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ON SPURIOUS NUMERICAL SOLUTIONS FOR NONLINEAR EIGENVALUE PROBLEMS

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1. Introduction. It is well known that discretization of the nonlinear eigenvalue problem

$$(1) \qquad \begin{aligned} u''(x) + \lambda f(u) &= 0 \\ u(-1) = u(1) &= 0 \end{aligned}$$

leads to a system of equations

$$(2) \qquad AU = \lambda F(U)$$

for which there may exist so-called "spurious solutions" in addition to the "numerically relevant solutions" (NRS) which approximate the solutions to (1). The branches of spurious or "numerically irrelevant solutions" (NIS) characteristically are located away from the origin in (λ, U)-space. As the mesh size of the discretization is decreased these branches will recede farther away from the origin while the branch of NRS will remain in approximately the same place. For practical computing purposes it if therefore not difficult to identify (or even avoid) spurious solutions. This, however, does not make the phenomenon any less interesting and it has been studied by various means. For example, in the recent work of Peitgen and Nussbaum sophisticated dynamical systems theory was brought to bear upon the problem of NIS for nonlinear elliptic eigenvalue problems as well as the closely related problem of special periodic solutions ($x(t+2) = x(-t) = -x(t)$) of the delay differential equation $\dot{x}(t) = -\alpha f(x(t-1))$ for odd f [**9**], [**10**], [**11**]. In what may be the eariliest reference to spurious solutions, Gaines [**6**] considered nonlinearities of the form $f(u, u')$. He showed that the discretized problem may have spurious solutions and proposed an algorithm which would find the numerically relevant solutions while avoiding the spurious ones. In another early analytical study Allgower [**1**] showed that spurious solutions may exist for as simple a nonlinearity as $f(u) = u^m, m > 1$. Later work includes

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that of Bohl [**3**], Peitgen, Saupe and Schmitt [**14**], Doedel and Beyn [**5**], Peitgen and Schmitt [**12**] [**13**]. The present work has primarily been motivated by the work of Beyn and Lorenz [**2**], who studied the problem for superlinear maps, in particular $f(u) = \exp u$, and showed the existence of (local) branches which bifurcate from infinity. Computational evidence shows that pairs of these local branches are in fact the ends of single branches which emanate from and return to infinity.

The main goal of the present work is to prove this global result for a class of functions which includes $\exp u$. Following Beyn and Lorenz NIS can be classified as those which exist for large λ and tend to solutions of $F(U) = 0$ as $\lambda \to \infty$ (Type I) and those solutions which exist for small λ and tend to infinity in the norm of U as $\lambda \to 0$ (Type II). In the papers [**13**] and [**14**] complete and precise information was obtained about the existence and disposition of branches of spurious solutions of type I. This paper will only be concerned with NIS of Type II for 1-dimensional problems, although the methods can also be used to study NIS of Type I as well as NIS arising in higher dimensional elliptic eigenvalue problems. The objective here has not been to obtain the greatest generality, but rather to propose a relatively simple method that can be used to obtain qualitative information about branches of NIS. This method relies on the homotopy invariance of the topological degree where the homotopy parameter describes the grid. As this parameter is increased the grid smoothly changes into a very simple one (that is, one with very few grid-points) for which the system of equations (2) can be analyzed in detail.

It is interesting that the homotopy process may also be reversed, thereby introducing more and more grid points. This shows that the NIS for (1)-(2) are in fact NRS for certain multi-point boundary value problems: Let χ be the characteristic function for $(-b,-a) \cup (a,b)$, $b = a + H$, and

$$\begin{aligned}
&u'' + \lambda(1-\chi)f(u) = 0, \quad -1 < x < 1,\\
&u(-1) = u(1) = 0,\\
\pm u'(\pm a) + H^{-1}(u(\pm a) - u(\pm b)) &= \frac{1}{2}\lambda H f(u(\pm a)),\\
\mp u'(\pm b) - H^{-1}(u(\pm a) - u(\pm b)) &= \frac{1}{2}\lambda H f(u(\pm b)).
\end{aligned}$$

This is a one-dimensional steady state diffusion problem in which the sources $f(u)$ within each interval $(-b,-a)$ and (a,b) have been divided

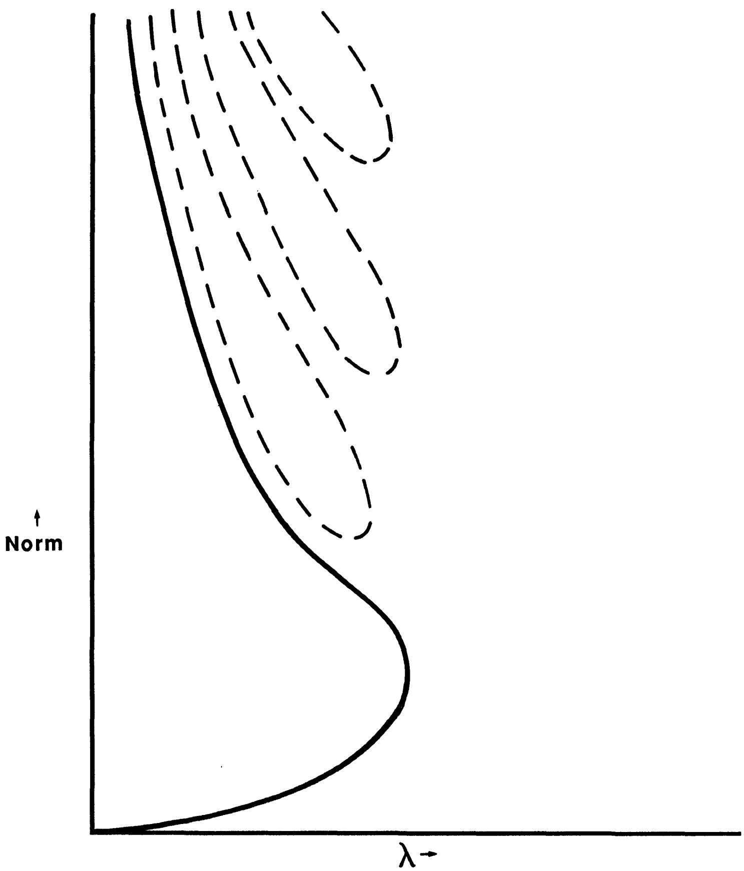

Figure 1. Typical NRS (solid) and NIS branches for $AU = \lambda F(U)$ with F positive and superlinear.

equally and moved to their respective boundaries. This problem has a branch of symmetric solutions as well as two branches of nonsymmetric solutions.

For some computational purposes it is desirable to find a reasonably convenient variable which parametrizes the spurious solution branches.

However, even for relatively simple problems (e.g., $f(u) = \exp u$ with a small number of grid points) this proved to be too difficult. Another difficulty, namely that local analysis is generally ineffectual in studying qualitative behavior of solutions to difference equations, is overcome by a main lemma which will be proved in §2. In §3 some necessary a priori bounds will be obtained and the main result will be proven in §4. In §5, a special situation will be studied, namely the case of a uniform grid which straddles the center. In this case branches of NIS bifurcate from the branch of NRS. Also, a qualitative picture will be presented which provides a simple way of seeing how the various branches may be related. Finally, in §6, some brief comments will be made on how one may apply the proposed method to other situations. In particular the case $f(u) = \exp[u(1+\varepsilon u)^{-1}]$ will be studied and it will be shown that for sufficiently small positive ε the corresponding problem will have *bounded* branches of NIS.

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2. Some preliminary results. In this paper it will be assumed that f is a continuous, real-valued function on $\mathbf{R}^1$ which satisfies $f(u) > 0$ for $u > 0$, and:

$H_0 : Lf(u)/u$ is monotone nonincreasing on $(0, \alpha)$ for some $\alpha > 0$.

The function f has a monotone majorant and monotone minorant which are defined respectively by $\overline{f}(u) = \max\{f(s) : s \leq u\}$ and $\underline{f}(u) = \min\{f(s) : s \geq u\}$.

It will be assumed that the grid on $[-1, 1]$ is symmetric with respect to the origin. The relevant solutions will then also be symmetric. It will be shown that there exist no nonsymmetric solutions (λ, U) with norm

$$\| U \| \equiv \max\{| U(x_i) | : | x_i | \leq 1\} \leq \alpha.$$

Let π be a *partition* of $[-1, 1]$ which determines a grid on that interval. That is to say:

$$\pi : \mathbf{Z}^0 \to [-1, 1], \mathbf{Z}^0 = \mathbf{Z} - \{0\},$$

is a nondecreasing map. Its image $\{x_i \mid x_i = \pi(i), i \varepsilon Z^0\}$, also denoted by π, is the *grid*. Note that $x_i = x_{i+1}$ is allowed. In fact if 0 is a grid-point then it will always be assumed that it corresponds to an even number of x_i's. This is done in order to allow homotopies which remove 0 from the range and at the same time preserve the symmetry. Of course, the symmetry requirement just says that π is an odd map. The reason for using $\mathbf{Z}^0$ rather than a finite set is to allow an influx of arbitrarily many grid points into $(-1,1)$. It will be assumed that there is a positive integer $K = K(\pi)$ such that $| x_i |= 1$ for $| i |> K$, while $| x_i |< 1$ for $| i |\leq K$. Assuming $x_i \neq x_j$ for $i \neq j$ and letting $x_{i+1} - x_i = h_i$, problem (1) may be approximated by

$$(3) \qquad \frac{-2U_{i+1}}{h_i(h_i + h_{i-1})} + \frac{2U_i}{h_i h_{i-1}} - \frac{2U_{i-1}}{h_{i-1}(h_i + h_{i-1})} = \lambda f(U_i)$$

for $| i |\leq K$ and

$$(4) \qquad U_{K+1} = U_{-K-1} = 0.$$

Letting $U = (U_{-K}, U_{-K+1}, \dots, U_K)^T$ and $F(U) = (f(U_{-K}), f(U_{-K+1}), \dots, f(U_K))^T$ this problem may be written as

$$(5) \qquad A_\pi U = \lambda F(U).$$

Since only positive solutions will be considered there is no loss of generality in setting $f(u) = f(0)$ for $u < 0$. An equivalent problem may be obtained by multiplying equation (5) by the matrix $D_\pi = (d_{ij})$, where $d_{ij} = -\frac{1}{2}h_{i-1}(h_j + h_{j-1})$ for $-K \leq i \leq j \leq i_0$ and $d_{ij} = \frac{1}{2}h_i(h_j + h_{j-1})$ for $i_0 + 1 \leq j \leq i \leq K$. This yields a system

$$(6) \qquad \hat{A}_\pi U = \lambda \hat{F}_\pi(U).$$

This system has removable singularities if one or more of the h_i's $i \neq i_0$, is zero and hence is well defined in such cases. Also it will change smoothly as the partition is changed smoothly. If h_j becomes zero (but h_{i_0} stays positive) then the j^{th} equation merely becomes $U_j - U_{j+1} = 0$. By relabeling and removing equations such as these the system (6) is obviously equivalent to $A_{\hat{\pi}} U = \lambda F(U)$ for some new partition $\hat{\pi}$ for which $x_i \neq x_j$ if $-1 < x_i, x_j < 1$ and $i \neq j$.

