y|^2)\{\int_{|x|\le 1} |f|^2 dx + \int_{|x|\ge \frac{1}{2}} |f|^2 \sum_{m=0}^{\log(1+|x|)} 2^{2m\alpha} dx\},
\end{aligned}
$$

We next notice that since $2^m \le |x| \le 2^{m+1}$ and $|t| \le 2^{m-1}$ here, we can view $K_{a,b}(x-t)$ as being supported in the annulus $2^{m-1} \le |x-t| \le 2^{m+2}$. Thus the kernel is supported between two squares. Denote this kernel by $K_{a,b}^{(m)}(t)$.

Then

$$
\begin{aligned}
II &\le c \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} \Big| \int_{|t|\le 2^{m-1}} K_{a,b+iy}^{(m)}(x-t) f(t)dt \Big|^2 dx \\
&\le c \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} |K_{a,b+iy}^{(m)} * f|^2 dx \\
&+ c \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} \Big| \int_{|t|\ge 2^{m-1}} K_{a,b+iy}^{(m)}(x-t) f(t)dt \Big|^2 dx \\
&= II_1 + II_2.
\end{aligned}
$$

The term II_2 can be estimated just as III was and so the earlier argument applied because of (2.8). For $\alpha = a + 2n - 2$, use Lemma 2.5 for the term II_1.

This gives the result for $\alpha = a + 2b - 2$, and because of Theorem A with $q = 2$ and $w = 1$, we get by change of measures all weights where $0 \le \alpha \le a + 2n - 2$. The proof is completed by duality.

3. Weighted L^q-estimates. Theorem 1 will be proved in this section. We need a generalization of Proposition 1.9 of [**1**] to n dimensions. But arguing as there one gets

PROPOSITION 3.1. *Let $a > 0, q \ge 2$, and let T be a linear operator satisfying*

(i) $(|x|^n)^{(2-a)/q} |Tf(x)| \le c_1 ||f||_1$, *and*

(ii) $||Tf||_{2,(|x|^n)^{(1-(\frac{2}{q}))(a-2)}} \le c_2 ||f||_2$.

Then, there is a constant, $c \le c_q \max(c_1, c_2)$, such that

$$||Tf||_q \le c||f||_{q,(|x|^n)^{q-2}}.$$

PROOF OF THEOREM 1. Arguing as in Theorem 2.6 (note it suffices to do the case where $\alpha = bq + a - 2$

$$\begin{aligned}
&\int |K_{a,b+iy} * f|^q w(x)dx \\
&\leq c\{\int_{|x|\leq 1} |K_{a,b+iy} * f|^q dx + \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} \\
&|\int_{|t|\leq 2^{m-1}} K_{a,b+iy}(x-t)f(t)dt|^q dx \\
&+ \sum_{m=0}^{\infty} 2^{2m\alpha} \int_{E_m} |\int_{|t|\geq 2^{m-1}} K_{a,b+iy}(x-t)f(t)|^q dx\} \\
&= I + II + III.
\end{aligned}$$

Since $2 \leq q \leq \frac{a}{1-b}$, and because of Theorem A, the arguments in Theorem 2.6 apply to terms I, III. In order to do the II_2 term (i.e., II_2 in Theorem 2.6 to the qth power), appeal to (2.9).

This leaves the term

$$\sum_{m=0}^{\infty} s^{2m\alpha} \int_{E_m} |K^{(m)}_{a,b+iy} * f(x)|^q dx.$$

Consider the linear operator

$$Tf(x) = (1 + |x|^2)^{b+\frac{(a-2)}{q}} \sum_{m=0}^{\infty} \chi_m(x)(K_{a,b+iy} * f)(x).$$

First notice that for $2^m \leq |x| \leq 2^{m+1}$,

$$\begin{aligned}
&(|x|^2)^{\frac{(2-a)}{q}} |Tf(x)| \leq c \int |f| dx, \text{ and} \\
&\int (|x|^2)^{(1-(2/q))(a-2)} |Tf(x)|^2 dx \\
&\leq c \sum_{m=0}^{\infty} \int_{E_m} |K^{(m)}_{a,b+iy} * f|^2 (1 + |x|^2)^{a+2b-2} dx \\
&\leq c(1 + |y|^2) \int |f|^2, \text{ by Lemma 2.5.}
\end{aligned}$$

Now since (i), (ii) of Proposition 3.1 are satisfied with $n = 2$,

$$\int |Tf(x)|^q dx \leq c(1+|y|^2) \int |x|^{2(q-2)} |f|^q dx.$$

Since $q-2 \leq bq+a-2$ for $q \leq \frac{a}{1-b}$, the result follows for $\alpha = bq+a-2$.

4. The proof of (0.2) in case $1 < a < 2$. In this section $S(R)$ is defined in (0.2). Also let $||\cdot||$ be the supremum norm as defined in §0.

LEMMA B$'$. *Let R be in the first quadrant. If f_1 satisfies the hypothesis of Lemma B, $|f_1(t)| \leq \frac{1}{c_1}$ for all $t \in R$, then*

$$S(R) \leq B(1+\alpha^2+\gamma^2)^{(a/2)-1} ||(\frac{\partial^2 \phi}{\partial t_2^2})^{-1}||^{1/2} [\frac{1}{c_1} + \int_R |\frac{\partial^2 f_1}{\partial t_2 \partial t_2}| dt].$$

LEMMA C. *If f_1 satisfies the hypothesis of Lemma B$'$ and $\frac{\partial^2 f_1}{\partial t_2 \partial t_1}$ does not change sign in R (R in the first quadrant), then*

$$S(R) \leq B(1+\alpha^2+\gamma^2_{\frac{(a/2)-1}{c_1}} ||(\frac{\partial^2 \phi}{\partial t_2^2})^{-1}||^{1/2}.$$

Both these lemmas follows from [**4**] and [**5**]. Note that here $1 < a < 2$. Also these lemmas both hold with f_2 in place of f_1 and $\frac{\partial^2 \phi}{\partial t_1^2}$ in place of $\frac{\partial^2 \phi}{\partial t_2^2}$.

Let us discuss our strategy. In order to estimate $S(R)$, we are concerned with when $\phi(t) - t \cdot v$ has zero partials, i.e., $\frac{\partial \phi}{\partial t_i} = v_i, i = 1, 2$. As explained earlier, we can suppose that all of our rectangles lie in one of the quadrants. And so we will begin working in the first quadrant.

