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ON SUPERLINEAR ELLIPTIC PROBLEMS WITH NONLINEARITIES INTERACTING ONLY WITH HIGHER EIGENVALUES

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1. Introduction. Consider the Dirichlet problem

$$-\Delta u = g(x,u) \text{ in } \Omega, \quad \mu = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded smooth domain in $R^N, N \geq 2$. For the purposes of this section we shall assume that $g : \overline{\Omega} \times R \to R$ is continuous, where $\overline{\Omega}$ denotes the closure of Ω. Grosso modo problem (1) is said to be *superlinear at* $+\infty$ if

$$\lim_{s \to +\infty} \frac{g(x,s)}{s} = +\infty. \tag{2}$$

In [**1**] Ambrosetti and Rabinowitz proved the existence of a solution for problem (1) under a set of conditions which we discuss next. First, growth assumptions at $\pm\infty$ on g were made so as to guarantee that the Euler-Lagrange functional

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int G(x,u), \quad G(x,s) = \int_0^s g(x,\xi)d\xi \tag{3}$$

is well defined in $H_0^1(\Omega)$. (Integrals $\int$ are supposed to be taken over the whole of Ω, unless indicated otherwise.) The critical points of Φ are then the H_0^1 solutions of (1). Also, in [**1**], the following condition is assumed in order to ensure that Φ satisfies the Palais-Smale condition:

$$\begin{aligned} &\text{There are numbers } \theta \in (0, 1/2) \text{ and } s_0 > 0 \text{ such that} \\ &\qquad 0 < G(x,s) \leq \theta s g(x,s) \quad \text{for } |s| \geq s_0, x \in \Omega. \end{aligned} \tag{4}$$

We remark that condition (4) implies that the function $g(x,s)$ is superlinear in both directions, that is, at $\pm\infty$. Namely, (4) implies

$$\lim_{s \to \pm\infty} \frac{g(x,s)}{s} = +\infty.$$

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An adequate condition at $s = 0$ guarantees that 0 is a local minimum of Φ, and a nontrivial critical point of Φ is then obtained in [**1**] using the Mountain Pass Theorem. In this way the authors of [**1**] have introduced in the literature a simple but extremely useful tool to treating problems in Nonlinear Partial Differential Equations.

Another instance of a superlinear problem appears in the so called Ambrosetti-Prodi problem

$$(5) \qquad -\Delta u = g(x, u) + t\varphi + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\varphi > 0$ is an eigenfunction associated with the first eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$, t is a real parameter and $\int h\varphi = 0$. In this problem, the nonlinearity g satisfies condition (2), but at $-\infty$ one has

$$\lim_{s \to -\infty} \frac{g(x, s)}{s} < \lambda_1.$$

This problem was studied in [**2, 4,** and **5**] by variational methods. Indeed, for $t < 0$ and large in absolute value, problem (5) possesses a solution which is a local minimum of the corresponding Euler-Lagrange functional. And again the Mountain Pass Theorem is invoked to get a second solution for problem (5).

The natural question now is: Suppose again that problem (1) is superlinear at $+\infty$, namely (2) is satisfied, but at $-\infty$ one has

$$(6) \qquad \lim_{s \to -\infty} \frac{g(x, s)}{s} = \lambda \quad \text{with } \lambda_k < \lambda < \lambda_{k+1},$$

where λ_k and λ_{k+1} are two consecutive eigenvalues of $(-\Delta, H_0^1(\Omega))$. As we shall see, the existence of one solution for this type of problem is not hard to establish. Nevertheless it may not be true that the associated Euler-Lagrange functional has a local minimum. This precludes the use of the Mountain Pass Theorem, as before, in order to obtain a second solution. However, the setting invites the application of the Generalized Mountain Pass Theorem, due to Rabinowitz [**8**]. Our present work on this problem was motivated by Ruf-Srikanth [**10**] which treated the nonlinearity $g(x, u) = \lambda u + (u^+)^p + t\varphi + h$, where λ is as in (6), t, φ, and h are as in problem (5), and $p > 1$ is restricted in a usual way, see below. In this paper we show that a result similar to the one in [**10**] can be established for a larger class of nonlinearities. Namely, we consider the following parametrized form of (1)