The set of all integers z in $\mathbf{Z}^0$ (or $\mathbf{Z}$) with $j - 1 < z < k + 1$ will be denoted by $\langle j, k\rangle$. Here $j = -\infty$ and $k = +\infty$ are allowed. Suppose X and Y are real-valued maps on $\langle j, k\rangle$. Then XY is a map on $\langle j, k\rangle : (XY)(i) = X(i)Y(i) \equiv X_i Y_i$. The expression $X \geq Y$ (resp

$X > Y$) means that $X_i \geq Y_i$ (resp $X_i > Y_i$) for $i \in \langle j,k \rangle$. The following lemma is a discrete analog of Sturm's First Comparison Theorem:

LEMMA 1. *Suppose X, Y, μ and ν are maps defined on $\langle M,N \rangle, N > M+1$, and that*

$$A_\pi X = \mu X, X \not\equiv 0, \quad X \geq 0 \text{ on } \langle M+1, N \rangle,$$
$$A_\pi Y = \nu Y, \quad Y_{M+1} \geq 0, \quad X_M Y_{M+1} \geq X_{M+1} Y_M \geq 0,$$

and $\mu \geq \nu \geq 0$ on $\langle M+1, N-1 \rangle$. Then $X > 0$ on $\langle M+1, N-1 \rangle$ and $Y \geq \left(\frac{Y_{M+1}}{X_{M+1}}\right) X$ on $\langle M, N \rangle$.

PROOF. Obviously it will be sufficient to prove this result for the case where $\mu > \nu > 0$. It is clear that X is concave in the sense that if $i \leq j \leq k$ then the point (x_j, X_j) lies on or above the line segment joining (x_i, X_i) to (x_k, X_k). It follows that $X > 0$ on $\langle M+1, N-1 \rangle$. If $Y_{M+1} = 0$ then $Y_M = 0$ and hence $Y \equiv 0$, leaving nothing else to prove. On the other hand, if $Y_{M+1} > 0$ then it may be assumed, without loss of generality, that $Y_{M+1} = X_{M+1}$. In this case one also has $X_M \geq Y_M \geq 0$. Suppose there is a smallest integer $J \in \langle M+1, N \rangle$ such that $Y_J < X_J$. Define $\overline{X}$ as follows

$$\overline{X}_i = \begin{cases} Y_M & \text{if } i = M \\ X_i & \text{if } M < i < J \\ Y_J & \text{if } i = J \end{cases}$$

It is intuitively clear that by increasing μ_{M+1} and μ_{J-1} one can obtain $\overline{\mu}$ such that $A_\pi \overline{X} = \overline{\mu} \overline{X}$. More precisely let $\overline{\mu}$ be as follows

$$\overline{\mu}_i = \begin{cases} \frac{2[(h_M + h_{M+1}) X_{M+1} - h_M X_{M+2} - h_{M+1} Y_M]}{[h_M h_{M+1} (h_M + h_{M+1}) X_{M+1}]} & \text{if } i = M+1 \\ \mu_i & \text{if } M+1 < i < J-2 \\ \frac{2[(h_{J-2} + h_{J-1}) X_{J-1} - h_{J-2} Y_J - h_{J-1} X_{J-2}]}{[h_{J-2} h_{J-1} (h_{J-2} - h_{J-1}) X_{J-1}]} & \text{if } i = J-1. \end{cases}$$

Then $\overline{\mu} \geq \mu > \nu \geq 0$ and $A_\pi \overline{X} = \overline{\mu} \overline{X}$ on $\langle M+1, J-1 \rangle$. Next let Λ be defined by

$$\Lambda_j = Y_M + (Y_J - Y_M)(x_j - x_M)/(x_J - x_M);$$

then clearly $A_\pi \Lambda = 0$ on $\langle M+1, J-1\rangle$ and $Y \geq \overline{X} \geq \Lambda$ on $\langle M, J\rangle$ with equality only at the end points x_M and x_J. Letting $\tilde{X} = \overline{X} - \Lambda, \tilde{Y} = Y - \Lambda$ and $\tilde{\mu}_i = \overline{\mu}_i + \overline{\mu}_i \Lambda_i/(X_i - \Lambda_i), \tilde{\nu}_i = \nu_i + \nu_i \Lambda_i/(Y_i - \Lambda_i)$ for $M+1 \leq i \leq J-1$, the following equations are satisfied

$$A_\pi \tilde{X} = \tilde{\mu}\tilde{X} \text{ on } \langle M+1, J-1\rangle, \tilde{X}_M = \tilde{X}_J = 0,$$
$$A_\pi \tilde{Y} = \tilde{\nu}\tilde{Y} \text{ on } \langle M+1, J-1\rangle, \tilde{Y}_M = \tilde{Y}_J = 0,$$

with $\tilde{Y} \geq \tilde{X} > 0$ and $\tilde{\mu} > \tilde{\nu}$ on $\langle M+1, J-1\rangle$. This means that 1 is an eigenvalue for both problems

$$A_\pi \tilde{X} = \lambda D_\mu \tilde{X} \text{ and } A_\pi \tilde{Y} = \lambda D_\nu \tilde{Y},$$

where D_μ andD_ν are diagonal square matrices with diagonal entries $\tilde{\mu}_i$ and $\tilde{\nu}_i$ respectively. Since $A_\pi^{-1}D_\mu$ and $A_\pi^{-1}D_\nu$ are indecomposable matrices with nonnegative entries the Perron-Frobenius theorem [**8**] may be applied to conclude the existence of a unique (up to scaling) positive eigenvector corresponding to the maximal real eigenvalue. Hence 1 must be that maximal real eigenvalue for both problems. But then the following characterization of this eigenvalue [**8**, p. 125] maybe used

$$\max_{Y>0} \min_{M<i<J} \frac{(A_\pi^{-1}D_\nu Y)_i}{Y_i} = 1 = \max_{x>0} \min_{M<i<J} \frac{(A_\pi^{-1}D_\mu X)_i}{X_i}$$

But this is impossible since $\tilde{\mu} > \tilde{\nu}$ on $\langle M+1, J-1\rangle$, a contradiction to the existence of an integer $J \in \langle M, N\rangle$ with $Y_J < X_J$.

THEOREM 2. *If $f(u)/u$ is monotone nonincreasing on $(0, \alpha)$ and if the grid π is symmetric, then all solutions of (5) of norm $\leq \alpha$ are symmetric.*

PROOF. Let $g(u) = f(u)/u$ on $(0, \alpha)$ and suppose that Y is a nonsymmetric solution of

$$A_\pi Y = \lambda g(Y) Y.$$

Let $\check{Y}$ be its reflection, i.e., $\check{Y}(x_i) = Y(-x_i)$, then $\check{Y}$ satisfies the same equation. By the uniqueness of solutions of the initial value problem the slopes of Y (or $\check{Y}$) on the outermost segments of the partition on

$[-1,1]$ must be different. Then by the symmetry of π neither $Y \leq \check{Y}$ nor $Y \geq \check{Y}$ are possible. Therefore, assuming $Y_{-K} < \check{Y}_{-K}$, there must exist a smallest integer $N \in \langle -K+1, K\rangle$ such that $Y_N > \check{Y}_N$. However, applying lemma 1, one may conclude that $\check{Y} \geq (\check{Y}_{-K}/Y_{-K})Y > Y$ on $\langle -K, N\rangle$, yielding a contradiction. For slightly stronger hypotheses, such as requiring that $f(u) + ku$ is increasing for some $k > 0$, this result can be proven using uniqueness results for u_0-concave operators [**7**].

The next lemma is a technical result which will only be used in a later section in connection with spurious solutions which bifurcate from the relevant solutions. The following definition is needed. If $S = \{\ldots, S_{-1}, S_0, S_1, \ldots\}$ is a doubly infinite sequence then $A_h(S)_i = (-S_{i-1} + 2S_i - S_{i+1})/h^2$.

LEMMA 3. *Suppose π is a uniform grid on $[-1,1]$ with step size h and that for $u > 0$: $f(\lambda,u) \geq 0, f_u(\lambda,u) \geq 0, f_\lambda(\lambda,u) > 0, f_{uu}(\lambda,u) \geq 0$. Suppose $Z = Z(\lambda,\tau)$ satisfies*

$$A_h Z = F(\lambda, Z) \equiv \{f(\lambda, Z_i)\} \tag{8}$$

with $Z_0 = R \geq 0$ and $Z_1 = R+\tau$. Let N_τ be a positive integer such that $(Z_{N-1} - Z_N)(Z_N - Z_{N+1}) \geq 0$ for all $N \in \langle 0, N_\tau - 1\rangle$. Let $\beta = \frac{1}{2}h^2$ if $f_{uuu}(\lambda, u) \geq 0$ for $0 \leq u \leq R$, otherwise let $\beta = h^2$. Then

(i) $\frac{\partial Z}{\partial \tau} > 0$ *on $\langle 0, N_\tau\rangle$ for $\tau \notin (-\beta f(\lambda,R), 0)$,*

(ii) $\frac{\partial Z}{\partial \lambda} < 0$ *on $\langle 2, N_\tau\rangle$.*

(iii) *Let R be a positive constant. Then $Z_n = 0$ together with $Z_0 = R, Z_1 = R+\tau$ and (8) defines λ as a function $\phi(\tau)$ on $[-R/N, 0]$ which is nondecreasing on $[-R/N, \overline{\tau}]$, where $\phi(-R/N) = 0$ and $\overline{\tau}$ is the smallest value $\geq -R/N$ such that $\overline{\tau} = -\beta f(\phi(\overline{\tau}), R)$.*