There are essentially two types of rectangles, the critical rectangle R where ρ is in the interior of R, and the non-critical rectangles, which are "far" from ρ. There are two cases to worry about: $\rho_1 \geq \rho_2$ and $\rho_2 \geq \rho_1$. But *due to symmetry* we need only concern ourselves with one of these cases. In fact, when $\rho_2 \geq \rho_1$ decompose the first quadrant as shown in figure 1 and when $\rho - 1 \geq \rho_2$decompose it as in figure 2.

Hence we shall suppose throughout that $\rho_2 \geq \rho_1$. In this case, the critical rectangle takes the form

$$R_I = [0, 2^{\mu_0}\rho_2] \times [c\rho_2, 2^{\mu_0}\rho_2],$$

with $c^{a-1} = \frac{1}{2^{2-a}}$. Since the kernel is bounded this rectangle 'disappears' in the case that both $\rho_1, \rho_2 \leq 1$. In the case that only $\rho_1 \leq 1$, replace ρ_1 by 1 in the critical rectangle. Hence we can suppose that both $\rho_1 \geq 1$ and $\rho_2 \geq 1$.

We shall begin by proving that $S(R_I) \leq B$. Following these ideas we need to decompose R_I into rectangles for which we can obtain 'good' lower bounds for $|v_i - \frac{\partial \phi}{\partial t_i}|$ as well as determine either the sign of $\frac{\partial^2 f_i}{\partial t_2 \partial t_1}$ or 'good' upper bounds for $|\frac{\partial^2 f_i}{\partial t_2 \partial t_1}|$. Since these rectangles are contained in R_I, it follows that they satisfy (1.1)

4.1. *Lower bounds for* $|v_i - \frac{\partial \phi}{\partial t_i}$ *in* R_I.

Now $\frac{\partial \phi}{\partial t_i} = v_i$ for $i = 1, 2$, defines t_2 in terms of t_1, denote this function by $y_i(t)$ for $i = 1, 2$ respectively. Note that

$$(4.1.1) \qquad (2-a)\frac{dy_1}{dx} = \frac{(a-1)x^2 + y_1^2}{xy_1}; \; \frac{dy_2}{dx} = \frac{(2-a)xy_2}{(a-1)y_2^2 + x^2}.$$

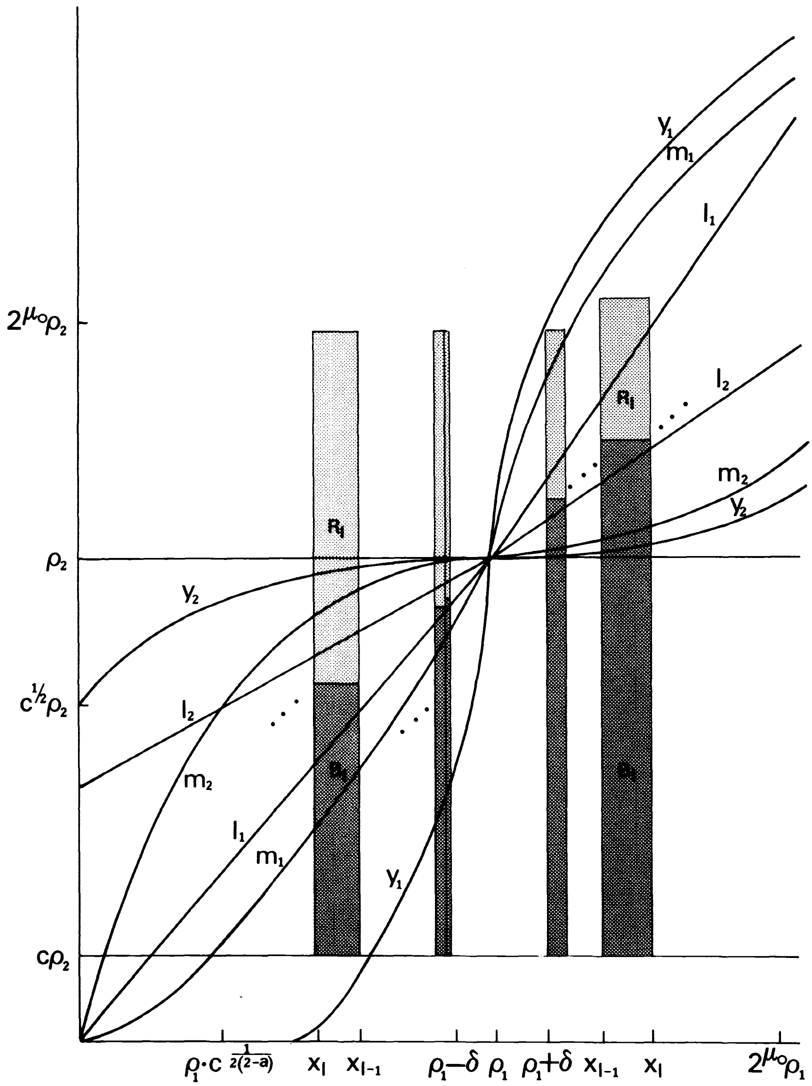
There are two cases, namely when $0 < x < \rho_1$ and$x > \rho_1$. Set $R'_{II} = [0, \rho_1] \times [c\rho_2, 2^{\mu_0}\rho_2], R_{II} = [\rho_1, 2^{\mu_0}\rho_2] \times [cp_2, 2^{\mu_0}\rho_2]$, so that $R_I = R_{II} \cup R', c^{a-1} = \frac{1}{2^{2-a}}$.

Define linear functions as follows,

$$(4.1.2) \qquad \begin{cases} \rho \cdot (\ell_1(x) - \rho_2) = \rho_2 \cdot (x - \rho_1) & \text{if } 0 < x < \rho_1, \\ \rho_1 \cdot (\ell_2(x) - \rho_2) = \lambda \cdot \rho_2 \cdot (x - \rho_1) & \text{if } 0 < x < \rho_1, \\ (2-a)\rho_1(\ell_1(x) - \rho_2) = \rho_2(x - \rho_1) & \text{if } x > \rho_1, \\ \rho_1(\ell_2(x) - \rho_2) = (2-a)\rho_2(x - \rho_1) & \text{if } x > \rho_1. \end{cases}$$

Take $\lambda = \frac{1-c^{1/2}}{1-c^{(1/2(2-a))}}$ and note that $\lambda < 1$. Let h_1, h_2 denote linear functions which are inverse to ℓ_1, ℓ_2 respectively. Using these ideas we obtain the next result.

