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ABSENCE OF EIGENVALUES OF THE ACOUSTIC PROPAGATOR IN DEFORMED WAVE GUIDES

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ABSTRACT. We prove that the acoustic propagator for deformed wave guides has no positive eigenvalues.

Introduction. The propagation of acoustic waves in a deformed wave guide with speed of propagation $c(x,y)$ is described by the equation

$$\frac{\partial^2 u}{\partial^2 t} - c^2(x,y)\Delta u = 0, \tag{1.1}$$

where $u(x,y,t)$ is a real valued function of $x \in \mathbf{R}^n, y \in \mathbf{R}, t \in \mathbf{R}$, where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial^2 x_i} + \frac{\partial^2}{\partial^2 y}, \tag{1.2}$$

and where $c(x,y)$ is a measurable real valued function of $\mathbf{R}^{n+1}$ that satisfies

$$0 < c_1 \leq c(x,y) \leq c_2, \tag{1.3}$$

for a.e., (x,y), and c_1, c_2 positive constants.

The deformed wave guide is a perturbation of a perfect wave guide whose velocity profile, $c_0(y)$, is a measurable real valued function of y only, and satisfies (1.3). The corresponding wave equation is

$$\frac{\partial^2}{\partial^2 t} u - c_0^2(y)\Delta u = 0. \tag{1.4}$$

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Let $\mathbf{H}$ be the Hilbert space consisting of the Lebesgue space $L^2(\mathbf{R}^{n+1})$ endowed with the scalar product

$$(1.5)\qquad (f,g)_{\mathbf{H}} = \int_{\mathbf{R}^{n+1}} f(x,y)\bar{g}(x,y)c^{-2}(x,y)dxdy,$$

for $f, g \in \mathbf{H}$.

The acoustic propagator, A, is the selfadjoint operator in $\mathbf{H}$ defined by

$$(1.6)\qquad Af = -c^2(x,y)\Delta f,$$

$$(1.7)\qquad D(A) = \{f \in \mathbf{H} : \Delta f \in \mathbf{H}\},$$

where the Laplacian is taken in distribution sense. $D(A)$ consists of the Sobolev space $H_2(\mathbf{R}^{n+1})$.

The acoustic propagator plays a fundamental role in the spectral and scattering theory for the pair of equations (1.1), (1.4). In [**7, 8**] the limiting absorption principle was proven and the scattering theory was developed. In [**8**] transmission problems and exterior domains were also considered. In [**9**] the limiting absorption principle is proven at thresholds (cutoff frequencies) and between spaces with radial weights.

In [**10**] and [**11**] the same results are obtained for three dimensional wave guides, for the vector Maxwell equations.

In this paper we prove the absence of positive eigenvalues of the acoustic propagator.

Let Ω be a connected exterior domain, i.e., a connected set that is the complement of a compact set.

In what follows the functions $c(x,y)c_0(y)$, are only defined in Ω.

THEOREM I. *Suppose that $c_0(y)$ is measurable, satisfies (1.3) in Ω, and that*

$$(1.8)\qquad c_0(y) = c_+, \quad y \geq h_+,$$

$$(1.9)\qquad |c_0(y) - c_-| \leq C(1+y^2)^{\frac{-1-\varepsilon}{2}}, \quad y \leq h_-,$$

for some positive constants, c_+, c_-, h_+, h_-*, and* $C, \varepsilon > 0$.

We assume that $c(x,y)$ *is measurable, satisfies (1.3) in* Ω*, and*

$$c(x,y) - c_0(y) = 0, \tag{1.10}$$

for a.e., $y > y_0$*, some* y_0*, and*

$$|c(x,y) - c_0(y)| \leq Ce^{-\alpha|x|} \tag{1.11}$$

in Ω *for some positive constants* C, α *and where*

$$|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

Then let $u(x,y) \in L^2(\Omega)$ *be a solution in the distribution sense in the* Ω *of the equation*

$$-c^2(x,y)\Delta u = \lambda u, \tag{1.12}$$

for $\lambda > 0$*. Then* $u(x,y) \equiv 0$ *for a.e.* $(x,y) \in \Omega$.

In this paper we develop our technique under simple conditions.

Theorem I generalizes in several directions. Condition (1.3) can be relaxed; $c(x,y)$ and $c_0(y)$ can have both zeros, and singularities. (1.8) only needs to holds asymptotically as $y \to \infty$. The decay condition (1.11) can be generalized. We only need decay in the complement of a proper cone in $\mathbf{R}^n$. Also more general equations can be considered.