PROOF. Since $f(\lambda, u) \geq 0$ it follows that if $\tau \leq 0$ then N_τ can be taken arbitrarily large. If $\tau > 0$ then N_τ corresponds to the turning point of Z. Let $Y = \frac{\partial Z}{\partial \tau}$ and let $X_i = Z_{i+1} - Z_i$ if $\tau > 0$ and $X_i = Z_{i-1} - Z_i$ if $\tau \leq -h^2 f(\lambda, R)$ (this implies $X_0 \geq 0$). Then on $\langle 0, N_\tau - 1\rangle$

$$\begin{aligned} A_h Y &= F_u(\lambda, Z)Y, \qquad Y_0 = 0, Y_1 = 1 \\ A_h X &= F_u(\lambda, W)X, \qquad X_0 \geq 0 \end{aligned} \tag{9}$$

with $Z_i \leq W_i \leq Z_{i+1}$ if $\tau > 0$ or $Z_i \leq W_i \leq Z_{i-1}$ if $\tau \leq 0$. This means $F_u(\lambda, Z) \leq F_u(\lambda, W)$ so that, by Lemma 1, $Y \geq 0$ on $\langle 0, N_\tau \rangle$. If $f_{uuu} \geq 0$ and $\tau \leq -\frac{1}{2}h^2 f(\lambda, R)$ (implying $Z_{-1} \geq Z_1$) then a similar argument can be employed by letting $X_i = Z_{i-1} - Z_{i+1}$. To prove ii) let $V = \frac{\partial Z}{\partial \lambda}$ to obtain

$$A_h V = F_u(\lambda, Z)V + F_\lambda(\lambda, Z), \quad V_0 = 0 = V_1. \tag{10}$$

Define ϕ_i by $V_i = \phi_i X_i$ with $\phi_0 = \phi_1 = 0$. This is legitimate since $X_i > 0$ on $\langle 1, N_\tau \rangle$. Equation (10) then yields

$$\begin{aligned}(\phi_{i+1} - \phi_i)X_{i+1} + \phi_i(X_{i+1} - 2X_i + X_{i-1}) + (\phi_{i-1} - \phi_i)X_{i-1} \\ = -h^2 f_u(\lambda, Z_i)\phi_i X_i - h^2 f_\lambda(\lambda, Z_i).\end{aligned}$$

Next letting $\psi_i = \phi_i - \phi_{i-1}$ with $\psi_1 = 0$, a recursion relation for ψ can be obtained:

$$\begin{aligned}X_{i+1}\psi_{i+1} &= X_{i-1}\psi_i + \phi_i h^2 (f_u(\lambda, W_i) - f_u(\lambda, Z_i))X_i - h^2 f_\lambda(\lambda, Z_i) \\ &= X_{i-1}\psi_i + (\sum_{j=2}^{i} \psi_j) h^2 f_{uu}(\lambda, \tilde{W}_i)(W_i - Z_i)X_i - h^2 f_\lambda(\lambda, Z),\end{aligned}$$

with $\psi_1 = 0$. The restriction $\tau \notin (-\beta f(\lambda, R), 0)$ needed in proving part i) is not needed here. This can be seen from the above equation. Because $\phi_1 = 0$ the relation $W_1 \geq Z_1$ is not needed and because $\psi_1 = 0$ the restriction $X_0 \geq 0$ is not needed. Since $W_i \geq Z_i$ and $X_i \geq 0$ on $\langle 1, N_\tau \rangle, f_{uu} \geq 0, f_\lambda > 0$, it follows by induction that $\psi_i < 0$ for all $2 \leq i \leq N_\tau$. Therefore $\phi_i < 0$ for all $2 \leq i \leq N_\tau$ and consequently $\frac{\partial Z}{\partial \lambda} = V = \phi X < 0$ on $\langle 2, N_\tau \rangle$. Considering $Z_N(\lambda, \tau) =$ constant, (iii) becomes a trivial consequence of (i) and (ii).

It is of some interest to note that the failure of (i) to hold for $\tau \in (-\beta f(\lambda, R), 0)$ is exactly what allows the existence of more than one spurious solution with the same maximum point (x_{i_0}, U_{i_0}).

3. Bounds and estimates. In the previous section it was shown that monotone nonincreasing behavior of $f(u)/u$ near zero excludes the existence of small nonsymmetric solutions. Most of this section will be devoted to obtaining a priori upper bounds for large positive solutions which are nonsymmetric. Beyn and Lorenz showed that, for a uniform grid, large unsymmetric solutions asymptotically are of two types. The first consists of two linear pieces which meet at a point (x_i, U_i) where $U_i = \| U \|$. The second type attains its maximum at two neighboring grid points. In other words, these solutions asymptotically consist of three linear segments, the first connecting $(x_{-K-1}, 0)$ to $(x_i, \| U \|)$, the third connecting $(x_{i+1}, \| U \|)$ to $(x_{K+1}, 0)$. More precisely, if U is a nonsymmetric solution with large norm $\| U \|$ then $U/ \| U \|$ must be approximately equal to W_p or W_{pq}, where p and q are consecutive grid points with $p \neq -q$ and

$$W_p(x_i) = \begin{cases} \frac{x_i+1}{p+1} & \text{if } x_i \leq p \\ \frac{1-x_i}{1-p} & \text{if } x_i \geq p \end{cases} \tag{11a}$$

$$W_{pq}(x_i) = \begin{cases} \frac{x_i+1}{p+1} & \text{if } x_i \leq p \\ \frac{1-x_i}{1-q} & \text{if } x_i \geq q. \end{cases} \tag{11b}$$

If $p < 0$ and U attains its maximum value at $x_{i_0} \leq p$, then $\rho(U) = U(q)/U(p)$ is a parameter which will have to satisfy $(1-q)(1-p)^{-1} \leq \rho \leq 1$.

In the following lemma π can be any grid on $[-1, 1]$ without symmetry restriction, and p and $q = p + H$ are assumed to be consecutive grid with $p < 0$. In this case $\| U \|$ can be bounded a priori by a constant depending only on ρ, p and H, provided f satisfies

$$H_\infty : \lim_{M\to\infty} \frac{f(tM)}{f(M)} = 0 \text{ uniformly for } t \text{ in compact subsets of } [0, 1).$$

$$\limsup_{M\to\infty} \overline{f}(M)/\underline{f}(M) < \infty.$$

LEMMA 4. *Let f satisfy H_∞ and suppose $A_\pi U = \lambda F(U)$, then there exists a continuous function $B(p, \rho, H)$ such that $\| U \| \leq B(p, \rho, H)$ whenever U attains its maximum value at $x_{i_0} \leq p$ and $\rho(U) = \rho > (1-p-H)(1-p)^{-1}$. Except for its dependence on p and H this bound is otherwise independent of π and is also independent of λ.*

PROOF. For convenience it will be temporarily assumed that the grid-points x_i are labeled such that $x_0 = p$ and $x_1 = q = p + H$. There is no loss of generality in assuming that $H_i \equiv x_i - x_{i-1} \neq 0$ for all i. Also, letting $\rho_i = U_i/M$ with $M = U_0$, the i^{th} equation in the system $A_\pi U = \lambda F(U)$ becomes

$$(i) : \frac{\rho_i - \rho_{i+1}}{H_{i+1}} - \frac{\rho_{i-1} - \rho_i}{H_i} = \frac{1}{2}(H_i + H_{i+1}) \frac{\lambda}{M} f(\rho_i M).$$

Of course $H_1 = H, \rho_0 = 1$ and $\rho_1 = \rho$. We note that since U attains its maximum at $x_{i_0} \leq p$ it follows that $\rho_i < 1$ for $i > 0$.

Equation (0) can be used to bound λ/M:

$$\begin{aligned}\lambda/M &= \frac{2}{(H_0 + H_1)f(M)}\Big(\frac{1-\rho_{-1}}{H_0} + \frac{1-\rho_1}{H_1}\Big)\\ &\leq 2((p+1)^{-1} + 1/H)/(Hf(M)).\end{aligned}$$

Equations (i) with i ranging from 1 to $N-1$ can be summed to provide

$$-(1-\rho)/H + (\rho_{N-1} - \rho_N)/H_N = (\lambda/2M)\sum_{i=1}^{N-1}(H_i + H_{i+1})f(\rho_i M).$$

Choosing N such that $x_N = 1$ and $U_N = 0$ and using the above estimate on λ/M this yields

$$\begin{aligned}&-(1-\rho)/H + \rho_{N-1}/H_N\\ &\leq ((1+p)^{-1} + 1/H)\overline{f}(\rho M)/(f(M)H)\sum_{i=1}^{N-1}(H_i + H_{i+1}).\end{aligned}$$

On the other hand

$$\begin{aligned}-(1-\rho)/H + \rho_{N-1}/H_N &\geq -(1-\rho)/H + (1-\rho)^{-1}\\ &= [H(1-p)^{-1} - (1-\rho)]/H\end{aligned}$$

and hence

$$f(M)/\overline{f}(\rho M) \leq 4((1+p)^{-1} + 1/H)[H(1-p)^{-1} - (1-\rho)]^{-1}.$$

Defining $T_\rho(x) = \sup\{M : f(M)/\overline{f}(\rho M) \leq x\}$,this yields

$$M \leq T_\rho(4((1+p^{-1}) + 1/H)[\rho - (1-p-H)(1-p)^{-1}]^{-1}) < \infty$$

whenever $\rho > (1-p-H)(1-p)^{-1}$. The proof is now finished by noting that $\| U \| \leq 2(1-p)^{-1}M$, and $B_0(p,\rho,h) = 2(1-p)^{-1}T_\rho(4(1+p)^{-1}+1/H)(\rho-1+H(1-p)^{-1})^{-1})$ is bounded on compact subsets of its domain. Hence B can be chosen to be continuous with $B(p,\rho,h) \geq B_0(p,\rho,h)$.

The above lemma can be used to provide some asymptotic results: If $(\lambda^{(j)}, U^{(j)}), U^{(j)} \geq 0$, is a sequence of solutions to $A_\pi U = \lambda F(U)$

with $\| U_j \| \to \infty$ and each $U^{(j)}$ attaining its maximum at $p < 0$, then the sequence $\{\rho(U^{(j)})\}$ has two possible accumulation points, namely $\rho = (1-p-H)(1-p)^{-1}$ and $\rho = 1$. In the next section this will be strengthened to be a global result.

To conclude this section some simple bounds will be obtained for the eigenvalues. This will require the following notation:

$$E_i^{(p)} = \begin{cases} 0 & \text{if } x_i \neq p \\ 1 & \text{if } x_i = p \end{cases}, \text{ and } \mathbf{1} = \sum_{i=-K}^{K} E_i^{(p)}.$$

LEMMA 5. *If* (λ, U) *is a solution of* (5) *then*

$$\frac{3}{2} \| U \| / \bar{f}(\| U \|) \leq \lambda \leq 12\beta \tag{12}$$

where $\beta = \sup\{u/f(u) : u > 0\}$.