PROPOSITION 4.1.1. *Let* $(2-a)\frac{m_1'(x)}{m_1(x)} = \frac{1}{x}, \frac{m_2'(x)}{m_2(x)} = \frac{(2-a)}{x}$ *with* $m_i(\rho_1) = \rho_2$ *for* $i = 1, 2$ *and* $\rho_2 \geq \rho_1$. *Then*



$2^{\mu_0}\rho_2$
ρ_2
$c^{1/2}\rho_2$
$c\rho_2$
y_1
m_1
l_1
l_2
m_2
y_2
R_l
B_l
$\rho_1 \cdot c^{\frac{1}{2(2-a)}}$
x_l
x_{l-1}
$\rho_1-\delta$
ρ_1
$\rho_1+\delta$
$2^{\mu_0}\rho_1$

Figure 1

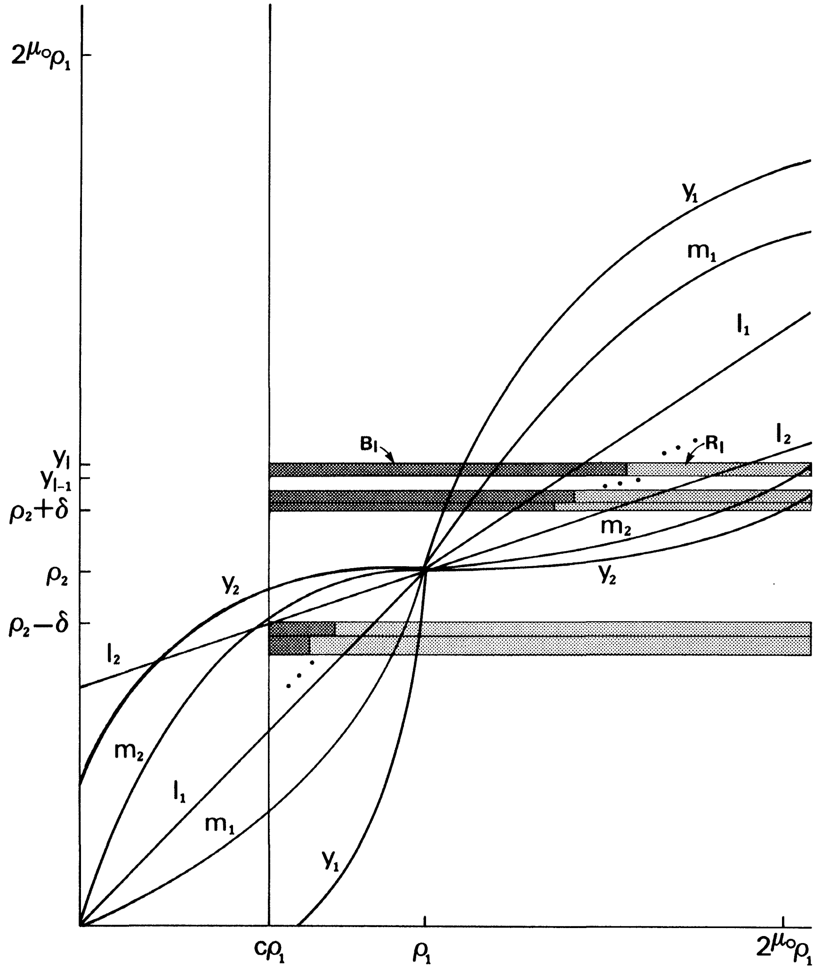


Figure 2

(i) $y_2(x) \le m_2(x) \le \ell_2(x) \le \ell_1(x) \le m_1(x) \le y_1(x)$ if $x > \rho_1$, and
(ii) $y_1(x) \le \ell_1(x) \le \ell_2(x) \le y_2(x)$ if $0 < x < \rho_1$.

PROOF. Start with (i). Form $F_i(x) = \log \frac{y_i(x)}{m_i(x)}$, $i = 1,2$ and note that $F_i(\rho_1) = 0$. Then

$$F_i'(x) = \frac{y_i'(x)}{y_i(x)} - \frac{m_i'(x)}{m_i(x)}.$$

By (4.1.1) $F_1'(x) \geq 0$ while $F_2'(x) \leq 0$. It follows that

$$\begin{aligned} &y_2(x) \leq m_2(x) \text{ and } m_1(x) \leq y_1 9x) \text{ for } x > \rho_1, \text{ and} \\ &y_2(x) \geq m_2(x) \text{ and } y_1(x) \leq m_1(x) \text{ for } 0 < x < \rho_1. \end{aligned}$$

Since $m_1(x) = \rho_2 \cdot (\frac{x}{\rho_1})^{\frac{1}{2-a}}$ and $m_2(x) = \rho_2 \cdot (x/\rho_1)^{2-a}$ on setting $F_i(x) = \ell_i(x) - m_i(x)$ it follows that $m_2(x) \geq \ell_2(x)$ and$m_1(x) \leq \ell_1(x)$ for $x > \rho_1$ and it is clear that $\ell_2(x) \leq \ell_1(x)$. This proves (i).

Next since $m_1(x)$ is concave and $m_2(x)$ is convex $\ell_2(x) \leq m_2(x)$ for $c^{\frac{1}{2(2-a)}} \cdot \rho_1 < x < \rho_1$ while $\ell_1(x) \geq m_1(x)$ for $0 < x < \rho - 1$. Since $\lambda < 1$, for $0 < x < \rho_1$ we get that $\ell_2(x) \geq \ell_1(x)$. Next note that $\ell_2(x) \leq \ell_2(\rho_1 \cdot \frac{1}{c^{2(2-a)}}) = c^{\frac{1}{2}}\rho_2$ for $0 < x < \rho_1 \cdot \frac{1}{c^{2(2-a)}}$ since ℓ_2 is increasing here. While for all $x, ay_2(x)(x^2 + y_2^2(x))^{\frac{a}{2}-1} = v_2$ and for $x = 0$ we get $y_2^{a-1}(0) = \frac{v_2}{a}$. Also $\rho_2(\rho_1^2 + \rho_2^2)^{\frac{a}{2}-1} \geq \frac{\rho_2^{a-1}}{2^{\frac{2-a}{2}}}$ or $y_2(0) \geq \rho_2 \cdot c^{\frac{1}{2}} \geq \ell_2(x)$ for $0 < x < \rho_1 \cdot c^{\frac{1}{2(2-a)}}$. This completes the proof of (ii).