Theorem I is proven by using the limiting absorption principle of a related problem, in an argument for absence of eigenvalues that seems to be new. In the case of the half space $y \geq 0$ with a boundary condition at $y = 0$, where there is only the asymptotic $y \to \infty$ to consider, the absence of eigenvalues was proven in [**3**] using a different argument (see also [**1**]).

After this work was completed and presented at the Conference on February 5-7, 1986, I learned at the International Conference in Differential Equations and Mathematical Physics, held at Birmingham, March 2-8, 1986, of results in the absence of eigenvalues for the Dirichlet Laplacian in the complement of a deformed cylinder, obtained by W. Littman, using a technique similar to ours [**4**].

Finally, in [9] we give a proof of absence of positive eigenvalues of the acoustic propagator for a different class of deformations by means of virial techniques that are quite different from the one in this paper.

2. Proof of the theorem. Suppose that $u(x,y)$ is a solution in $L^2(\Omega)$ of (1.12). Let $\Omega_i = \mathbf{R}^{n+1}\backslash\Omega$ be contained in a ball of radius R. Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, satisfy)$0 \le \phi \le 1, \phi = 0$ in the ball of radius $R+1$, and $\phi = 1$, outside the ball of radius $R+2$. Then

(2.1) $$v(x,y) = \phi(x,y)u(x,y) \in L^2(\mathbf{R}^{n+1}),$$

and satisfies the equation

(2.2) $$-\Delta v - \frac{\lambda}{c_0^2(y)} v = f(x,y)$$

in the distribution sense in $\mathbf{R}^{n+1}$, for some $f(x,y)$ that satisfies

(2.3) $$f(x,y) = 0, \; y > M,$$

for some M, and

(2.4) $$e^{\alpha|x|} f(x,y) \in L^2(\mathbf{R}^{n+1}).$$

We will prove that $v(x,y) = 0$, for $y > M$.

Let F be the Fourier transform in the x variables. It follows from a simple argument using the separability of the test space $C_0^\infty(\mathbf{R}^{n+1})$ that, for almost every $k \in \mathbf{R}^n$,

(2.5) $$-\frac{d^2}{dy^2}\hat{v}(k,y) + \left(k^2 - \frac{\lambda}{c_0^2(y)}\right)\hat{v}(k,y) = \hat{f}(k,y)$$

in the distribution sense in R, where $\hat{v}(k,y)$ and$\hat{f}(k,y)$ are, respectively, the Fourier transform of $v(x,y)$, and $f(x,y)$.

Suppose that $k^2 < \lambda/c_+^2$. Then the homogeneous equation

(2.6) $$\frac{-d^2\phi(y)}{dy^2} + \left(k^2 - \frac{\lambda}{c_0^2(y)}\right)\phi(y) = 0$$

has two linearly independent solutions for $y > h_+$:

$$e^{\pm\sqrt{\lambda/c_+^2 - k^2}\,y}. \tag{2.7}$$

By elementary techniques in O.D.E. (variation of parameters for example), we construct a solution $\psi(k,y)$ of (2.5) that satisfies

$$\psi(k,y) = 0, \quad y > M, \qquad \text{for } k^2 < \lambda/c_+^2. \tag{2.8}$$

Since $\hat{v}(k,y) \in L^2(\mathbf{R})$ for a.e. k, it follows that $\psi(k,y) - \hat{v}(k,y)$ is a solution in $L^2(\mathbf{R}^+)$ of the homogeneous equation (2.6). By (2.7) it follows that

$$\psi(k,y) - \hat{v}(k,y) = 0, \quad y > h_+,$$

and then

$$\hat{v}(k,y) = \psi(k,y) = 0, \quad y > M, \tag{2.9}$$

for a.e. $k^2 < \lambda/c_+^2$.

We will prove below that, also, $\hat{v}(k,y) = 0, y > M$, for a.e. $k^2 \geq \lambda/c_+^2$. It will follow then by Fourier transform that $v(x,y) = 0, y > M$, for a.e. x. Then, by (2.1), $u(x,y) = 0$, for $y > M$ and (x,y) outside the ball of radius $R+2$. Then, by (1.12) and unique continuation [**6**],

$$u(x,y) = 0, \quad \text{a.e. } (x,y) \in \Omega. \tag{2.10}$$

To complete the proof we will show that (2.9) also holds for $k^2 \geq \lambda/c_+^2$, by means of the limiting absorption principle for the operator

$$h = -\frac{d^2}{dy^2} - \frac{\lambda}{c_0^2(y)} + \frac{\lambda}{a^2}, \tag{2.11}$$

where $a = \min(c_-, c_+)$.