PROOF. Suppose U attains its maximum at x_k and let $Z = A_\pi^{-1} E^{(x_k)}$. It is easily seen that

$$Z(x_i) = \begin{cases} \frac{1}{4}(1+x_i)(x_{k+1}-x_{k-1})(1-x_k) & \text{if } x_i \leq x_k \\ \frac{1}{4}(1-x_i)(x_{k+1}-x_{k-1})(1+x_k) & \text{if } x_i \geq x_k \end{cases}$$

with $\| Z \| = \frac{1}{4}(x_{k+1}-x_{k-1})(1-x_k^2)$. Now, since $f(U) \leq \bar{f}(\| U \|)\mathbf{1}$ and A_π^{-1} is monotone, it must be true that $\| U \| \leq \frac{1}{4}\lambda \bar{f}(\| U \| \times \sum_{i=-K}^{k}(x_{i+1}-x_{i-1})(1-x_1^2)$. The summation is merely twice the trapezoid quadrature rule applied to $y = 1-x^2$ on $[-1,1]$, and hence it is bounded by twice the area under this curve which equals 8/3. This yields the first inequality in (12). Let $\phi = 1/6 - x^2/3 + | x |^3 /6$. A straightforward calculation yields

$$(A_\pi \phi)_i = 2/3 - | x_i | + \begin{cases} (1/3)(h_i - h_{i-1}) & \text{if } x_i < 0 \\ (1/6)(h_i + h_{i-1}) & \text{if } x_i = 0 \\ (1/3)(h_{i-1} - h_i) & \text{if } x_i > 0. \end{cases}$$

Therefore $A_\pi^{-1} W_0 \geq \phi$ (see (11)) and $\| A_\pi^{-1} W_0 \| \geq 1/6$. Consequently $U = \lambda A_\pi^{-1} F(U) \geq \lambda A_\pi^{-1}(\beta^{-1} U) \geq \frac{1}{2}\lambda\beta^{-1} \| U \| A_\pi^{-1} W_0$ and $\| U \| \geq (1/12)\lambda\beta^{-1} \| U \|$ which yields the second inequality in (12).

Using the notation of Lemma 4 and adding the equations (i) for $i = i_0, i_0 + 1, \ldots, -1$, on obtains

$$-\frac{U_{i_0} - U_{i_0+1}}{H_{i_0+1}M} + \frac{1-\rho}{H} = \frac{\lambda}{2M} \sum_{i_0+1}^{-1} (H_i + H_{i+1}) f(U_i) \geq$$

$$\frac{\lambda}{2M}\underline{f}(M) \sum_{i=i_0+1}^{0} H_i \geq \frac{3}{4}\frac{\| U \|}{M}[\overline{f}(M)/\underline{f}(M)] \sum_{i=i_0+1}^{0} H_i$$

or

$$(13) \qquad | p - x_{i_0} | \leq \frac{4M(1-\rho)}{3 \| U \| H}[\overline{f}(M)/\underline{f}(M)], \| U \| = U(x_{i_0}).$$

4. Branches of Nonsymmetric Solutions. In this section the grid π will be allowed to depend on a homotopy parameter t. The following restrictions will be assumed:

(i) π_t (i) depends continuously on t,

(ii) π_t is odd for each t (i.e., the grid is symmetric) and nondecreasing,

(iii) as t varies, pairs of neighboring grid points are allowed to coalesce,

(iv) $\pi_t(i_0) \equiv x_{i_0} < 0, \pi_t(i_0 + 1) \equiv x_{i_0} + H = x_{i_0+1}$.

During the homotopy the system of equations are of the form (6) in order to allow grid points to coalesce and still provide a well defined system of equations which changes smoothly with t. However, at each fixed t the system is equivalent to one of the form (5).

The grid π_0 is the one which is part of the given problem. As t increases this grid can be made more simple. Due to the requirements ii) and iv) the simplest allowed grid is one with four points: $x_{i_o}, x_{i_0} + H, -x_{i_0} - H, -x_{i_0}$. It is easily seen that one can design a homotopy which will smoothly change π_0 into this particular simple grid while satisfying requirements i) - iv). At this time it may be noted that these requirements made it necessary to use $\mathbf{Z} - \{0\}$ as an index set. If 0 is a grid point then it must correspond to an even number of integers, including -1 and 1. Now, as t is increased x_1 and x_{-1} separate and eventually merge with x_{i_0+1} and x_{-i_0-1} respectively. For any subset S of $[\rho_1, \rho_2] \times \mathbf{R}^m$: let S_ρ denote $S \cap \{\rho\} \times \mathbf{R}^m$. Suppose that G is a relatively open subset of $[\rho_1, \rho_2] \times \mathbf{R}^m$ such that G_ρ is bounded for

$\rho_1 < \rho < \rho_2$. Let $\Phi : G \to \mathbf{R}^m$ be a continuous map such that for each $\rho \in (\rho_1, \rho_2)$ there are no solutions of $\Phi(\rho, x) = 0$ with $x \in (\partial G)_\rho$. It is then well known that the topological degree $d = \deg(\Phi(\rho, \cdot), G_\rho, 0)$ is constant on (ρ_1, ρ_2). If $d \neq 0$ then in fact there must exist a branch, or closed connected set, S_1 of solutions which joins $\mathbf{R}_1^* = \{\rho_1\} \times \mathbf{R}^*$ to $\mathbf{R}_2^* = \{\rho_2\} \times \mathbf{R}^*$, where $\mathbf{R}^*$ is the one-point compactification of $\mathbf{R}^m$. More precisely $\mathbf{R}_1^* \cup S \cup \mathbf{R}_2^*$ is a compact connected set, or continuum, in $[\rho_1, \rho_2] \times \mathbf{R}^*$. To see this let T be the set of all solutions of $\Phi(\rho, x) = 0$ in $\overline{G}$ and let $T^* = \mathbf{R}_1^* \cup T \cup \mathbf{R}_2^*$. Then T^* is a compact metric space. If this space does not contain a continuum which contains $\mathbf{R}_1^* \cup \mathbf{R}_2^*$ then [**15**, p.15] there must exist a separation $T^* = T_1^* \cup T_2^*$ into two disjoint compact sets with $T_1^* \supset \mathbf{R}_1^*$ and $T_2^* \supset \mathbf{R}_2^*$. This means there must exist an open set $G^* \subset [\rho_1, \rho_2] \times \mathbf{R}^*$ containing T_1^* but not intersecting T_2^*. But then $\sup\{\rho : (\rho, x) \in G^*$ for some $x \in X^*\} < \rho_2$ and hence G_ρ^* is empty for ρ close to ρ_2. Let $H = G^* \cap G$. Since there are no solutions on $\partial H_\rho \subset \partial G_\rho^* \cup \partial G_\rho$ for any $\rho \in (\rho_1, \rho_2), \deg(\Phi(\rho, \cdot), H_\rho, 0)$ is constant for $\rho \in (\rho_1, \rho_2)$. Since $H_\rho = \emptyset$ for ρ sufficiently large this implies that $\deg(\Phi(\rho, \cdot), H_\rho, 0) = 0$ for ρ close to ρ_1. However G^* must contain a slice $[\rho_1, \rho_1 + \delta) \times \mathbf{R}^*$ for some $\delta > 0$ and hence $\partial H_\rho = \partial G_\rho$ for ρ sufficiently small. But this implies that $\deg(\Phi(\rho, \cdot), H_\rho, 0) \neq 0$ for ρ close to ρ_1, a contradiction. This idea can be applied to the map $\Phi_t(\sigma, \lambda, U) = (\sigma - \rho(U), A_{\pi_t} U - \lambda F(U))$ where, it will be recalled, $\rho(U) = U(q)/U(p)$ with p and q neighboring grid points. The set G must be an appropriately chosen subset of $\mathbf{R}^1 \times \mathbf{R}^{2K}$. Using the a priori bounds it is known that

$$G = U_{\rho \in J} G_\rho, G_\rho = \left(\frac{3}{2} M/f(M), 12\beta\right) \times (0, M)^{2K},$$
$$J = ((1-q)(1-p)^{-1}, 1),$$

where $M = B(p, \rho, H)$, has the property of not having solutions on ∂G_ρ. Also, this is true for each family of partitions π_t satisfying i) - iv). Consequently, if it can be shown that $\deg(\Phi_1(\rho, \cdot), G_\rho, 0)$, corresponding to the simple grid π_1 is not zero then the following theorem holds.

THEOREM 6. *Suppose f is continuous and positive on $(0, \infty)$ and satisfies: $f(u)/u$ is monotone nonincreasing on $(0, \alpha), \alpha > 0, \lim_{M \to \infty} f(tM)/f(M) = 0$ uniformly for t in compact subsets of $[0, 1)$. Let π be a symmetric grid on $[-1, 1]$, and let $p < 0$ and $q = p + H$ be two* *neighboring grid points of* π. *Then the problem* $A_\pi U = \lambda f(U)$ *has a branch of positive solutions* (λ, U) *which attain their maximum at or to the left of* p. *On this branch* $\rho(U) = U(q)/U(p)$ *takes on all values in* $((1-q)/(1-p)^{-1}, 1)$. *Moreover, for* $\rho(U)$ *sufficiently close to* 1, U *attains its maximum at* p

COROLLARY 7. *Assume the hypothesis of the theorem and the additional assumption that* π *is a uniform grid. Corresponding to each pair of neighboring grid points* p *and* q *with* $-1 < p < q$ *there exists a branch of positive solutions* (λ, U) *on which* $\| U \| = U(p)$ *and* ρ *attains all values in* $((1-q)(1-p)^{-1}, 1)$. *For each* $M > 0$ *there exists a* $0 < \delta < \frac{1}{2}(q-p)(1-p)^{-1}$ *such that* $\| U \| > M$ *whenever* $\rho(U) \notin ((1-q)(1-p)^{-1} + \delta, 1-\delta)$.

The corollary follows from the theorem and the fact that if U_i is a sequence of solutions with either $\rho(U_i) \uparrow 1$ or $\rho(U_i) \downarrow (1-q)(1-p)^{-1}$ then $\| U \| \to \infty$. To prove this fact suppose the $\| U_i \|$ stayed bounded. Then, by continuity there would exist a solution U with either $\rho(U) = 1$ or $\rho(U) = (1-q)(1-p)^{-1}$. But if $\rho(U) = 1$, then by symmetry $U(q + | p |) = 0$, which is impossible since $q + | p | < 1$. If $\rho(U) = (1-q)(1-p)^{-1}$ then U would have to be linear on $[q, 1]$ and hence $\lambda = 0$, an impossibility. Clearly, since U attains its maximum at p for $\rho(U)$ near 1, and since there are no solutions (λ, U) for which U attains its maximum at more than one-grid-point, it follows that $U(p) = \| U \|$ along the entire branch.