Now decompose the rectangle R_{II} as follows. Let $d = \frac{1}{4}(3 + \frac{1}{(2-a)^2})$, $x_0 = \rho_1 + \delta$, $x_\ell = \rho + \delta \cdot d^\ell$ and $y_\ell^* = \frac{1}{2}(\ell_1(x_{\ell-1}) + \ell_2(x_{\ell-1}))$ for $\ell = 1, 2, \ldots$. Note that $d > 1$. Next define subrectangles of R_{II},

(4.1.3)
$$\begin{cases} R_\ell = & [x_{\ell-1}, x_\ell] \times [yj_\ell, 2^{\mu_0}\rho_2], B_\ell = [x_{\ell-1}, x_\ell] \times [c\rho_2, y_\ell^*] \text{ and} \\ & R_0 = [\rho_1, \rho_1 + \delta] \times [c\rho_2, 2^{\mu_0}\rho_2]. \end{cases}$$

See Figure 1, which is constructed using Proposition 4.1.1 (i). Now

$$R_{II} = \cup_{\ell=0}^{\infty} R_\ell \cup \cup_{\ell=1}^{\infty} B_\ell. \tag{4.1.4}$$

In case of the rectangle R_{II}',take $d = \frac{1+(\frac{\lambda+1}{2\lambda})}{2}$, $x_0 = \rho_1 - \delta$, $x_\ell = \rho_1 - \delta \cdot d^\ell$ and $y_\ell^* = \frac{1}{2}(\ell - 1(x_{\ell-1}) + \ell_2(x_{\ell-1}))$ and define the rectangles R_ℓ', B_ℓ' in a similar fashion as above, keeping in mind that this time $x_\ell < x_{\ell-1}$.

Here we stop the construction till we get to that N whereby $y_N^* \leq c\rho_2$ and take $R'_N = [0, x_{N-1}] \times [c\rho_2, 2^{\mu_0}\rho_2]$ and $R'_0 = [\rho_1 - \delta, \rho_1] \times [c\rho_2, 2^{\mu_0 \rho_2}]$. Then

$$R'_{II} = \cup_{\ell=1}^{N-1} R'_\ell \cup B'_\ell \cup R'_0 \cup R'_0 \cup R'_N. \tag{4.1.5}$$

This time see Figure 1, which is constructed by employing Proposition 4.1.1 (ii). Again notice that here also $d > 1$.

Observe that for fixed $t_2 > 0$ there is a unique $w > 0$ such that $\frac{\partial\phi}{\partial t_1}(w, t_2) = v_1$ and similarly for fixed $t_1 > 0$ there is a unique $w > 0$ such that $\frac{\partial\phi}{\partial t_2}(t_1, w) = v_2$. Furthermore, note that $\frac{\partial\phi}{\partial t_1}(t_1, t_2)$ increases as a function of $t_1$and decreases as a function of t_2. Similarly for $\frac{\partial\phi}{\partial t_2}(t)$. And also for $t \in R_I$ note that

$$\frac{\partial^2\phi}{\partial t_i^2} \geq B\delta^{-2}, \text{ for } i = 1, 2. \tag{4.1.6}$$

It follows that for $t \in R = [\alpha, \beta] \times [\gamma, \tau] \subseteq R_I$,

$$\begin{cases} \frac{\partial\phi}{\partial t_2} - v_2 & \geq \frac{\partial\phi}{\partial t_2}(\beta, \gamma) - \frac{\partial\phi}{\partial t_2}(\beta, w) \\ & \geq B\delta^{-2}(\gamma - w) \text{ if } \gamma \geq w, \\ \frac{\partial\phi}{\partial t_1} - v_1 & \geq \frac{\partial\phi}{\partial t_1}(\alpha, \tau) - \frac{\partial\phi}{\partial t_1}(w, \tau) \\ & \geq B\delta^{-2}(\alpha - w) \text{ if } \alpha \geq w. \end{cases} \tag{4.1.7}$$

We get similar estimates for $v_i - \frac{\partial\phi}{\partial t_i}$. In particular, for $t \in B'_\ell$

$$\begin{aligned} v_2 - \frac{\partial\phi}{\partial t_2} &\geq \frac{\partial\phi}{\partial t_2}(x_\ell, w) - \frac{\partial\phi}{\partial t_2}(x_\ell, y_\ell^*) \geq B\delta^{-2}(w - y_\ell^*) \\ &\geq B\delta^{-2}(\ell_2(x_\ell) - y_\ell^*) \end{aligned} \tag{4.1.8}$$

where the last of this string of inequalities follows from Proposition 4.1.1 (ii) since $w = y_2(x_\ell) \geq \ell_2(x_\ell)$.

Next note that if R lies in one of the quadrants, then $\frac{\partial f_i}{\partial t_1}, \frac{\partial f_i}{\partial t_2}$ do not change signs in R if and only if f_i does not change sign in R.

THEOREM 4.1.2. *If $\rho_2 \geq \rho - 1$, then for $\ell = 1, 2, \ldots$ there is a $d > 1$ such that*

$$(4.1.9) \qquad \begin{cases} \frac{\partial\phi}{\partial t_2} - v_2 \geq B\delta^{-1} d^\ell \frac{\rho_2}{\rho - 1} & \text{for } t \in R_\ell, \\ v_2 - \frac{\partial\phi}{\partial t_2} \geq B\delta^{-1} \frac{\rho_2}{\rho_1} & \text{for } t \in B'_\ell, \\ \frac{\partial\phi}{\partial t_1} - v_1 \geq B\delta^{-1} d^\ell & \text{for } t \in B_\ell, \\ v_1 - \frac{\partial\phi}{\partial t_1} \geq B\delta^{-1} d^\ell & \text{for } t \in R'_\ell, \text{ and} \\ v_1 - \frac{\partial\phi}{\partial t_1} \geq B\rho_1 \cdot \delta^{-2} & \text{for } t \in R'_N. \end{cases}$$

PROOF. Using (4.1.7) throughout the argument, we get for $t \in R_\ell$ that

$$\frac{\partial\phi}{\partial t_2} - v_2 \geq B\delta^{-2}(y^*_\ell - w).$$

But for $t \in R_\ell$ by (i) of Proposition 4.1.1 $w \leq \ell_2(x_\ell)$. Thus

$$y^*_\ell - w \geq y^*_\ell - \ell_2(x_\ell) \geq \delta \cdot \frac{\rho_2}{\rho_1} \cdot d^\ell.$$

For $t \in B_\ell$ note that $w \leq x^*$ with $\ell_1(x^*) = y^*_\ell$ and then argue as above.