$$(7)_t. \qquad -\Delta u = \lambda u + f(x, u) + t\varphi + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\lambda_k < \lambda < \lambda_{k+1}, t$ is a real parameter, $\varphi > 0$ is an eigenfunction associated with the first eigenvalue of $(-\Delta, H_0^1(\Omega)), h \in C^\nu(\overline{\Omega}), 0 < \nu \leq 1, \int h\varphi = 0$. In order to prove existence of two solutions for $(7)_t$ we assume that the nonlinearity f satisfies the following set of conditions:

(F1) $f : \overline{\Omega} \times R \to R$ is a C^1 function.

(F2) There exists $0 < \alpha < 1$ such that $\lim_{s\to-\infty} f(x,s)|s|^{-\alpha} = 0$.

(F3) $\lim_{s\to-\infty} f'_s(x,s) = 0$.

(F4) There are $0 < \theta < 1/2$ and $s_0 > 0$ such that $0 < F(x,s) \leq \theta s f(x,s)$, for $s > s_0$ and all $x \in \Omega$.

(F5) $\lim_{s\to+\infty} f(x,s)s^{-\sigma} = 0$, where $\sigma \leq (N+2)/(N-2)$ if $N \geq 3$ or $1 < \sigma < \infty$ if $N = 2$.

(F6) $f'_s(x,s) \geq -\mu$ where $\mu < \lambda - \lambda_k$.

(F7) $\sigma\theta < \min\{\frac{1}{1+\alpha}, \frac{N+2}{2N}\}$.

THEOREM. *Under assumptions* (F1) *through* (F7), *there exists a* $\hat{t} > 0$ *such that, for all* $t \geq \hat{t}$, *problem* $(7)_t$ *has (at least) two solutions.*

REMARKS ON THE ABOVE ASSUMPTIONS ON THE NONLINEARITY. (1) All the above limits are supposed to be uniform with respect to $x \in \Omega$.

(2) Conditions (F1), (F2) and (F5) imply that there is a constant $c > 0$ such that $|f(x,s)| \leq C|s|^\sigma + C$. Consequently, the Euler-Lagrange functional associated with problem $(7)_t$ is well defined in $H_0^1(\Omega)$.

(3) Condition (F4) implies that $F(x,s) \geq Cs^{1/\theta} - C$ for $s > 0$, which in its turn implies that f is superlinear at $+\infty$: $f(x,s) \geq Cs^{(1/\theta)-1} - C$, for $s > 0$. This last inequality, together with (F5), implies that $\sigma > (1/\theta) - 1$. It then follows that

$$(*) \qquad 1/\theta < 2N/(N-2) \Rightarrow (N-2)/2N < \theta < 1/2 \text{ for } N \geq 3.$$

(4) Either condition (F2) or (F3) implies

$$\lim_{s\to-\infty} f(x,s)s^{-1} = 0,$$

which implies that problem (7) is not resonant at $-\infty$. And condition (F4) implies

$$\lim_{s\to+\infty} f(x,s)s^{-1} = +\infty$$

(5) Condition (F3) does not imply (F2). Indeed, if $f(x,s)$ is equal to $s|\ln s|^{-1}$ for $s < -1$, then f satisfies (F3) but not (F2). It is also easy to see that (F2) puts no restriction on the growth of $f'_s(x,s)$ at $-\infty$; so (F2) and (F3) are independent.

(6) Condition (F2) is verified, for instance, if $f(x,s)$ is bounded for $s < 0$. We recall that the case treated in [**10**] is $f(x,s) \equiv 0$ for $s < 0$, which gives also (F3) trivially.