PROOF OF THEOREM 6. The grid π_1 consists of the points $p < p+H \le -p-H < -p$, where $p = x_{i_0}$ and $p+H = -M \le 0$. Equation (5) essentially has been reduced to a system of four equations in four unknowns: $\lambda, U(p), U(-p-H)$ and $U(-p)$. This system of equations is still considerably complicated and therefore will be simplified even more by introducing another homotopy parameter m.. The grid π_{1+m} will consist of the points $p < p+H+m \le -p-H-m < -p, 0 \le m \le M \equiv | p+H |$. When the grid points are moved in this fashion, no solutions will appear on ∂G_ρ for $\rho \in ((1-p-H)(1-p)^{-1}, 1)$. Before showing this it should be noted that for each of the values of m, ρ will remain within the permissible range; in fact $\rho > (1-p-H)(1-p)^{-1} > (1-p-H-m)(1-p)^{-1}$. Next, it follows from the last line of the proof of Lemma 4 that the a priori upper bound for $\| U \|$ remains valid even if H is replaced by the larger value $H + m$ as long as $H + m \leq 1 - p$; equality occurs if $m = M$. The lower bound for $\| U \|$ remains independent of the grid. The a priori bounds provided by Lemma 5 now obviously also remain valid for the indicated range of values of $H + m$. This means $\deg(\Phi_t(\rho, \cdot), G_\rho, 0)$ remains constant for $t \in [0, 1+M]$. The problem for the grid π_{1+M} can be stated as follows.

Let $r = \rho M$ and let $-ar$ and $-cr$ be the slopes of the piecewise linear interpolant of the solution U on the segment $[p, 0]$ and $[0, -p]$ respectively. Let $k = 1 + p$ and $h = | p |$. This situation can be tabulated as follows

gridpoint	-1		p		0		$-p$		1
gridspacing		k		h		h		k	
value of U	0		$r(1+a)$		r		$r(1-c)$		0
slope		$(1+ah)r/k$		$-ar$		$-cr$		$(ch-1)r/k$	

The equations for U are then

$$r(1+ah)/k + ar = \frac{1}{2}(k+h)\lambda f(r(1+ah)) \tag{14a}$$

$$cr - ar = h\lambda f(r) \tag{14b}$$

$$r(1-ch)/k - cr = \frac{1}{2}(k+h)\lambda f(r(1-ch)). \tag{14c}$$

Equation (14a) can be solved for λ in terms of r. This value of λ can be substituted into (14b) to give c. Finally from (14c) one obtains the equation

$$\phi(r) \equiv r(1 - c(r)h) - kc(r)r - \frac{1}{2}k(k+h)\lambda(r) f(r(1 - c(r)h)) = 0.$$

Since $\| U \| \leq 2r$ and it is known from Theorem 2 that there are no nonsymmetric solutions of norm less than α, it suffices to restrict r to the interval $[\alpha/2, \infty]$. Evaluating $\phi(\alpha/2)$, using (14a), on obtains

$$\phi(\alpha/2) = (\alpha/2)[1 - c(\alpha/2)h - kc(\alpha/2) - (1 + ah + ak)R]$$

where $R = f(\alpha(1-c(\alpha/2)h)/2) f(\alpha(1+ah)/2) \geq (1-hc(\alpha/2))/(1+ha)$.

Hence it follows that $\phi(\alpha/2) < (\alpha/2)(-kc(\alpha/2)) < 0$. On the other hand, as $r \to \infty$, one has

$$\frac{1}{2}k(k+h)\lambda f(r(1-c(r)h)) = r(1+ah+ak)f(r(1-ch))/f(r(1+ah))$$

which tends to 0 as $r \to \infty$. Hence $\lim_{r\to\infty} \inf u(r) \geq \lim_{r\to\infty} \inf r(1-c(r)h) - c(r)kr \geq \lim_{r\to\infty} r(1-c(r)) = \infty$. This means that $\phi(r)$ has an odd number of simple roots (or at least so after an arbitrarily small perturbation) and hence $\deg(\Phi_{1+M}(\rho,\cdot), G_\rho, 0)$ is odd.

5. Bifurcation from the branch of symmetric solutions. Throughout this section, it will be assumed that π is a uniform grid on $[-1,1]$ with spacing $h = 2/N$. If N is an even integer, then 0 will be one of the grid points while if N is odd, the grid points nearest to 0 are $\pm h/2$. For this second case, it will be shown that there are unbounded branches Γ_+ and Γ_- of solutions (λ, U) with U attaining its maximum at $+h/2$ and $-h/2$ respectively, and that these two branches bifurcate from a point on the branch of symmetric solutions.

As before, $F(\lambda, U)$ and $\lambda F(U)$ will denote the vectors with entries $f(\lambda, U_i)$ and $\lambda f(U_i)$ respectively.

THEOREM 8. *Suppose* $f(0,u) = 0$ *and that in the region* $(\lambda, u) \in (0,\infty) \times (0,\infty) f \geq 0, f_u \geq 0, f_\lambda > 0, f_{uu} \geq 0$. *Also, in case* $2/h$ *is even, suppose in addition that* $f_{uuu} \geq 0$. *Then* (5) *has exactly one symmetric solution* (λ, U) *for each value of* $\| U \|$ *in* $[0, \infty)$.

PROOF. The proof follows directly from Lemma 3. If $N = 2/h$ is even, then define $p = 0$ and $\beta = \frac{1}{2}h^2$. If, on the other hand, N is odd, then $p = h/2$ and $\beta = h^2$. Let V_τ satisfy $A_\pi V_\tau = F(\lambda, V_\tau)$ with $V_\tau(p) = M$ and $V_\tau(p+h) = M + \tau, \tau \leq 0$. By Lemma 3 (iii) this, together with the restriction $V_\tau(1) = 0$, (defines λ as a function Λ of τ on $[-hM\ (1-p)^{-1}, 0]$ which is nondecreasing whenever $\tau \leq -\beta f(\lambda, M)$. Moreover, equation (8) at $x_i = p$ yields $\tau = -\beta f(\lambda, M)$. This means that the graphs of $\lambda = \Lambda(\tau)$ and $\tau = -\beta f(\lambda, M)$ will intersect exactly once to provide a unique solution of norm M. To see this one notes that the first intersection will occur at $\tau = \overline{\tau}$ and a second intersection would require Λ to be decreasing at that intersection, thus violating Lemma 3.

For the rest of this section, it will be assumed that $N = 2/h$ is odd and that $f(\lambda, u) = \lambda(u)$ with $f \geq 0, f' \geq 0$ and $f'' \geq 0$. The basic idea used to prove the bifurcation result will be to consider two problems: equation (3) on $[-1, h/2]$ and equation (3) on $[-h/2, 1]$ and to look for pairs of solutions which can be spliced together to form a solution for (5) on $[-1,1]$. Consider a solution V of (3) with $V_{-1} = V(-h/2)$, $V_0 = V(h/2)$, $V_1 = V(3h/2)$, $V_2 = V(5h/2), \ldots, V_n = V(1-h), V_{n+1} = V(1)$. Now, let $M = \frac{1}{2}(V_{-1} + V_0), \sigma = \frac{1}{2M}(V_0 - V_{-1})$ and require that $V_{n+1} = 0$. Let $V = (V_1, V_2, \ldots, V_n)^T$, then finding such a solution V is tantamount to solving the system of equations

$$(15) \qquad \Phi(\lambda, V, \sigma, M) \equiv TV - \lambda h^2 F_*(M, \sigma, V) + MG(\sigma) = 0$$

where T is the $(n+1) \times n$ matrix

$$T = \begin{bmatrix} -1 & & & & & & \cdots & & 0 \\ 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & \ddots & \vdots \\ & -1 & 2 & -1 & & & & & \\ & & & \cdots & & & & & \\ & & & & \cdots & & & & \\ \vdots & \ddots & & & & \cdots & & & \\ & & & & & & -1 & 2 & -1 \\ 0 & & \cdots & & & & & -1 & 2 \end{bmatrix}$$

$F_*(M, \sigma, U)^T = (f(M(1+\sigma)), f(U_1), \ldots, f(U_n))$ and $G(\sigma)^T = (1 + 3\sigma, -1 - \sigma, 0, \ldots, 0)$. Now suppose that $\Phi(\lambda, V, \sigma, M) = 0$ and $\Phi(\lambda, W, -\sigma, M) = 0$, then

$$(16) \quad U = (W_n, W_{n-1}, \ldots, W_1, M(1-\sigma), M(1+\sigma), V_1, V_2, \ldots, V_n)^T$$

is a solution of (3) with boundary conditions $U(-1) = U(1) = 0$. Conversely, if U is a solution to (3) with $U(-1) = U(1) = 0$, then defining $M = \frac{1}{2}(U(h/2) + U(-h/2))$ and $\sigma = \frac{1}{2M}(U(h/2) - U(-h/2))$ and $V_i = V(h/2 + ih), i = 1, 2, \ldots, n$, one obtains a solution of $\Phi(\lambda, V, \sigma, M) = 0$ and similarly, a solution of $\Phi(\lambda, W, -\sigma, M) = 0$. An investigation of the map Φ will lead to a proof of the bifurcation result. However, before proceeding with this, it may be of some interest to obtain a more graphical description of the situation under the present hypotheses. Let $N = M(1+ \mid \sigma \mid)$ and

$$\Psi(\lambda, V, \sigma, N) \equiv \Phi(\lambda, V, \sigma, N(1+ \mid \sigma \mid)^{-1}).$$

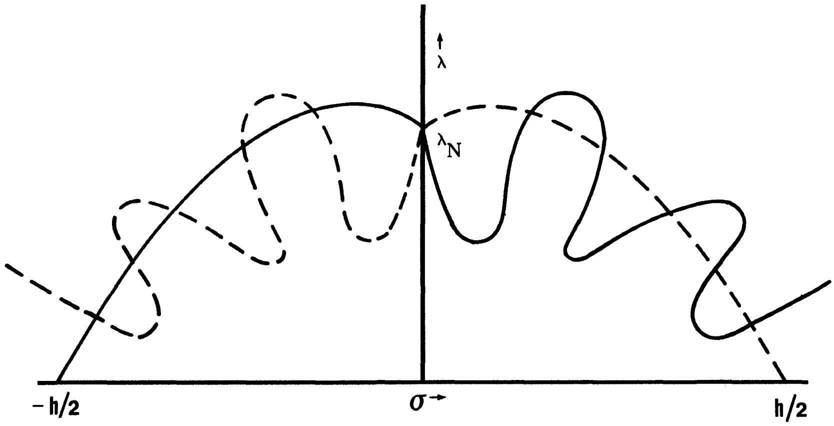

Figure 2. The curves C_N (solid) and $\check{C}_N$.