In order to argue the case where $t \in R'_N$, note that $y^*_N \leq c\rho_2$ implies that $x_{N-1} \leq \frac{2c-(1-\lambda)}{1+\lambda}\rho_1$, while $w \geq c\rho_1$ and proceed as above. The other cases are similar and will be omitted here.

4.2. *The sign of $\frac{\partial^2 f_i}{\partial t_2 \partial t_1}$ for R_I.* We begin with (also see lemma 4 of [**5**]),

LEMMA 4.2.1. *If $0 \leq v_i \leq \frac{\partial\phi}{\partial t_i}$ and R is in the first quadrant then $\frac{\partial^2}{\partial t_2 \partial t_1}(v_i \pm \frac{\partial\phi}{\partial t_i})^{-1}$ remains one sign in R.*

PROOF. Note that for $i = 2$

$$\frac{\partial^2}{\partial t_2 \partial t_1}(v_2 - \frac{\partial\phi}{\partial t_2})^{-1} =$$

$$a(a-2)f_2^3 \cdot |t|^{a-6} \cdot [(t_1^2 + (a+1)t_2^2)(\frac{\partial\phi}{\partial t_2} - v_2) + 2v_2(t_1^2 + (a-1)t_2^2)].$$

The proofs of all the other cases are similar and will be omitted here.

Using (4.1.9) for $t \in R_\ell \cup B_\ell$ by Lemma 4.2.1 the hypothesis of Lemma C is satisfied by f_i for each $\ell = 1, 2, \ldots$. By Lemma C ($i = 2$ for $R_\ell, i = 1$ for B_ℓ)

(4.2.1) $\quad S(R_\ell \cup B_\ell) \leq B/d^\ell$, where d is defined above (4.1.3).

For the rectangles $R\prime_\ell$, we need the next result.

PROPOSITION 4.2.2. *R is in the first quadrant, $1 < a < 2$.*

(i) *If for each $t \in R_l$ there is an η so that*

$$0 \leq v_2 - \frac{\partial \phi}{\partial t_2} = \frac{\partial^2 \phi}{\partial t_2 \partial t_1}(\eta, t_2) \cdot (w - t_1) \text{ with } 0 \leq w < \eta < t_1,$$

then $\frac{\partial}{\partial t_2 \partial t_1} f_2$ remains one sign for all $t \in R$.

(ii) *If for each $t \in R$, there is an η so that*

$$0 \leq v_2 - \frac{\partial \phi}{\partial t_2} = \frac{\partial^2 \phi}{\partial t_2^2}(t_1, \eta) \cdot (w - t_2), \rho/2 < t_2 < \eta < w \leq \rho_2,$$

then $\frac{\partial^2}{\partial t_2 \partial t_1} f_2$ remains one sign for all $t \in R$. Also a similar result holds for f_1.

PROOF. Using the fact that $x(x^2 + t_2^2)^{(\frac{a}{2})-1} \uparrow$ as a function of x and $\eta < t_1$, we get that

$$\begin{aligned} A &= 2|t|^{a-2}(t_1^2 + (a-1)t_2^2)(-t_1 - w) \cdot (a-2) \cdot \eta(\eta^2 + t_2^2)^{\frac{a}{2}-2}(t_1^2 + (a-3)t_2^2) \\ &\geq |t|^{a-2}(2(t_1^2 + (a-1)t_2^2) - (t_1 - w)(2-a)(3-a)t_1 t_2^2(\eta^2 + t_2^2)^{-1}) \end{aligned}$$

but for $1 < a < 2$ this term is non-negative and hence $A \geq 0$. Note that the sign of $\frac{\partial^2}{\partial t_2 \partial t_1} f_2$ is determined by A because of (i).

To see (ii), since $F(x) = (t_1^2 + (a-1)x^2) \cdot (t_1^2 + x^2)^{\frac{a}{2}-2} \downarrow$ and t_2, η,

$$A = \\ 2t_2|t|^{a-2}(t_1^2 + (a-1)t_2^2) + (w - t_2)(t_1^2 + \eta^2)^{\frac{a}{2}-2}(t_1^2 + (a-1)\eta^2)(t_1^2 + (a-3)t_2^2)$$

and so $A \geq |t|^{a-4}(t_1^2+(a-1)t_2^2)t_2^2((5-a)t_2-(3-a)w)$, but $(3-a)w \leq (3-a)\rho_2 \leq \frac{(5-a)}{2}\rho_2 \leq (5-a)t_2$ since $1 < a < 2$. This gives the result.

REMARK 4.2.1. Let $c^{a-1} = 2^{a-2}, \rho_2 \geq \rho_1, 1 < a < 2$.

(i) If $t_1 > c^{\frac{1}{2}}\rho_1$ then $\frac{\partial^2 f_1}{\partial t_2 \partial t_1}$ stays one sign for $t \in R'_\ell$.

(ii) If $\rho_2/2 \leq t_2$ then $\frac{\partial^2 f_2}{\partial t_2 \partial t_1}$ stays one sign for $t \in B'_\ell$.

