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INFLOW BOUNDARY CONDITIONS FOR STEADY FLOWS OF VISCOELASTIC FLUIDS WITH DIFFERENTIAL CONSTITUTIVE LAWS

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ABSTRACT. We prove an existence result for steady flows through a strip which can be regarded as a perturbation of rigid motion. The fluid is viscoelastic with a differential constitutive law, e.g., an upper convected Maxwell model. Particular emphasis is focussed on the question which boundary conditions need to be imposed at the inflow boundary to make the problem well-posed.

1. Introduction. Steady flows of non-Newtonian fluids can not be uniquely determined by imposing boundary conditions only for the velocities as in the Newtonian case. The reason for this is that the fluids have memory, and therefore the flow inside the domain under consideration is affected by the fluid motion that occurred before the fluid entered the domain. This leads to the need for extra boundary conditions at inflow boundaries, which must contain some information about the flow history outside the domain. The precise nature of such inflow boundary conditions is not understood; it is certainly dependent on the constitutive law of the fluid.

In this paper, we consider differential constitutive models, which relate the extra stress tensor to the velocity gradient by an evolution equation along streamlines. For the sake of concreteness, we consider the special case of an upper convected Maxwell fluid, we emphasize, however, that the analysis in this paper does not depend on the special form of the nonlinearities and also applies to other differential models (for examples, see [**3**], [**5**], [**7**]). In a steady flow, the extra stress tensor $\mathbf{T}$ in an upper convected Maxwell fluid satisfies the equation

$$(1)\qquad (u\cdot\nabla)\mathbf{T}-(\nabla u)\mathbf{T}-\mathbf{T}(\nabla u)^T+\lambda\mathbf{T}=\eta\lambda(\nabla u+(\nabla u)^T).$$

Here u is the velocity of the fluid, and η and λ are positive constants. The Newtonian fluid is recovered in the limiting case $\lambda\to\infty$. This

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equation for the stress is to be solved in conjunction with the equation of motion

$$\rho(u \cdot \nabla)u = \operatorname{div} \mathbf{T} - \nabla p + f, \qquad (2)$$
$$\operatorname{div} u = 0.$$

The fact that (1) describes an evolution of the stress along streamlines may suggest that appropriate boundary conditions for (1) and (2) are given by prescribing $\mathbf{T}$ at an inflow boundary, in addition to the Dirichlet conditions for the velocity which are required in the Newtonian case. In fact this seems to be what rheologists generally believe (see, e.g., [**2**], p. 31). The study of characteristics [**4**], [**6**], [**9**], however, suggests otherwise. We may regard (1) and (2) as a first order system for p and the components of u and $\mathbf{T}$. In this system, it turns out that the streamlines of the flow are double characteristics in two dimensions and quadruple characteristics in three dimensions, while the number of independent components of the symmetric tensor $\mathbf{T}$ is three and six, respectively. From this it appears that prescribing the stress at an inflow boundary may overdetermine the problem.

In an earlier paper [**8**], the author has proved an existence result for slow steady flows of differential non-Newtonian fluids. These flows were in a bounded domain with prescribed homogeneous Dirichlet conditions on the boundary. In this case, there are no inflow boundaries, and no extra boundary conditions are needed. The existence proof was based on an iteration scheme that alternates between solving an elliptic system and a hyperbolic system which has the streamlines as characteristics (a similar approach was recently used by Beirão da Veiga [**1**] to prove the existence of steady flows of compressible Newtonian fluids).

In the present paper, we consider transverse flows through a strip which are small perturbations of a flow with constant velocity and zero stress. If we want to use the iteration scheme of [**8**], we have to impose boundary conditions at inflow in order to be able to solve the hyperbolic part. If this is done, it turns out that in fact the iteration converges, but in general not to a solution of the equations (1) and (2). An extra conditions has to be satisfied for this to be the case. We shall then modify the iteration scheme to accommodate this condition. If this is done, the number of inflow conditions required is as the analysis of charactersitics suggests.