Clearly, there is a one-to-one correspondence between solutions of (3) with $\max(U(-h/2), U(h/2)) = N$ and intersections of the sets C_N and $\check{C}_N$ where

$$C_N = \{(\sigma, \lambda) : \sigma \geq -h/2, \quad \lambda \geq 0, V \geq 0, \Psi(\lambda, V, \sigma, N), = 0\}$$

and $\check{C}_N = \{(\sigma, \lambda) : (-\sigma, \lambda) \in C_N\}$. The set C_N is in fact a simple curve emanating from $(-h/2, 0)$. If $(\sigma, \lambda) \in C_N$ with $\sigma > -h/2$ then $\lambda > 0$. This curve passes through $(0, \lambda_N)$ where λ_N is the eigenvalue corresponding to the (unique) symmetric solution of norm N. The curve C_N is smooth except at $(0, \lambda_N)$. To see this, one may apply Lemma 3(ii). For $-h/2 \leq \sigma \leq 0$, this immediately yields λ as a function of σ and hence for these values of σ the curve C_N is the graph of this (continuously differentiable) function. By Theorem 8 there exists a unique value λ_N such that $(0, \lambda_N) \in C_N$. Next, consider $\sigma \geq 0$. Let $V(3h/2) = N + \tau, \tau \leq 0$, so that the condition $V(1) = 0$ determines a function $\lambda = \Lambda_N(\tau)$ on $[-hN(1-h/2)^{-1}, 0]$ with $\Lambda_N(-hN(1-h/2)^{-1}) = 0$ and $\Lambda_N(0) = \tilde{\lambda}_N$, the eigenvalue corresponding to the unique symmetric solution of norm N on an interval of length $2 - h$ (with mesh-size h). Now, since equation (3) also has to be satisfied at the point $(h/2, N)$ one also needs $2\sigma(1+\sigma)^{-1}N - \tau = h^2\lambda f(N)$ or $\sigma = [\tau + h^2\lambda f(N)]/[2N - \tau - h^2\lambda f(N)]$. Hence, each point on the graph of Λ_N corresponds to exactly one solution of $\Psi(\lambda, V, \sigma, N) = 0$. This means that one may, for example, use the arc length along the graph of Λ_N to parametrize the curve of solutions (λ, V, σ) for $\sigma \geq 0$. The part of C_N for $\sigma \geq 0$ is just the projection of this curve onto the

(σ, λ)-plane. One easily verifies this curve is simple. Obviously C_N may, and in general will, fail to be smooth at $(0, \lambda_N)$. It is intuitively clear that as one changes N, the curves C_N and $\check{C}_N$ change smoothly. However, they always intersect at $\sigma = 0$ (there is a symmetric solution for each value of the norm N). As N varies nonsymmetric solutions, i.e., solutions which correspond to other intersections of C_N and $\check{C}_N$, may appear and disappear in pairs as "loops" of $\check{C}_N$ dip below and rise above C_N in the region $-h/2 < \sigma < 0$. The intersections of C_N and $\check{C}_N$ closest to the λ-axis correspond to solutions on Γ_+ and Γ_-. Incidentally, for these solutions N represents the norm. Whether or not Γ_+ and Γ_- bifurcate from the branch of symmetric solutions therefore depends on the local behavior of the curves C_N (or equivalently the maps Ψ or Φ) at $\sigma = 0$.

LEMMA 9. *Consider the solutions of* $\Phi(\lambda, V, \sigma, M) = 0$, *with* $V = (V_1, V_2, \ldots, V_n)^T$*:*

(i) *There exist constants* $M_0 > 0, 0 < \rho < 1$, *and* $\varepsilon > 0$ *such that* $V_1 < \rho M$ *whenever* $M \geq M_0$ *and* $| \sigma | < \varepsilon$.

(ii) *There exist positive constants* c_1 *and* C_1 *such that*

$$c_1 M/\bar{f}(M(1+\sigma)) \leq \lambda \leq C_1 M/f(M(1+\sigma))$$

whenever $| \sigma | < \varepsilon$.

PROOF. To prove (i), it suffices to prove that there exists a constant $\rho, 0 < \rho < 1$, such that $V_1 \leq \rho(1+\sigma)M$ whenever $| \sigma | < \varepsilon$ and $M \geq M_0$. One can then decrease ε sufficiently such that $0 < \rho(1+\sigma) < 1$ for $| \sigma | < \varepsilon$. If (i) were false, then there would exist a sequence of solutions $(\lambda^{(i)}, V^{(i)}, \sigma^{(i)}, M^{(i)})$ with $\sigma^{(i)} \to 0, M^{(i)} \uparrow \infty$ and $V_1^{(i)}/[M^{(i)}(1+\sigma^{(i)})] \uparrow 1$. Now, letting $Z^{(i)} = [M^{(i)}(1+\sigma^{(i)})]^{-1} V^{(i)}$ and $\mu^{(i)} = \lambda^{(i)} h^2 f(M^{(i)}(1+\sigma^{(i)}))/[M^{(i)}(1+\sigma^{(i)})]$ one arrives at the system of equations

$$\begin{aligned} &TZ^{(i)} - \mu^{(i)} f(M^{(i)}(1+\sigma^{(i)}))^{-1} \\ &\quad F_*(M^{(i)}, \sigma^{(i)}, M^{(i)}(1+\sigma^{(i)})Z^{(i)}) + G(\sigma^{(i)}) = 0. \end{aligned}$$

The first equation of this system shows that $\mu^{(i)} \to 0$. This means $Z^{(i)} \to Z$ where $TZ = (-1, 1, 0, \ldots, 0)^T \notin$ Range (T), a contradiction.

To prove (ii), one may use an argument similar to that used to prove Lemma 5. First, let $\overline{U} = (M(1+\sigma), U_1, \ldots, U_n)^T$ and let T_σ be the $(n+1)\times(n+1)$ matrix with tridiagonal entries $-1, 2-1$, modified such that the entry in the upper left hand corner is not 2 but $(1+3\sigma)/(1+\sigma)$. Then

$$T_\sigma \overline{U} = \lambda F(\overline{U}).$$

Again, letting $\mathbf{1}$ denote the vector of whose entries are 1, one sees that the above equation implies that $M(1+\sigma)\mathbf{1} \leq \lambda \overline{f}(M(1+\sigma))T_\sigma^{-1}\ \mathbf{1}$. Since T_σ is easily seen to be nonsingular, it is true that $\|\ T_\sigma^{-1}\ \|$ is uniformly bounded for all σ in some interval $[-\varepsilon, \varepsilon]$ and hence, $\lambda \geq c_1 M/\overline{f}(M(1+\sigma))$ for some positive constant c_1. To obtain the second inequality, one simply observes that the first component of (15) yields

$$\lambda f(M(1+\sigma)) = M(1+3\sigma) - U_1 \leq M(1+3\sigma) - (1-3h/2)(1-h/2)^{-1}M(1+\sigma)$$

and hence, since $|\ \sigma\ | \leq h/2, \lambda M(3h)/f(M(1+\sigma))$.

In order to show that Γ_+ and Γ_- bifurcate from the branch of symmetric solutions the hypothesis H_∞ will be replaced by

H'_∞ $\lim_{M\to\infty} \frac{Mf'(M)f(tM)}{f(M)^2} = 0$ uniformly for t in compact subsets of $[0,1)$ and is not a constant function.

LEMMA 10. *Suppose* $f \geq 0, f' \geq 0, f'' \geq 0$, *then*

(i) H'_∞ *implies* H_∞.

(ii) H'_∞ *implies*

$$\lim_{x\to\infty} xf'(x)/f(x) = \infty, \tag{17}$$

$$\lim_{x\to\infty} \frac{xf'(tx)}{f(x)} = 0 \text{ uniformly for } t \text{ in compact subsets of } [0,1). \tag{18}$$

PROOF. Since $f'' \geq 0, xf'(x) \geq f(x) - f(0)$ and hence (i) follows from the fact that $xf'(x)/f(x) \geq 1 - f(0)/f(x_0) > 0$ for some $x_0 > 0$ and $x > x_0$. To prove (17), one notes that for any $t \in [0,1), f(x) - f(tx) \leq (1-t)xf'(x)$ and hence $\liminf_{x\to\infty} xf'(x)/f(x) \geq (1-t)^{-1}$, where t can be chosen arbitrarily close to 1. Similarly, since

$f(x(1+t)/2) \geq f(tx) + f'(tx)(1-t)x/2$, where $(1+t)/2 < 1$, one has $0 \leq \lim_{x\to\infty} xf'(tx)/f(x) \leq \lim_{x\to\infty} 2(1-t)^{-1} f(x(1+t)/2)/f(x) = 0$.

Now let $f_i = 2 - \lambda f'(U_i)$ and let Φ_* denote the partial derivative of Φ with respect to (U, σ):

$$\Phi_* = \begin{bmatrix} -1 & 0 & & \cdots & & 0 & 3M - \lambda h^2 M f'(M(1+\sigma)) \\ f_1 & -1 & & & & & -M \\ -1 & f_2 & -1 & \ddots & & \vdots & 0 \\ 0 & & & & & & \\ & & \ddots & & & & \vdots \\ \vdots & \ddots & & & & 0 & \\ & & & -1 & f_{n-1} & -1 & \\ 0 & \cdots & & 0 & -1 & f_n & 0 \end{bmatrix}$$

Lemmas 9 and 10 can now be used to conclude that for large M, Φ_* will be approximately equal to the matrix obtained from Φ_* by setting $\lambda f'(U_i) = 0$ in the subdiagonal. Also, since $3 - h^2 f'(M(1+\sigma)) \to -\infty$ as $M \to \infty$, it is clear that for large M $\det \Phi_* \approx (-1)^n M f'(M(1+\sigma)) \det B_n$ where B_n is the $n \times n$ tridiagonal matrix with entries $-1, 2, -1$. This means that M_0 may be increased and ε decreased to ensure that for any fixed $M \geq M_0$ the values (σ, λ) corresponding to solutions of $\Phi(\lambda, U, \sigma, M) = 0$ with $|\sigma| < \varepsilon$ constitute a curve passing through the point $(0, \lambda_M)$. Letting $\mu = \lambda M^{-1} f(M(1+\sigma))$, this provides a function $\phi_M(\sigma)$ such that if $\phi(\lambda, U, \sigma, M) = 0$ and $|\sigma| < \varepsilon$ then $\mu = \phi_M(\sigma)$.