(7) Condition (F7) is not as restrictive as it may appear at first sight. For instance, if $f(x,s)$ is a *pure power* at $+\infty$, say s^σ with $1 < \sigma < (N+2)/(N-2)$ if $N \geq 3$, then one could take $\theta = 1/(\sigma+1)$. Thus $\sigma\theta = 1 - \theta$, and from (*) above we see that $\sigma\theta < (N+2)/2N$. So (F7) would be satisfied for all $\alpha \leq (N-2)/(N+2)$. If $N = 2$ and again the case of a pure power we observe that $\sigma\theta = \sigma/(\sigma+1) < 1$, and in this case one could take $\alpha < 1/\sigma$. Summarizing we see that, in the case when f is a pure power at ∞, condition (F7) establishes a *balance* between the growth σ at $+\infty$ and the growth α at $-\infty$. So (F7) seems a reasonable condition to be assumed.

(8) We do not know if conditions (F2) and (F6) are indeed necessary. (F2) enters in an apparently essential way to get the PS condition in §2. And(F6) permits one to do required "linking"in the Generalized Mountain Pass Theorem, done in §4.

REMARK. It is natural to ask if this problem has more than two solutions. In the ODE case a result of Ruf-Srikanth [**11**] shows that (for $f(x,u) = (u^+)^p, h = 0$) the problem has $2k+2$ solutions. This result has been extended to the PDE case when Ω is a ball and for a large class of nonlinearities by Padua [**7**]. Both the ODE case and the case when Ω is a ball use bifurcation arguments that are not applicable to the case of a general domain. It is also interesting to remark that if f is asymptotically linear at $+\infty$, then the work of Lazer and McKenna [**6**] shows that there are more solutions.

The Palais-Smale condition. In this section we show that the Euler-Lagrange functional associated with an equation from a certain class of superlinear elliptic problems satisfies the Palais-Smale condition (in short, the PS condition). We recall that a C^1 functional $\Phi : H_0^1 \to R$ satisfies the PS condition if every sequence (u_n) in H_0^1 with $|\Phi(u_n)| \leq$ const and $\Phi'(u_n) \to 0$ contains a convergent subsequence. If Φ' is of the form identity + compact, then, in order to check PS, it suffices to prove that $||u_n||_H 1 \leq$ const. Let us consider problem (1) and let us assume the following conditions on the nonlinearity g:

(G1) $g : \overline{\Omega} \times R \to R$ is continuous.

(G2) There are $\lambda \neq \lambda_j$ for $j = 1, 2, \ldots$ and $0 \leq \alpha < 2$ such that

$$\lim_{s \to -\infty} \frac{g(x,s) - \lambda s}{|s|^\alpha} = 0$$

(G3) $\lim_{s \to +\infty} g(x,s)s^{-\sigma} = 0$, where $\sigma \leq (N+2)/(N-2)$ if $n \geq 3$ or $1 < \sigma < \infty$ if $N = 2$.

(G4) There are $0 < \theta < 1/2$ and $s_0 > 0$ such that $0 < G(x,s) \leq \theta s g(x,s)$, for $s > s_0$ and all $x \in \Omega$.

(G5) $\sigma\theta < \min\{\frac{1}{1+\alpha}, \frac{N+2}{2N}\}$.

Remarks on the above conditions. (1) All limits in the above conditions are supposed to hold uniformly with respect to $x \in \Omega$.

(2) Conditions (G1), (G2) and (G3) imply that there is a constant $c > 0$ such that $|g(x,s)| \leq C|s|^\sigma + C$. Consequently the Euler-Lagrange functional Φ in (3) is well defined in $H_0^1(\Omega)$.