To see (i) note that $y_1(t_1) = 0$ implies $v_1 = t_1^{a-1} \cdot a$and so if $t_1^{a-1} \geq \rho_1(\rho_1^2+\rho_2^2)^{\frac{a}{2}-1}$, then there is a w so that $v_1 = \frac{\partial \phi}{\partial t_1}(t_1, w)$. Also, $\rho_1|\rho|^{a-2} \leq \frac{\rho_1^{a-1}}{2^{1-(\frac{a}{2})}}$, while,

$$v_1 - \frac{\partial \phi}{\partial t_1} = \frac{\partial \phi}{\partial t_1}(t_1, w) - \frac{\partial \phi}{\partial t_1}(t_1, t_2) = \frac{\partial^2 \phi}{\partial t_1 \partial t_2}(t_1, \eta) \cdot (w - t_2).$$

Now by (4.1.9) and (i) of Proposition 4.2.2 the result holds.

For (ii) note that for $t \in B'_\ell$ and $t_2 \geq \rho_2/2$

$$0 \leq v_2 - \frac{\partial \phi}{\partial t_2} = \frac{\partial^2 \phi}{\partial t_2^2}(t_1, \eta) \cdot w - t_2) = \frac{\partial \phi}{\partial t_2}(t_1, w) - \frac{\partial \phi}{\partial t_2}(t_1, t_2)$$

and $\frac{\rho_2}{2} \leq t_2 < \eta < w \leq \rho_2$. Now (ii) of Proposition 4.2.2 applies.

4.2. *Upper bounds for $S(R_I)$.* We begin with,

PROPOSITION 4.3.1. *If $\rho_j \geq \rho_i$, then*

$$\int_{|t_i - \rho_i| \leq \delta} dt_i | \in_{\rho_j}^{2_{\rho_j}} \frac{e^{i\phi} e^{iv \cdot t}}{(1+|t|^2)^{1-(\frac{a}{2})}} dt_j | \leq B, \text{ and } i, j, \in \{1, 2, \}.$$

PROOF. Notice that for $|t_i - \rho_i| \leq \delta$ and $\rho_j \geq \rho_i$ that

$$|\int_{\rho_j}^{2_{\rho_j}} \frac{e^{i\phi} e^{-iv \cdot t}}{(1+|t|^2)^{1-(\frac{a}{2})}} dt_j | \leq \frac{B}{(\rho_j^2)^{1-(\frac{a}{2})} \rho_j^{(\frac{a}{2})-1}}$$

by Van der Corput's lemma since $\frac{\partial^2 \phi}{\partial t_j^2} \geq B\rho_j^{a-2}$. This gives the result.

It follows from Proposition 4.3.1 that

$$S(R_0) + S(R_0') \leq B. \tag{4.3.1}$$

PROPOSITION 4.3.2. *Fix $i \in \{1,2,\}$ and take $R = R_{II}'$ for $i = 1$ and $R = R_I$ in case $i = 2$. If $v_i - \frac{\partial \phi}{\partial t_i} \geq B\rho_i \cdot \delta^{-2}$ for $t \in R$, then*

$$S(R) \leq B.$$

PROOF. In case $i = 2$,

$$|\frac{\partial^2 f_2}{\partial t_2 \partial t_1}| \leq \frac{B}{\rho_2^{a+1}} \text{ and by}$$

Lemma B$'$, the result follows.

In case $i = 1$, if $\rho_1 \leq \rho_2^{1-\frac{a}{2}}$, then $\rho_1 \leq \delta$ and since $R \subseteq R_{II}'$, then by Proposition 4.3.1 with $i = 1$ and $j = 2$

$$S(R) \leq B.$$

Otherwise $\rho_1 \geq \rho_2^{1-\frac{a}{2}}$ and since $\rho_2 \geq \rho_1$ this implies

$$|\frac{\partial^2}{\partial t_2 \partial t_1} f_1| \leq \frac{B}{\rho_2^{a-1}\rho_1^2}.$$

And now by Lemma B$'$

$$S(R) \leq \frac{B\rho_2^{1-\frac{a}{2}}}{\rho_1} \leq B$$

and hence the result is true.

THEOREM 4.3.3. *If $\rho_2 \geq \rho_1$, then $S(R_I) \leq B$.*

PROOF. By (4.2.1)

$$\sum_{\ell=1}^{\infty} S(R_\ell \cup B_\ell) \leq B \text{ and by (4.3.1) } S(R_0) \leq B.$$

By (4.1.4) $S(R_{II}) \le B$. It remains to show that $S(R'_{II}) \le B$.

By remark 4.2.1 if $t \in R'_\ell$ and $t_1 > c^{1/2}\rho_1$ then f_1 satisfies Lemma C. And by Lemma C and (4.1.9) there is a d¿1 so that $S(R'_\ell) \le B/d^\ell$, hence

$$\sum_{\ell \in A_1} S(R'_\ell) \le B, \tag{4.3.2}$$

$A_1 = \{\ell : t \in R'_\ell, t_1 > c^{1/2}\rho - 1\}$.

Now in case $t \in R'_\ell$ and $\ell \notin A_1$ then $t_1 < c^{1/2}\rho_1$ and this implies there is at most a finite number of ℓ's so that $x_\ell \le c^{1/2}\rho_1$, say M. M only depends on a. Thus,

$$\cup R'_\ell \subseteq \cup_{\ell \in A_1} R'_\ell \cup \cup_{\ell \notin A_1} R'_\ell \cup R'_N.$$

Then

$$S(\cup R'_\ell) \le \sum_{\ell \in A_1} S(R') + \sum_{\ell \notin A_1} S(R'_\ell) + S(R'_N). \tag{4.3.3}$$

For $\ell \notin A_1$ since $\delta \cdot d^\ell \ge (1 - c^{1/2})\rho_1$ we get by (4.1.9) and Proposition 4.3.2 that

$$S(R'_N) + \sum_{\ell \notin A_1} S(R'_\ell) \le M \cdot B$$

where M is the number of terms.