REMARK. Suppose

$$\begin{aligned} TZ(\sigma, M) = {} & m(\sigma, M)(0, \alpha_1(\sigma, M), \ldots, \alpha_n(\sigma, M))^T \\ & + r(\sigma, M)(1, 0, \ldots, 0)^T + (0, \beta_1(\sigma, M), \ldots, \beta_n(\sigma, M))^T \end{aligned}$$

where $m(\sigma, M)$ is uniformly bounded for $M \geq M_0$ and $|\sigma| \leq \varepsilon, \alpha_i(\sigma, M) \to 0$ and $\beta_i(\sigma, M) \to 0$ uniformly for $|\sigma| \leq \varepsilon$ as $M \to \infty$. Then $Z(\sigma, M) \to 0$ and $r(\sigma, M) \to 0$ uniformly for $|\sigma| \leq \varepsilon$ as $M \to \infty$. The proof of this follows immediately from writing down the solution:

$$Z_i = (n+1)^{-1}(n+1-i)\sum_{j=1}^{n} j\gamma_j - \sum_{j=1+i}^{n} (j-i)\gamma_j$$

$$r = -Z_1 = -n(n+1)^{-1}\sum_{j=1}^{n} \gamma_j, \text{ where } \gamma_j = m(\sigma, M)\alpha_j(\sigma, M) + \beta_j(\sigma, M)$$

THEOREM 11. *Suppose* $f > 0, f' \geq 0, f'' \geq 0$ *on* $(0, \infty)$ *and satisfies* H_0 *and* H'_∞. *Consider problem* (5) *for a uniform grid on* $[-1, 1]$ *with gridspacing* h *such that* $2/h$ *is odd. There exist two unbounded branches,* Γ_+ *and* Γ_-, *of nonsymmetric solutions* (λ, U) *which bifurcate from a point on the branch of symmetric solutions. If* $(\lambda, U) \in \Gamma_+$ *(resp.* Γ_-*) then* U *attains its maximum value at* $h/2$ *(resp.* $-h/2$*).*

PROOF. The hypotheses of Theorem 6 are satisfied. Therefore, by the arguments used in Corollary 7, there exists a branch Γ_- of positive solutions (λ, U) along which $\| U \| = U(p)$ and the parameter $\rho = \rho(U) = U(h/2)/U(-h/2)$ takes on all values in $(\rho_0, 1), \rho_0 = (1 - h/2)/(1 + h/2)$. Again, as in the proof of Theorem 6, for ρ sufficiently close to ρ_0 the values of $\| U \|$ become arbitrarily large. However, in the present case, it may be true that for some sequence of solutions $(\lambda^{(i)}, U^{(i)}), \| U^{(i)} \|$ stays bounded while $\rho(U^{(i)}) \to 1$. In this case, Γ_- must be connected to a symmetric solution. Therefore, it suffices to prove that there exist positive values M_0 and δ such that there are no solutions $(\lambda, U) \in \Gamma_-$ with $\| U \| \geq M_0$ and $\rho(U) > 1 - \delta$. This is equivalent to showing that there are no solutions of (15) with $M \geq M_0$ and $\sigma < \delta$. But, since a solution of (15) with $| \sigma | < \varepsilon$ corresponds to an intersection of graphs of $\mu = \phi_M(\sigma)$ and $\mu = \phi_M(-\sigma)$ it will suffice to prove that the functions $\phi_M(\sigma)$ converge to a strictly increasing function ϕ in $C^1[-\varepsilon, \varepsilon]$ as $M \to \infty$. Here $\mu = \phi_M(\sigma)$ is a solution of

$$TV - \mu \tilde{F}(M, V, \sigma) - \tilde{G}(\sigma) = 0 \tag{19}$$

where $\tilde{F}^T = (1, f(M(1 + \sigma))^{-1} F(MV)^T)$ and $\tilde{G} = (-1 - 3\sigma, 1 + \sigma, 0, \ldots, 0)^T$. Letting represent derivatives with respect to σ this yields

$$T\dot{V} - \dot{\mu} \tilde{F}(M, V, \sigma) - \mu H - (-3, 1, 0, \ldots, 0)^T \tag{20}$$

where $H^T = (0, H_1, \ldots, H_n)$ with

$$H_i = Mf(M(1+\sigma))^{-2}[f(M(1+\sigma))f'(MV_i)\dot{V}_i - f(MV_i)f'(M(1+\sigma))].$$

Letting $M \to \infty$ in (19) and using Lemma 9, one obtains

$$T\tilde{V} - \tilde{\mu}(1, 0, 0, \ldots, 0)^T + (1 + 3\sigma, -1 - \sigma, 0, \ldots, 0)^T = 0 \tag{21}$$

Next, letting $\Delta = V - \tilde{V}$ yields

$$T\Delta = \mu(0, \eta_1, \ldots \eta_n)^T + (\mu - \tilde{\mu})(1, 0, \ldots, 0)^T$$

where $0 \le \eta_i \equiv f(MV_i)/f(M(1+\sigma)) \le f(\rho M)/f(M)$. By the remark above the statement of the theorem $\Delta \to 0$ and $\mu \to \tilde{\mu}$ uniformly on $[-\varepsilon, \varepsilon]$. Similarly, letting $\dot{\Delta} = \dot{V} - \dot{\tilde{V}}$

$$T\dot{\Delta} = \dot{\mu}(0, \eta_1, \ldots, \eta_n)^T + \mu(0, \dot{\eta}_1, \ldots, \dot{\eta}_n) + (\dot{\mu} - \dot{\tilde{\mu}})(1, 0, \ldots, 0)^T,$$

where $| \dot{\eta}_i |=| f(M(1+\sigma))^{-2} f(MV_i) f'(M(1+\sigma))M | \le f(M)^{-2} f(\rho M)$ $f'(M)M \to 0$ as $M \to \infty$. This implies that $\dot{\Delta} \to 0$ and $\dot{\mu} \to \dot{\tilde{\mu}}$ uniformly on $[-\varepsilon, \varepsilon]$. It remains to verify that $\tilde{\mu}$ is a strictly increasing function of σ. But this presents no problem at all since (21) can be solved explicitly: $\tilde{\mu} = (n+1)^{-1}[1 + (2n+3)\sigma]$.

6. The case $f(u) = \exp[u(1+\varepsilon u)^{-1}]$ and others. It has been proved that the global bifurcation diagrams observed in numerical solutions of the problem $u'' + \lambda f(u) = 0, f(u) = e^u, u(-1) = u(1) = 0$, describe, at least qualitatively, the situation for a class of problems involving nonlinearities $f(u)$ which satisfy $f \ge 0, f' \ge 0, f'' \ge 0$ with f satisfying H'_∞ and H_0. However, one may use these results or methods to obtain information for other problems. For example, suppose one removes the hypothesis H_0, but also considers a related problem corresponding to a nonlinearity $\tilde{f}$ obtained by altering $f(u)$ for small values of u, say $0 \le u \le \delta_0$, such that $\tilde{f}$ does satisfy H_0. Obviously, large solutions to the discretized problem for $\tilde{f}$ are also solutions for the same discretized problem with $\tilde{f}$ replaced by f. For example, if the grid is uniform, of step size h, and if the solution to the discretized problem $A_\pi U = \lambda h^2 \tilde{F}(U)$ satisfies $|| U || \ge \delta_0/h$ then U is also a solution to $A_\pi U = \lambda h^2 F(U)$. Hence, if one removes the hypothesis H_0, then one still has the same branches of solutions, but now a branch coming from $|| U || = \infty$ and returning to $|| U || = \infty$ may also connect to the λ-axis, either at some finite value of λ or "at $\lambda = \infty$".

Another observation is that one may have isolas, or isolated continua, of spurious solutions. One can easily design a specimen $f(u)$ which will lead to a problem exhibiting isolas as follows. Let $f(u) = e^u$ for $u < M$. Choose M so large that there are spurious solutions of

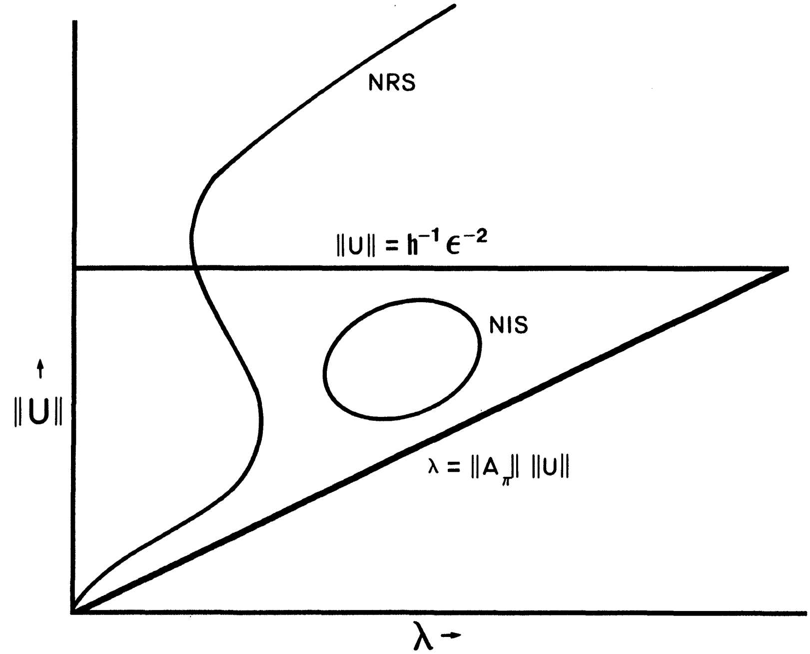

Figure 3. The region to which spurious solutions for $f(u) = \exp[u(1 + \varepsilon u)^{-1}]$ are confined.

norm $< M$. For $u > 2M$ let $f(u) = e^{2M}u$ and for $M \leq u \leq 2M$ define f in such a manner that f will be a monotone increasing continuously differentiable functions. By the argument used above, any solution of $A_\pi U = \lambda f(U)$ with $\| U \| \geq 2M/h$ will be a solution of the linear problem $A_\pi U = \lambda e^{2M} U$ and hence there will be no spurious solutions of norm greater than M/h. A more interesting problem is considered in the following theorem

THEOREM 12. *Let p and $q, q = p + h$, be two neighboring grid points of a uniform grid on $[-1, 1]$ with $q < 0$. Let $\rho_0 = (1 - q)(1 - p)^{-1}$ and $\rho_0 < \rho_1 < \rho_2 < 1$. There exists a positive number $\varepsilon_0 = \varepsilon_0(p, h, \rho_1, \rho_2)$ such that for each $0 < \varepsilon < \varepsilon_0$ the discretized problem* (5) *corresponding to the nonlinearity $f(u) = \exp[u/(1 + \varepsilon u)]$ has a continuum of* *positive solutions* (λ, U) *on which* $\| U \| = U(p)$ *and on which* $\rho(U)$ *assumes all values in* $[\rho_1, \rho_2]$. *Moreover all positive nonsymmetric solutions (i.e. spurious solutions) lie in the set* $\{(\mu, V) : \| V \| \leq h^{-1}\varepsilon^{-2}$ *and* $\mu \leq \| A_\pi \| \| V \|\}$.