(3) Conditions (G1) and (G2) imply that there is a constant C such that

$$|g(x,s)s - 2G(x,s)| \leq C + C|s|^{1+\alpha}, \quad s < 0, \tag{8}$$

$$|g(x,s)s - \lambda s^2| \leq C + C|s|^{1+\alpha}, \quad s < 0. \tag{9}$$

(4) Condition (G2) implies that

$$\lim_{s \to -\infty} g(x,s)s^{-1} = \lambda$$

and condition (G4) implies that

$$\lim_{s \to +\infty} g(x,s)s^{-1} = +\infty.$$

Lemma 1. *If $g(x,s)$ satisfies conditions* (G1) *through* (G5), *then the functional Φ, as defined in* (3), *satisfies the Palais-Smale condition.*

PROOF. Let (u_n) be a sequence in $H_0^1(\Omega)$ for which

$$|\Phi(u_n)| = \left|\frac{1}{2}\int |\nabla u_n|^2 - \int G(x,u_n)\right| \leq C, \tag{10}$$

$$|\langle \Phi'(u_n), v\rangle| = \left|\int \nabla u_n \nabla v - \int g(x,u_n)v\right| \leq \varepsilon_n ||v||_{H^1}, \tag{11}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $H_0^1, \varepsilon_n \to 0$ as $n \to \infty$ and v is any element in H_0^1. As we remarked above to prove that Φ satisfies the PS condition it suffices to prove that $||u_n||_{H^1}$ is bounded uniformly with respect to n. Taking $v = u_n$ in (11) and using (10) we obtain

$$\int [g(x,u_n)u_n - 2G(x,u_n)] \leq C + \varepsilon_n ||u_n||_{H^1}. \tag{12}$$

The integral in (12) is broken into three parts. The first is estimated using (8) as follows:

$$\left|\int_{u_n<0} [g(x,u_n)u_n - 2G(x,u_n)]\right| \leq C + C\int |u_n^-|^{1+\alpha},$$

where $u^- = \max\{0, -u\}$. The second integral taken over $0 \leq u_n \leq s_0$ is obviously bounded uniformly with respect to n. And the third one is estimated using (G4):

$$\int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n)) \geq \left(\frac{1}{\theta} - 2\right)\int_{u_n>s_0} G(x,u_n).$$

Altogether, one obtains the estimate

$$\int_{u_n>s_0} G(x,u_n) \leq C + \varepsilon_n ||u_n||_{H^1} + C||u_n^-||_{L^{1+\alpha}}^{1+\alpha}. \tag{13}$$

Next let us take $v = u_n^-$ in (11) to obtain

$$\left|\int |\nabla u_n^-|^2 - \int_{u_n<0} g(x,u_n)u_n\right| \leq \varepsilon_n ||u_n^-||_{H^1}. \tag{14}$$

Using (8) and (9), it follows from (14) that

$$\left| \int |\nabla u_n^-|^2 - 2\int_{u_n<0} G(x,u_n) \right| \le C + \varepsilon_n ||u_n^-||_{H^1} + C||u_n^-||_{L^{1+\alpha}}^{1+\alpha}, \tag{15}$$

$$\left| \int |\nabla u_n^-|^2 - \lambda \int |u_n^-|^2 \right| \le C + \varepsilon_n ||u_n^-||_{H^1} + C||u_n^-||_{L^{1+\alpha}}^{1+\alpha}. \tag{16}$$

Now we consider the following alternatives: either (i) $||u_n^-||_{H^1} \le$ const. or (ii) $||u_n^-||_{H^1} \to \infty$, passing to a subsequence if necessary. In case (i) holds, estimate (15) implies that $|\int_{u_n<0} G(x,u_n)| \le$ const. Consequently it follows from (10) and (13) that

$$||u_n^+||_{H^1} \le C + \varepsilon_n ||u_n^+||_{H^1},$$

which implies that $||u_n^+||_{H^1}$ is bounded. So $||u_n||_{H^1} \le C$ and the proof of the lemma would be complete in case (i). Next let us consider case (ii) and prove that it cannot hold, completing in this way the proof of the lemma. It follows from (10) using (13) and (15) that

$$\int |\nabla u_n^+|^2 \le C + C\varepsilon_n ||u_n^-||_{H^1} + C||u_n^-||_{L^{1+\alpha}}^{1+\alpha}. \tag{17}$$

Now we obtain from (11) that

$$\left| \int \nabla u_n^- \nabla v - \lambda \int u_n^- v \right| \le I_1 + I_2 + I_3 + \varepsilon_n ||v||_{H^1}, \tag{18}$$

where the integrals I_1, I_2 and I_3 are explicitly written below and then estimated.