2. An iterative procedure for steady flows perturbing flow with constant velocity. We consider transverse flow through the strip $0 \leq x \leq 1$ with periodic boundary conditions imposed in the y and z directions. The periods are denoted by L_y and L_z. For $f = 0$, a solution of (1) and (2) is given by a flow with constant velocity $U > 0 : u = (U, 0, 0), p = 0, \mathbf{T} = \mathbf{0}$. We are looking for flows which are small perturbations of such a flow. I.e., we prescribe a small body force f and boundary conditions for the velocity which are small perturbations of the constant velocity: $u(0, y, z) = (U, 0, 0) + v_1(y, z), u(1, y, z) = (U, 0, 0) + v_2(y, z)$. These boundary conditions must be consistent with incompressibility,

$$\int_0^{L_z} \int_0^{L_y} e_x \cdot v_1(y, z) dy\, dz = \int_0^{L_z} \int_0^{L_y} e_x \cdot v_2(y, z) dy\, dz. \tag{3}$$

In addition, we shall need conditions on the stresses at the inflow boundary $x = 0$. The nature of these conditions will now be discussed.

Following [8], we apply the divergence operator to the constitutive equation (1), and obtain

$$(u \cdot \nabla)\operatorname{div} \mathbf{T} - (\nabla u)\operatorname{div} \mathbf{T} + \lambda \operatorname{div} \mathbf{T} = \mathbf{T} : \partial^2 u + \eta\lambda\Delta u. \tag{4}$$

Here we have set $\mathbf{T} : \partial^2 = \sum_{j,k} T_{j,k} \frac{\partial^2}{\partial y^j \partial y^k}$. If we substitute div $\mathbf{T}$ from the equation of motion, we find

(5)
$$\begin{aligned} &\nabla[(u \cdot \nabla)p + \lambda p] - [\nabla u + (\nabla u)^T]\nabla p - (u \cdot \nabla)f + (\nabla u)f - \lambda f \\ &= \mathbf{T} : \partial^2 u + \eta\lambda\Delta u - \rho(u \cdot \nabla)(u \cdot \nabla)u + \rho(\nabla u)(u \cdot \nabla)u - \lambda\rho(u \cdot \nabla)u. \end{aligned}$$

In [8], the construction of a solution was based on an iterative method, which alternates between solving a "Stokes-like" problem and a hyperbolic equation whose characteristics are streamlines. Similar iterations are in fact used in numerical calculations [2]. Since we are looking for small perturbations of the flow with constant velocity, we can take this flow as a starting point for the iteration. The iteration scheme of [8] is the following (we introduce the new variable $q = (u \cdot \nabla)p + \lambda p$)

$$u^0 = (U, 0, 0), p^0 = q^0 = 0, \mathbf{T}^0 = \mathbf{0}, \tag{6a}$$

(6b)
$$\mathbf{T}^n : \partial^2 u^{n+1} + \eta\lambda\Delta u^{n+1} - \rho(u^n \cdot \nabla)(u^n \cdot \nabla)u^{n+1} - \lambda\rho(u^n \cdot \nabla)u^{n+1} - \nabla q^{n+1}$$

$$= -[(\nabla u^n)+(\nabla u^n)^T]\nabla p^n-(u^n\cdot\nabla)f+(\nabla u^n)f-\lambda f-\rho(\nabla u^n)(u^n\cdot\nabla)u^n,$$

$$\operatorname{div} u^{n+1} = 0, u^{n+1} = (U,0,0)+v_1 \text{ on } x = 0, u^{n+1} = (U,0,0)+v_2 \text{ on } x = 1,$$

$$\int_0^1 \int_0^{L_y} \int_0^{L_z} q^{n+1} dz\, dy\, dx = 0,$$

$$(u^{n+1}\cdot\nabla)p^{n+1} + \lambda p^{n+1} = q^{n+1}, \tag{6c}$$

$$\begin{aligned}(u^{n+1}\cdot\nabla)\mathbf{T}^{n+1} - (\nabla u^{n+1})\mathbf{T}^{n+1} - \mathbf{T}^{n+1}(\nabla u^{n+1})^T + \lambda\mathbf{T}^{n+1}\\ = \eta\lambda[(\nabla u^{n+1}) + (\nabla u^{n+1})^T].\end{aligned} \tag{6d}$$