PROOF. Let δ be any constant such that $0 < \delta < 1$ and choose $0 < \eta < 1 - \rho_2$ such that $(1 - \rho_2)(1 - \rho_2 - \eta)^{-1} > 4$. Define $g(u) = \max(\delta e^u, \delta e^{u(1+\varepsilon u)^{-1}})$ and let $C(\varepsilon, \delta)$ be the unique positive solution of $\delta e^u = e^{u(1+\varepsilon u)^{-1}}$. One easily verifies that $C(\varepsilon, \delta) < \varepsilon^{-1} - \ln \delta$. Next let ε be chosen such that $0 < \varepsilon < [\sqrt{(1 - \rho_2)(1 - \rho_2 - \eta)^{-1}} - 2]/\ln \delta^{-1}$. The following fact will be needed: if $\rho = \rho_2$ or if

$$M \leq \min(C(\varepsilon, \delta), \frac{1}{\varepsilon}[\sqrt{(1 - \rho_1)(1 - \rho_1 - \eta)^{-1}} - 1]) \tag{22}$$

then

$$g(M)/g(\rho M) < \delta e^{(1-\rho-\eta)M}. \tag{23}$$

First suppose (22) is satisfied, then since $M \leq C(\varepsilon, \delta) : g(M)/g(\rho M) = e^{(1-\rho)M(1+\rho\varepsilon M)^{-1}(1+\varepsilon M)^{-1}} \geq e^{(1-\rho)M(1+\varepsilon M)^{-2}}$. On the other hand $M \leq \frac{1}{\varepsilon}[\sqrt{(1 - \rho_1)(1 - \rho_1 - \eta)^{-1}} - 1]$ implies that $(1 + \varepsilon M)^2 \leq (1 - \rho_1)(1 - \rho_1 - \eta)^{-1} \leq (1 - \rho)(1 - \rho - \eta)^{-1}$ if $\rho_1 \leq \rho$, and hence $g(M)/g(\rho M) \geq e^{(1-\rho-\eta)M}$. Next suppose $\rho = \rho_2$. There are three cases to consider, namely $M \leq C(\varepsilon, \delta), \rho M < C(\varepsilon, \delta) < M$ and $\rho M \geq C(\varepsilon, \delta)$. In the first case one has $g(M)/g(\rho_2 M) \geq \exp[(1 - \rho_2)M(1 + \varepsilon M)^{-2}]$, where $\varepsilon^{-1}[\sqrt{(1 - \rho_2)(1 - \rho_2 - \eta)^{-1}} - 1] \geq \varepsilon^{-1} + \ln \delta^{-1} \geq C(\varepsilon, \delta) > M$, so that $1 + \varepsilon M < \sqrt{(1 - \rho_2)(1 - \rho_2 - \eta)^{-1}}$ and consequently $g(M)/g(\rho_2 M) > e^{(1-\rho_2-\eta)M}$. Next if $\rho_2 M < C(\varepsilon, \delta) < M$, then $g(M)/g(\rho_2 M) = \delta \exp[(1 - \rho_2)M + \rho^2 \varepsilon M^2] \geq \delta e^{(1-\rho_2-\eta)M}$. If $\rho_2 M \geq C(\varepsilon, \delta)$ then one simply has $g(M)/g(\rho_2 M) = e^{(1-\rho_2)M} \geq \delta e^{(1-\rho_2-\eta)M}$. By Lemma 4, or more precisely by its proof, one may conclude that whenever (λ, U) is a solutions of $A_\pi U = \lambda F(U)$ with $\rho(U) = \rho \in [\rho_1, \rho_2]$ and $M = \| U \|$ satisfying (23) or with $\rho = \rho_2$ then

$$\| U \| \leq B(\rho) \equiv (1-\rho-\eta)^{-1} \ln[4\delta^{-1}(1+\rho)^{-1} + h^{-1})\{\rho - (1-q)(1-p)^{-1}\}^{-1}].$$

Now let $C > \max\{B(\rho) : \rho_1 \leq \rho \leq \rho_2\}$ and, if need be, decrease ε such that $C \leq C(\varepsilon, \delta)$ and $\varepsilon C \leq \sqrt{(1 - \rho_1)(1 - \rho_1 - \eta)} - 1$. This means that whenever $\rho = \rho_2$ or $\| U \| \leq C$ and $\rho_1 \leq \rho(U) \leq \rho_2$ then

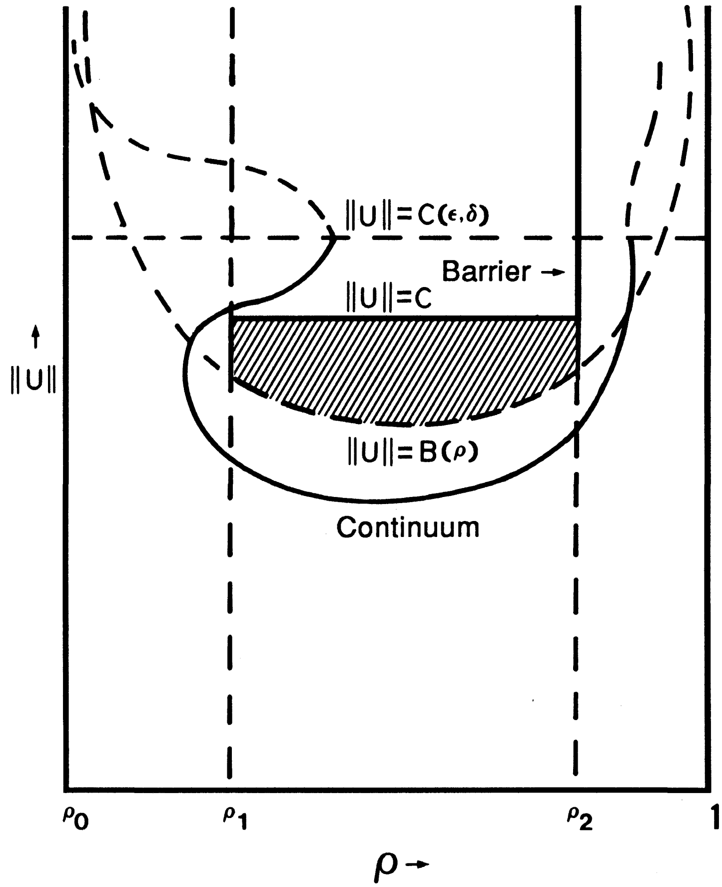

Figure 4.

in fact $\| U \| \leq B(\rho) \leq C$. Corollary 7 shows that for $A_\pi U = \lambda G(U)$ there exists a continuum of positive solutions (λ, U) with $\| U \| = U(p)$ on which $\rho(U)$ assumes all values in $(\rho_0, 1)$. This continuum (see Fig. 4) can not intersect the barrier $\{(\lambda, U) : \rho(U) = \rho_2 \text{ and } \| U \| > B(\rho_2)\} \cup \{(\lambda, U) : \rho_1 \leq \rho(U) \leq \rho_2 \text{ and } B(\rho(U)) < \| U \| \leq C\}$. A straightforward connectedness argument using, for example, the result in [**15**, p. 15] which was used in the proof of Theorem 6, shows that there must exist a subcontinuum on which $\rho(U)$ assumes all values in $[\rho_1, \rho_2]$ and on which $\| U \| \leq B(\rho(U) \leq C(\varepsilon, \delta)$. The last inequality implies that this subcontinuum is in fact also a continuum of solutions for the problem $A_\pi U = \lambda F(U)$. In order to prove the last assertion of the theorem one notes that $u^{-1}e^{u(1+\varepsilon u)^{-1}}$ is a decreasing function for $u \geq \varepsilon^{-2}$ and that if (λ, U) is a positive solution with $\| U \| \geq h^{-1}\varepsilon^{-2}$ then $U_i \geq \varepsilon^{-2}$ at each interior grid-point. This means that (λ, U) is also a solution of the problem $A_\pi U = \lambda \tilde{F}(U)$ where $\tilde{f}(u) = f(u)$ for $u \geq \varepsilon^{-2}$ and $\tilde{f}(u) = f(\varepsilon^{-2})$ for $u \leq \varepsilon^{-2}$. But by Theorem 2 this modified problem has no spurious solutions. Hence all spurious solutions (λ, U) of the original problem must have $\| U \| \leq h^{-1}\varepsilon^{-2}$. Next, since $U \geq \| U \| E^{(p)}$, one has $\| U \| \geq \lambda \exp[\| U \| (1 + \varepsilon \| U \|)^{-1}] \| A_\pi^{-1} E^{(p)} \| A_\pi \|^{-1} \lambda$.

Finally, a question about the hypothesis H_∞ and H'_∞: It was seen that H'_∞ implies H_∞ for function f with $f \geq 0, f' \geq 0$ and $f'' \geq 0$; is the converse true? It seems unlike, but the author has not found a counterexample.

REFERENCES

1. E. Allgower, *On a discretization of $y'' + \lambda y^k = 0$*, Proc. Conf. Roy. Irish Acad., J.J.H. Miller, Editor, 1-15, Academic Press, 1975

2. W.J. Beyn and J. Lorenz, *Spurios solutions for discrete superlinear boundary value problems,* Computing, **28** (1982), 43-51.

3. E. Bohl, *On the bifurcation diagram of discrete analogues for ordinary bifurcation problems,* Math Meth. Appl. Sci. 1 (1979), 566-571.

4. S.N. Chow and J.K. Hale, *Methods of Bifurcation Theory,* Springer, New York, 1982.

5. E.J. Doedel and W.J. Beyn, *Stability and multiplicity of solutions to discretizations of nonlinear ordinary differential equations,* SIAM J. Sci. Stat. Comp. **2** (1981), 107-120.

6. R. Gaines, *Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations,* SIAM J. Numer. Anal. **11** (1974), 411-434.

7. M.A. Krasnoselskii, *Positive Solutions of Operator Equations,* Noordhoff, Groningen, 1964.

8. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities,* Allyn and Bacon, Boston, 1964.

9. R. Nussbaum and H.O. Peitgen, *Special and spurious solutions of* $x'(t) = -\alpha F(x(t-1))$. Memoirs AMS **51** (1984).

10. H.-O. Peitgen, *Phase transitions in the homoclinic regime of area preserving diffeomorphisms,* Proc. Intern. Symp. on Synergetics, H. Haked, Editor, Springer

Series in Synergetics, **17** (1982), 197-214.

11. H.-O. Peitgen, *A mechanism for spurious solutions of nonlinear boundary value problems,* Forschumgsschwerpunkt Dynamische Systeme, Report No. **94**, Universität Bremen, 1983.

12. H.-O. Peitgen and K. Schmitt, *Global topological perturbations of nonlinear eigenvalue problems,* Math. Meth. Appl. Sci., **5** (1983), 376-388.

13. H.-O. and K. Schmitt, *Positive and spurious solutions of nonlinear eigenvalue problems,* Numerical Solution of Nonlinear Equations, A. Dold and B. Eckmann, Editors, pp. 276-324, Lecture Notes in Math. **878**, Springer, New York, 1981.

14. H.-O. Peitgen, D. Saupe and K. Schmitt, *Nonlinear elliptic boundary value problems versus their finite difference approximations: numerically irrelevant solutions,* J. Reine Angen. Math. **322** (1981), 74-117.

15. G.T. Whyburn, *Analytic Topology,* A.M.S. Colloquim Publ., Vol. **28**, Providence, 1963.

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