For z in the resolvent set of h we denote

$$r(z) = (h - z)^{-1}. \tag{2.12}$$

By $\sigma_d(h)$ we denote the discrete spectrum of h. See [**5**] for definitions.

For any $\alpha \in \mathbf{R}$ we denote by L^2_α the weighted L^2 space of all measurable complex valued functions on $\mathbf{R}$ such that

$$(1+y^2)^{\alpha/2} f(y) \in L^2(\mathbf{R}), \tag{2.13}$$

with norm

$$||f||_{L^2_\alpha} = ||(1+y^2)^{\alpha/2} f(y)||_{L^2(\mathbf{R})}. \tag{2.14}$$

By $H_{2,\alpha}(\mathbf{R}), \alpha \in \mathbf{R}$, we denote the weighted Sobolev space of function $f(y) \in L^2_\alpha$ such that $\frac{d}{dy} f(y) \in L^2_\alpha$ and $\frac{d^2}{d^2y} f(y) \in L^2_\alpha$, with norm

$$||f||_{H_{2,\alpha}} = \left(||f||^2_{L^2_\alpha} + ||\frac{d}{dy} f||^2_{L^2_\alpha} + ||\frac{d^2}{d^2y} f||^2_{L^2_\alpha} \right)^{1/2}. \tag{2.15}$$

LEMMA 2.1. *The essential spectrum of h consists of $[0,\infty)$, h has no positive eigenvalues.*

For every $\mu > 0, \mu \neq |\lambda/c_+^2 - \lambda/c_-^2|$, the limits

$$r(\mu \pm i0) = \lim_{\varepsilon \downarrow 0} (h - \mu \mp i\varepsilon)^{-1} \tag{2.16}$$

exist in the uniform operator topology in $L(L^2_\alpha; H_{2,-\alpha})$. The functions

$$r^{\pm}(z) = \begin{cases} r(z), & \operatorname{Im} z \neq 0, \\ r(z \pm i0), & \operatorname{Im} z = 0, \end{cases} \tag{2.17}$$

defined for $z \in D^{\pm} = (\mathbf{C}^{\pm} \cup \mathbf{R}^+) \backslash (0 \cup |\lambda/c_+^2 - \lambda/c_-^2|)$, are analytic for $\operatorname{Im} z \neq 0$ and are Hölder continuous with exponent $\gamma \leq 1, \gamma < \alpha - 1/2$, for $\lambda \in \mathbf{R}^+ \backslash 0 \cup |\lambda/c_+^2 - \lambda/c_-^4|$. Furthermore, since $\mathbf{R}^- \backslash \sigma_d(h)$ is contained in the resolvent set of h, $r^+(z) = r^-(z)$ for $z \in \mathbf{R}_- \backslash \sigma_d(h)$, and the common value is analytic.

We give below a simple proof of this Lemma. We handle the different limiting values of the potential in the left and the right by adding a Dirichlet boundary condition at zero. It is clear from the proof that the lemma is true if only $c_0(y)$ is asymptotic to c_+ when $y \to +\infty$.

Before we prove Lemma 2.1 let us use it to complete the proof of Theorem I.

By (2.5) and Lemma 2.1,

$$
\begin{aligned}
r^{\pm}(\frac{\lambda}{a^2} - k^2)\hat{f}(k,y) &= \lim_{\varepsilon \downarrow 0} r\left(\frac{\lambda}{a^2} - k^2 \pm i\varepsilon\right)\hat{f}(k,y) \\
&= \lim_{\varepsilon \downarrow 0} r\left(\frac{\lambda}{a^2} - k^2 \pm i\varepsilon\right)\left(h - \left(\frac{\lambda}{a^2}\right) + k^2\right)\hat{v}(k,y) \\
&= \hat{v}(k,y) \pm \lim_{\varepsilon \downarrow 0} i\varepsilon\, r\left(\frac{\lambda}{a^2} - k^2 \pm i\varepsilon\right)\hat{v}(k,y) = \hat{v}(k,y),
\end{aligned}
\tag{2.18}
$$

for a.e. k such that $\lambda/a^2 - k^2 \in \mathbf{R}\backslash\Big(0 \cup |\lambda/c_+^2 - \lambda/c_-^2| \cup \sigma_p(h)\Big)$.