In setting up this iteration, it is essential that the singularly perturbed operator $(u\cdot\nabla)+\lambda$ is inverted rather than evaluated. As long as $U^2 < \lambda\eta/\rho$, and u^n and $\mathbf{T}^n$ are small perturbations of $(U,0,0)$ and $\mathbf{0}$, equation (6b) is an elliptic system for u^{n+1} and q^{n+1}. If $U^2 > \lambda\eta/\rho$, a change of type occurs [**4**], [**6**], [**10**], and this case will not be considered here. Equations (6c) and (6d) are hyperbolic equation with the streamlines as characteristics. If there are no in-and outflow boundaries as in [**8**], characteristics do not cross the boundary, and (6c) and (6d) can be solved without any boundary conditions imposed. In the present situation, however, we have to prescribe p and $\mathbf{T}$ at the inflow $x = 0$ in order to solve (6c) and (6d).

It can in fact be shown along lines similar to [**8**] that with such inflow boundary conditions the iteration will converge, provided that the data are sufficiently small. However, it will in general not converge to a solution of the original problem. The "solution" obtained from the iteration will satisfy (1) and (5), but the original problem is (1) and (2). Let us recall that (5) was obtained by substituting $\rho(u\cdot\nabla)u+\nabla p-f = \operatorname{div}\mathbf{T}$ into (4), and (4) was obtained by differentiating (1). Hence the fact that (5) is satisfied means the following: If we set

$$\rho(u\cdot\nabla)u + \nabla p - f - \operatorname{div}\mathbf{T} = h, \tag{7}$$

then

$$(u\cdot\nabla)h - (\nabla u)h + \lambda h = 0. \tag{8}$$

If streamlines do not cross the boundary as in [**8**], then (8) does in fact lead to $h = 0$, but in the present case we need to know that $h = 0$ at the inflow boundary in order to make that conclusion. This condition has to be viewed as a constraint on the possible inflow data for p and $\mathbf{T}$, and the iteration (6) has to be modified in order to accommodate this constraint. If we simply count numbers, we see that in three dimensions $\mathbf{T}$ has six components, p has one and h has three, thus suggesting that four inflow boundary conditions can be prescribed. In two dimensions, $\mathbf{T}$ has three components, p has one and h has two, suggesting that one needs two inflow boundary conditions. This is also suggested by the analysis of characteristics [**4**], [**6**], [**9**].

We want to restrict the choice of inflow boundary conditions in (6c), (6d) by the constraint

$$\operatorname{div} \mathbf{T}^{n+1} - \nabla p^{n+1} = \rho(u^{n+1} \cdot \nabla) u^{n+1} - f \text{ at } x = 0 \tag{9}$$

On the left hand side of (9), the x-derivatives of p^{n+1} and $\mathbf{T}^{n+1}$ can be expressed using (6c) and (6d), e.g.,

$$\frac{\partial p^{n+1}}{\partial x} = -\frac{\lambda}{U} p^{n+1} + \frac{q^{n+1}}{U} + \text{ nonlinear terms.} \tag{10}$$

When doing this, (9) assumes the form

$$\begin{aligned} &\frac{\lambda}{U}(p^{n+1} - T_{11}^{n+1}) + T_{12,y}^{n+1} + T_{13,z}^{n+1} = \dots \\ &-\frac{\lambda}{U} T_{12}^{n+1} + (T_{22}^{n+1} - p^{n+1})_y + T_{23,z}^{n+1} = \dots \\ &-\frac{\lambda}{U} T_{13}^{n+1} + T_{23,y}^{n+1} + (T_{33}^{n+1} - p^{n+1})_z = \dots \end{aligned} \tag{11}$$

The right hand sides indicated by dots contain terms involving f, u^{n+1}, q^{n+1} and nonlinear terms which also involve $\mathbf{T}^{n+1}$ and p^{n+1}.