By (4.3.2) and (4.3.3)

$$S(\cup R'_\ell) \le B. \tag{4.3.4}$$

Next let $A_2 = \{\ell : y^*_\ell \ge \frac{3}{4}\rho_2\}$. First note that

$$B'_\ell = [x_\ell, x_{\ell-1}] \times [\rho_2/2, y^*] \cup [x_\ell, x_{\ell-1}] \times [c\rho_2, \rho_2/2] = B''_\ell \cup B'''_\ell,$$

where $B'_\ell = B''_\ell x_\ell, x_{\ell-1} \times [c\rho_2, y^*_\ell]$ in case $y^*_\ell < \rho_2/2$. By Remark 4.2.1 and Lemma C, (4.1.9), for f_2 and $t \in B''_\ell$ there is a $d > 1$ such that $S(B''_\ell) \le B/d^\ell$ and hence

$$\sum S(B''_\ell) \le B. \tag{4.3.5}$$

Arguing as in (4.1.8) it follows that for $t \in B_\ell'''$

$$v_2 - \frac{\partial \phi}{\partial t_2} \geq B\delta^{-2}(\ell_2(x_\ell) - \frac{\rho_2}{2}).$$

For $\ell \in A_2$ and $t \in B_\ell'''$ since $\ell_2(x_\ell) \geq y_\ell^* \geq \frac{3}{4}\rho_2$

$$v_2 - \frac{\partial \phi}{\partial t_2} \geq B\delta^{-2}\rho_2. \tag{4.3.6}$$

If $\ell \notin A_2$ then $y_\ell^* \leq \frac{3}{4}\rho_2$ and thus $\delta d^\ell \geq B\rho - 1$. This implies there is at most a finite number, say M, of rectangles such that $\ell \notin A_2$. By (4.1.9)

$$v_2 - \frac{\partial \phi}{\partial t_2} \geq B\delta^{-2}\rho_2 \text{ for } t \in B_\ell''') + \sum_{\ell \notin A_2} S)B_\ell''' \subseteq B_\ell', \ell \notin A_2. \tag{4.3.7}$$

By (4.3.6), (4.3.7) and Proposition 4.3.2

$$\sum_{\ell \in A_2} S(B_\ell''') + \sum_{\ell \notin A_2} S(B_\ell''') = S(\cup_{\ell \in A_2} B_\ell''') + M \cdot B \leq B,$$

where M is the number of terms in the second sum. Now by (4.3.5)

$$\sum S(B_\ell' \leq B.$$

(4.3.1) and (4.3.4) yield the result.

4.4. *Estimates of $S(R)$ for the remaining R.* We shall begin with the next set of results.

LEMMA 4.4.1. *Let $R = [k, 2k] \times [\ell, 2\ell], \tau = \max(k, \ell)$ and f_i satisfy Lemma B for $t \in R_j$ where $i = 1$ when $k \geq \ell$ and $i = 2$ when $\ell \geq k$. If for $k \geq \ell, |v_1 \pm \frac{\partial \phi}{\partial t_1}| \geq Bk^{a-1}$ and if for $\ell \geq k$ $|\frac{\partial \phi}{\partial t_2} \pm v_2| > B\ell^{a-1}$, then*

$$S(R) \leq B\tau^{-a/2}.$$

PROOF. Let $\eta = \min(k,\ell)$. Then

$$|\frac{\partial^2}{\partial t_2 \partial t_1} f_i| \leq \frac{B \cdot \eta}{\tau^{a+2}} \text{ and hence}$$

by Lemma B′

$$S(R) \leq B \frac{(1+\eta^2+\tau^2)^{(a/2)-1}}{(\eta^2+\tau^2)^{\frac{1}{2}((\frac{a}{2})-1)}} [\frac{1}{\tau^{a-1}} + \frac{\eta^2 \cdot \tau}{\tau^{a+2}}].$$

The result follows

As an immediate consequence of Lemma 4.4.1 we get

PROPOSITION 4.4.2. *Let $\rho_2 > 0, m \geq 1, a$ dn $R_\ell = [k,2k] \times [2^\ell \rho_2, 2^{\ell+1}\rho_2], \ell = 0,1,2,\ldots$. If $R = \cup_{\ell=0}^{\infty} R_\ell$ and the hypothesis of Lemma 4.4.1 is satisfied for each R_ℓ, then*

$$S(R) \leq B \begin{cases} \frac{m}{(2^m \rho_2)^{a/2}}, & \text{if } 2^m \rho_2 \leq k \leq 2^{m+1}\rho_2, \text{ and} \\ \rho_2^{-a/2}, & \text{if } k \leq 2\rho_2. \end{cases}$$

REMARK 4.4.1. Note Proposition 4.4.2 handles the case $[k,2k] \times [1,\infty)$, i.e., $\rho_2 = 1$ and also with a similar hypothesis it handles the case where $R = [\rho_1,\infty) \times [\ell, 2\ell]$.

THEOREM 4.4.3. *The first quadrant can be decomposed $[0,\infty) \times [0,\infty) = \cup_m R_m$, so that the hypothesis of Proposition 1.1 is satisfied, each R_m satisfies (1.1) and $\sum_m S(R_m) \leq B$.*

PROOF. To complete the first quadrant, we begin by obtaining estimates of $|v_i - \frac{\partial \phi}{\partial t_i}|$. Recall that $\mu_0(a-1) \geq 3(2-a)$, $1 \leq \rho_1 \leq \rho_2$ and the increasing and decreasing properties of $\frac{\partial \phi}{\partial t_i}$.

Set, (with $c^{a-1} = 2^{a-2}$)

$$U_0 = S_1 \cup S_2 \cup \cup_{k=\mu_0}^{\infty} R_k, U_\ell = \cup_{k=\mu_0}^{\infty} R_{k,\ell}, \ell = 1,2,3,$$

where, $S_1 = [0,\rho_1]\times[0,c\rho_2]$, $S_2 = [\rho_1, 2^{\mu_0}\rho_2]\times[0,c\rho_2]$, $R_k = [2^k\rho_2, 2^{k+1}\rho_2]$ $\times[0,c\rho_2]$, $R_{k,1} = [2^k\rho_2, 2^{k+1}\rho_2]\times[c\rho_2, 2^{\mu_0}\rho_2]$, ${}_{k,2} = [0,\rho_1]\times[2^k\rho_2, 2^{k+1}\rho_2]$, and $R_{k,3} = [\rho_1, 2^{\mu_0}\rho_2]\times[2^k\rho_2, 2^{k+1}\rho_2]$. We get that

$$|v_2 - \frac{\partial\phi}{\partial t_2}| \geq B\begin{cases} \rho_2^{a-1} & \text{for } t \in S_1 \cup S_2, \text{ and} \\ (2^k\rho_2)^{a-1} & \text{for } t \in R_{k,2} \cup R_{k,3}, \end{cases} \tag{4.4.1}$$

and

$$\frac{\partial\phi}{\partial t_1} - v_1 \geq B(2^k\rho_2)^{a-1} \text{ for } t \in R_k \cup R_{k,1}. \tag{4.4.2}$$

Since $\rho_2 \geq \rho_1$ this implies $v_2 \geq \frac{a}{2^{1-\frac{a}{2}}}\rho_2^{a-1}$ and for $t \in S_1$ $\frac{\partial\phi}{\partial t_2} \leq \frac{\partial\phi}{\partial t_2}(0, c\rho_2)$. Hence (4.4.1) is true fort $\in S_1$. For $t \in S_2 \frac{\partial\phi}{\partial t_2} \leq \frac{\partial\phi}{\partial t_2}(\rho - 1, c\rho_2)$ and (4.4.1) holds for $t \in S_2$. The other inequalities in (4.4.1) and (4.4.2) follow in a similar fashion.