Then

$$
\hat{v}(k,y) = r^{\pm}\Big(\lambda/a^2 - k^2\Big)\hat{f}(k,y). \tag{2.19}
$$

By (2.4) $\hat{f}(k,y)$ has an analytic extension as a function of $|k|$ to the strip $-\alpha < \operatorname{Im}|k| < \alpha$. By Lemma 2.1 and (2.19), $\hat{v}(k,y)$ is Hölder continuous as a function of $|k|$ for $\lambda/a^2 - k^2 \in \mathbf{R}\backslash\big(0 \cup |\lambda/c_+^2 - \lambda/c_-^2| \cup \sigma_p(h)\big)$, and it has analytic extensions to the strips $0 < \operatorname{Im}|k| < \alpha$, $-\alpha < \operatorname{Im}|k| < 0$. By (2.19) the analytic extensions above and below the real axis coincide on the real axis with $\hat{v}(k,y)$. Then $\hat{v}(k,y)$ is also analytic in $|k|$ for $\lambda/a^2 - k^2 \in \mathbf{R}\backslash\Big(0 \cup |\lambda/c_+^2 - \lambda/c_-^2| \cup \sigma_p(h)\Big)$.

Note that, since (2.5) is not true for a set of measure zero of exceptional values, we have to redefine $\hat{v}(k,y)$ as given by (2.19) at those points.

Note that at this point it is enough to have that $\hat{v}(k,y)$ is analytic in one side only and continuous up to the boundary. In this way Theorem I holds true if (1.11) only holds in the complement of a proper cone.

PROOF OF LEMMA 2.1. This elementary result can be proven in many ways. We give a simple proof based on the techniques of [**8**].

Let $h_{\pm}$ be the selfadjoint realization of $-\frac{d^2}{dy^2}$ in $L^2(\mathbf{R}^{\pm})$ with Dirichlet boundary condition at zero.

The limiting absorption principle for $h_{\pm}$ follows easily by, for example, as in the proof of Lemma A.7 of [**7**], using the fact that the sine transform gives us a spectral representation for $h_\pm$. The existence of the trace operators is elementary in this case. Let

$$r_\pm^+(z) = \begin{cases} (h_+ - z)^{-1}, & \operatorname{Im} z \neq 0 \\ \lim_{\varepsilon \downarrow 0}(h_+ - \mu \mp i\varepsilon)^{-1}, & z = \mu \in \mathbf{R}^+, \end{cases} \tag{2.20}$$

be the extended resolvents of h_+, where the limit is in the uniform operator topology in $\mathbf{L}(L^2_\alpha, H_{2,-\alpha}), \alpha > 1/2$, and $r_\pm^+(z)$ are defined in $C^\pm \cup R^+$. The extended resolvents $r_\pm^-(z)$ of h_- are similarly constructed.

We define

$$q(y) = \begin{cases} \max(0, \lambda/c_-^2 - \lambda/c_+^2), & y \geq 0, \\ \max(0, \lambda/c_+^2 - \lambda/c_-^2), & y < 0. \end{cases} \tag{2.21}$$

Let h_D be the selfadjoint realization of $-\frac{d^2}{dy^2}$ in $L^2(\mathbf{R})$ with Dirichlet boundary condition at zero. We denote

$$m = h_D + q \equiv (h_- + q_-) \oplus (h_+ + q_+), \tag{2.22}$$

where $q_\pm$ are the restrictions of q to $\mathbf{R}^\pm$.

By (2.20), and the corresponding statement for h_- the limiting absorption principle is true for m. We denote by $p_\pm(z)$ the extended resolvents of m for $z \in \mathbf{C}^\pm \cup (\mathbf{R}^+ \backslash 0 \cup |\lambda/c_+^2 - \lambda c_-^4|)$.

Let h_0 be the selfadjoint realization of $-\frac{d^2}{dy^2}$ in $L^2(\mathbf{R})$ with domain $H_{2,0}$. Let $b > 0$ be such that $h_D + q + b > 0$, and $(h_0 + q + b) > 0$. We denote

$$V = (h_0 + q + b)^{-1} - (h_D + q + b)^{-1}. \tag{2.23}$$

As in Lemma 2.4 of [**8**] we prove that V extends to a compact operator from $L^2_{-\alpha}$ into L^2_β, for any $\alpha, \beta \in \mathbf{R}$ (this problem is clearly much simpler). The limiting absorption principle for $h_0 + q$ follows as in the proof of Theorem I of [**8**]. Finally, since

$$h = h_0 + q(y) + p(y), \tag{2.24}$$

where $p(y)$ satisfies

$$|p(y)| \leq (1 + y^2)^{\frac{-1-\varepsilon}{2}}, \tag{2.25}$$

the limiting absorption principle for h is obtained from the one for $h_0 + q(y)$, for example, as in [**7**].

The absence of positive eigenvalues for h follows from [**2**].

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