We have to use equation (11) in order to express some stress components at $x = 0$ in terms of others. To obtain a convergent iteration, we want to do this in such a way that no loss of regularity occurs when solving for the undetermined stress components. Unfortunately, this is not possible in such a way that certain components of stress are prescribed at inflow and others are left to be determined. Rather, we have to prescribe stress components partially. For example, we can do the following. Let each component of the stress be expanded in a Fourier series in y and z, e.g.,

$$T_{11}(0, y, z) = \sum_{k,l} t_{11}^{kl} e^{2\pi i(ky/L_y + lz/L_z)}. \tag{12}$$

We can then prescribe the Fourier components of T_{11}, T_{22}, T_{13} and T_{33} for $|k| \geq |l|, k \neq 0$ and solve for those of p, T_{12} and T_{23}. For $|\ell| > |k|$, we prescribe the Fourier components of T_{11}, T_{12}, T_{22} and T_{33} and solve for those of p, T_{13} and T_{23}. For $k = l = 0$, we prescribe the Fourier components of T_{11}, T_{22}, T_{23} and T_{33} and solve for those of p, T_{12} and T_{13}. In the two-dimensional case, this amounts to prescribing T_{11} and T_{22} and solving for p and T_{12}.

The solution procedure is now as follows. We start the iteration with initial data (6a). Then at each step of the iteration we compute a new u and q using (6b). Then we compute a new p and $\mathbf{T}$ from (6c) and (6d) with inflow boundary conditions which are in part prescribed and in part computed from (11) in the manner outlined above. We show in the next section that such an interation converges under appropriate smallness conditions for the body force f, the velocity boundary data v_1 and v_2 and the prescribed part of the stress boundary data.

3. Proof of convergence. As usual, we denote by H^s the space of all functions on the strip $0 \leq x \leq 1$, which are periodic with periods L_y and L_z in the y and z directions and have s derivatives which are square integrable over one period. Sobolev spaces of periodic functions living on one of the boundaries $x = 0$ or $x = 1$ are denoted by $H^{\langle s \rangle}$. The corresponding norms are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{\langle s \rangle}$.

In the following, let s be any integer ≥ 1. We assume that the body force and the velocity boundary data satisfy the bound

$$\|f\|_{s+1} \leq \gamma, \|v_1\|_{\langle s+3/2 \rangle} \leq \gamma, \|v_2\|_{\langle s+3/2 \rangle} \leq \gamma, \tag{13}$$

where γ is a positive number which will later be chosen small. The stress at the inflow boundary consists of a prescribed part $\mathbf{T}_p$ and an unknown part $\mathbf{T}_u$, which must be determined from (11) at each stage of the iteration. We assume that

$$\|\mathbf{T}_p\|_{\langle s+1 \rangle} \leq \gamma. \tag{14}$$

The convergence proof consists of two parts: First we show that all iterates remain bounded in a certain norm, and then we use this fact to prove convergence in a weaker norm. This procedure is typical in dealing with hyperbolic problems, and the iteration (6) involves the hyperbolic part (6c), (6d). In order to carry out the first part, let us assume that

$$\|u^n - (U, 0, 0)\|_{s+2} \leq \varepsilon, \|p^n\|_{s+1} \leq \varepsilon, \|\mathbf{T}^n\|_{s+1} \leq \varepsilon, \tag{15}$$

where ε is small. As long as $\rho U^2 < \eta\lambda$ and ε is sufficiently small, (6b) is an elliptic system for u^{n+1} and q^{n+1}, and a standard argument shows that there is a constant C_1 such that

$$||u^{n+1} - (U,0,0)||_{s+2} + ||q^{n+1}||_{s+1} \le C_1(\gamma + \varepsilon^2) =: \delta. \tag{16}$$

In the next step, $\mathbf{T}_u^{n+1}$ and p^{n+1} on the inflow boundary are determined from (11). Using the trace theorem, we see that q^{n+1} and ∇u^{n+1} are in $H^{\langle s+1/2\rangle}$. By using this in (11), it is easy to show that for small δ there is a constant C_2 such that

$$||\mathbf{T}_u^{n+1}||_{\langle s+1\rangle} + ||p^{n+1}||_{\langle s+1\rangle} \le C_2(\gamma + \delta) =: \sigma. \tag{17}$$