Now to complete the first quadrant set $U_4 = \cup_{\ell=\mu_0}^{\infty} \cup_{k=\mu_0}^{\infty} R_{\ell,k}$ where $R_{\ell,k} = [2^\ell\rho_2, 2^{\ell+1}\rho_2] \times [2^k\rho_2, 2^{k+1}\rho_2]$.

If $\ell \geq k$, then

$$\frac{\partial\phi}{\partial t_1} - v_1 \geq \frac{a(2^\ell\rho_2)^{a-1}}{5^{1-(\frac{a}{2})}} - a\rho_1^{a-1} \geq B(2^\ell\rho_2)^{a-1} \text{ for } t \in R_{\ell,k}, \tag{4.4.3}$$

while for $k \geq \ell$

$$\frac{\partial\phi}{\partial t_2} - v_2 \geq B(2^k\rho_2)^{a-1} \text{ for } t \in R_{\ell,k}. \tag{4.4.4}$$

Now by (4.4.1-4) and Proposition 4.4.2

$$\sum_{i=0}^{4} S(U_i) \leq B.$$

Putting these results together with Theorem 4.3.3 gives the result.

We are in a position to show,

THEOREM 4.4.4. *Let* $1 < a < 2$. *There is a decomposition of the plane* $R^2 = \cup_m R_m$ *that satisfies the hypothesis of Proposition 1.1 where each* R_m *satisfies (1.1) and*

$$\sum_m S(R_m) \leq B.$$

PROOF. If R is in the fourth quadrant, we can shift everything to the first quadrant by changing variables. This time we need to estimate $v_2 + \frac{\partial\phi}{\partial t_2}$ and $|v_1 - \frac{\partial\phi}{\partial t_1}|$ from below. We get that all the non-critical rectangles will go as before.

The only cases that are not clear are for the critical rectangles $R_I = [0, 2^{\mu_0}\rho_2] \times [c\rho_2, 2^{\mu_0}\rho_2]$ with $\rho_2 \geq \rho_1$ and $R_I' = [c\rho_1, 2^{\mu_0}\rho_1] \times [0, 2^{\mu_0}\rho_1]$ with $\rho - 1 \geq \rho_2$. In case $\rho_2 \geq \rho - 1$ since $v_2 + \frac{\partial\phi}{\partial t_2} \geq B\rho_2|\rho|^{a-2}$ by Proposition 4.3.2

$$S(R_I) \leq B.$$

In case $\rho_1 \geq \rho_2$, set

$$\begin{aligned} R_I' &= [c\rho_1, 2^{\mu_0}\rho_1] \times [0, \delta] \cup \cup_k R_k = R \cup \cup_k R_k \\ R_k &= [c\rho_1, 2^{\mu_0}\rho - 1] \times [2^k\delta, 2^{k+1}\delta] \text{ and } 2^{k+1}\delta \leq 2^{\mu_0}\rho_1. \end{aligned}$$

Then

$$\frac{\partial\phi}{\partial t_2} + v_2 \geq B(2^k\delta)\rho_1^{a-2} \text{ and } |\frac{\partial^2}{\partial t_2 \partial t_1} f_s| \leq \frac{B}{(2^k\delta)^2 \rho_1^{a-1}} \text{ for } t \in R_k.$$

Using Lemma B', since $2^{k+1}\partial \leq 2^{\mu_0}\rho_1$ for all k, we get

$$\sum_k S(R_k) \leq B \sum_k \frac{\rho_1^{a-\frac{a}{2}}}{2^k\delta} \leq B.$$

By Proposition 4.3.1 $S(R) \leq B$. Since all other quadrants can be dealt with in a similar fashion the theorem follows.

COROLLARY 4.4.5. *Let* $a > 1, a \neq 1$. *Then for any rectangle* R *with sides parallel to the coordinate axis*

$$|\int_R K_{a,1-(\frac{a}{2})+iy}(t)e^{-it\cdot v}dt| \leq B(1 + |y|^2),$$

where B is a positive constant independent of R, y and v.

PROOF. In case $a \geq 2$, we refer the reader to [**5**] and in case $1 < a < 2$ use Theorem 4.4.4. Now apply Proposition (1.1) and (0.2′).

REFERENCES

1. S. Chanillo, D.S. Kurtz, and G. Sampson, *Weighted L^p estimates for escillatingkernels,* Arkiv för Mat., **21**(1983), 233-257.

2. R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis,* Bull. of Amer. Math. Soc. **83**(1977), 569-545.

3. W.B. Jurkat and G. Sampson, *The complete solution to the (L^p, L^q) mapping problem for a class of oscillating kernels,* Ind. U. Math. J. **30** (1981), 403-413.

4. G. Sampson, *A note on oscillating kernels in Two Dimensions,* J. Math. Anal. Appl. **93** (1983), 418-435.

5. ———, *Fourier transform estimates for oscillating kernels,* Springer-Verlag series, Lecture Notes 908, *Harmonic Analysis,* Minneapolis 1981, 247-273.

6. ———, *Oscillating kernels that map H^1 into L^1,* Ark. förMat. **18** (1980), 125-144.

7. P. Sjölin, *Convolution with oscillating kernels,* Ind. U. Math. J. **30** (1981), 47-55.

8. ———, *Convolution with oscillating kernels on H^p spaces,* J. London Math. Soc. (2) **23** (1981), 442-454.

9. A. Zygmund, *Trigonometric Series,* 2nd Ed., vols. 1 and 2, Cambridge U. Press, New York, 1959.

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