We now turn to the hyperbolic equations (6c) and (6d). The solution to these equations can be obtained by the integrating along streamlines (since u^{n+1} is Lipschitz, streamlines exist and are unique). To obtain estimates for the solution, let us multiply equation (6c) by p and integrate over the domain. This yields

$$\begin{aligned}&\int_0^1\int_0^{L_y}\int_0^{L_z} p^{n+1}(u^{n+1}\cdot\nabla)p^{n+1} + \lambda(p^{n+1})^2 dz\,dy\,dx\\ &= \int_0^1\int_0^{L_y}\int_0^{L_z} p^{n+1}q^{n+1} dz\,dy\,dx\,.\end{aligned} \tag{18}$$

The left hand side is equal to

$$\begin{aligned}&\frac{1}{2}\int_0^{L_y}\int_0^{L_z} e_x\cdot u^{n+1}(1,y,z)(p^{n+1}(1,y,z))^2 dz\,dy\\ &-\frac{1}{2}\int_0^{L_y}\int_0^{L_z} e_x\cdot u^{n+1}(0,y,z)(p^{n+1}(0,y,z))^2 dz\,dy\\ &+\lambda\int_0^1\int_0^{L_y}\int_0^{L_z}(p^{n+1})^2 dz\,dy\,dx\,.\end{aligned} \tag{19}$$

The first term in this is positive, and hence we obtain

$$\begin{aligned}&\lambda\int_0^1\int_0^{L_y}\int_0^{L_z}(p^{n+1})^2 dz\,dy\,dx \le \int_0^1\int_0^{L_y}\int_0^{L_z} p^{n+1}q^{n+1} dz\,dy\,dx\\ &+\frac{1}{2}\int_0^{L_y}\int_0^{L_z} e_x\cdot u^{n+1}(0,y,z)(p^{n+1}(0,y,z))^2 dz\,dy.\end{aligned} \tag{20}$$

That is, we have an estimate of the L^2-norm for p in terms of theL^2-norm of q and the L^2-norm of the inflow boundary data. By differentiating (6c) with respect to y or z, we can obtain estimates for derivatives of p in these directions, and estimates for x-derivatives can be obtained from the equation itself. Similarly, we can deal with (6d). If δ is sufficiently small, this argument leads to estimates of the form

$$(21) \qquad ||p^{n+1}||_{s+1} + ||\mathbf{T}^{n+1}||_{s+1} \leq C_3(\sigma + \gamma + \delta).$$

If we now choose ε small enough and γ sufficiently small relative to ε, we will have $\delta \leq \varepsilon$ and $C_3(\sigma + \gamma + \delta) \leq \varepsilon$. This implies that (15) holds with n replaced by $n+1$. By induction, we see that (15) holds for all values of n.

The fact that all the iterates are bounded and in fact small can now be used to show convergence of the iteration in a weaker norm. More specifically, we can show that u^n converges in H^{s+1} and that $p^n, q^n, \mathbf{T}^n$ converge in H^s. The argument is fairly routine and we shall just show a typical step. Let us take (6c) at step $n+1$ and step n and take the difference. We find

(22)
$$(u^{n+1} \cdot \nabla)(p^{n+1} - p^n) + \lambda(p^{n+1} - p^n) + ((u^{n+1} - u^n) \cdot \nabla)p^n = q^{n+1} - q^n$$

From this we find

(23)
$$||p^{n+1} - p^n||_s \leq C(||q^{n+1} - q^n||_s + ||u^{n+1} - u^n||_{s+1}||p^n||_{s+1} + ||p^{n+1} - p^n||_{\langle s \rangle}).$$

From above, we already have a bound on the term $||p^n||_{s+1}$. Similar arguments are applied to (6b), (6d) and (11). By putting the resulting estimates together, we obtain an inequality of the form

(24)
$$\begin{aligned} ||u^{n+1} - u^n||_{s+1} + ||q^{n+1} - q^n||_s + ||p^{n+1} - p^n||_s + ||\mathbf{T}^{n+1} - \mathbf{T}^n||_s \\ \leq K(\varepsilon, \gamma)[||u^n - u^{n-1}||_{s+1} + ||q^n - q^{n-1}||_s \\ + ||p^n - p^{n-1}||_s + ||\mathbf{T}^n - \mathbf{T}^{n-1}||_s]. \end{aligned}$$

The constant $K(\varepsilon, \gamma)$ tends to zero as γ and ε tend to zero. Hence a contraction is obtained if we choose ε and γ sufficiently small.

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