$$I_1 \equiv \left| \int \nabla u_n^+ \nabla v \right| \le ||u_n^+||_{H^1} ||\nu||_{H^1}$$

$$I_2 \equiv \left| \int_{u_n>0} g(x,u_n) v \right| \le (C + C||u_n^+||_{L^{p\sigma}}^{\sigma}) ||v||_{L^q},$$

where we have used (G3) and $p = 2N/(N+2), q = 2N/(N-2)$ if $N \ge 3$. If $N = 2$ we take $1 < p \le 1/\sigma\theta$; this can be done because (G5) in this case implies $\sigma\theta < 1$.

$$I_3 \equiv \left| \int_{u_n<0} g(x,u_n)v + \lambda u_n^- v \right| \le \int_{u_n<0} |\delta(x,u_n)||v|,$$

where $\delta(x,s) = g(x,s) - \lambda s$. Using (G2) we obtain the estimate

$$I_3 \leq (C + C\|u_n^-\|_{H^1}^\alpha)\|v\|_{H^1}.$$

Altogether, the left side of (18) is estimated by

$$(C + \|u_n^+\|_{H^1} + C\|u_n^+\|_{L^{p\sigma}}^\sigma + C\|u_n^-\|_{H^1}^\alpha)\|v\|_{H^1}.$$

Our purpose now is to show that the expression in parenthesis goes to zero when divided by $\|u_n^-\|_{H^1}$. Firstly, we see that (17) implies

$$\text{(19)} \qquad \|u_n^+\|_{H^1}/\|u_n^-\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Secondly we claim that

$$\text{(20)} \qquad \|u_n^+\|_{L^{p\sigma}}^\sigma/\|u_n^-\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

We remark that this is the only point in the present proof where condition (G5) is used. It follows from the fact that $G(x,s) \geq Cs^{1/\theta} - C$, for $s > 0$, and from estimate (13) that

$$\text{(21)} \qquad \int (u_n^+)^{1/\theta} \leq C + \varepsilon_n\|u_n\|_{H^1} + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}.$$

Recalling the statement at (19) and that the $L^{1+\alpha}$ norm is bounded by the H^1 norm, we obtain from (21) that the $L^{1/\theta}$ norm of $u_n^+/\|u_n^-\|_{H^1}^{\theta(1+\alpha')}$ goes to zero, where α' is any number with $\alpha < \alpha' < 1$. Our claim (20) is that the $L^{p\sigma}$ norm of $u_n^+/\|u_n^-\|^{1/\sigma}$ goes to zero. So this will be achieved if $p\sigma \leq 1/\theta$ and $1/\sigma \geq \theta(1+\alpha')$. But this is guaranteed by(G5). So (20) is established. Next let us define $w_n = u_n^-/\|u_n^-\|_{H^1}$ and recall that we are treating case (ii), that is $\|u_n^-\|_{H^1} \to \infty$. Let us assume that $w_n \to w_0$ weakly in H_0^1 and strongly in L^2. It follows from (16) that $\int w_n^2 \to \lambda^{-1}$. So $\|w_0\|_{L^2}^2 = \lambda^{-1}$ and $w_0 \not\equiv 0$. Now, dividing (18) through by $\|u_n^-\|_{H'}$ and passing to the limit using (19) and (20), we get

$$\int \nabla w_0 \nabla v - \lambda \int w_0 v = 0 \ \text{ for all } v \in H_0^1.$$

This means that w_0 is an eigenfunction ($\neq 0$) corresponding to λ, which is impossible.

3. Existence of a first solution of $(7)_t$**.** In this section we consider problem $(7)_t$ as stated in the introduction. In order to prove the existence of one solution of $(7)_t$ we do not have to assume all conditions (F1) through (F7) stated in the Introduction. In fact the following result is true.

LEMMA 2. *Suppose that* $f(x, s)$ *is continuous and*
(F8) $\lim_{s \to -\infty} f(x, s)s^{-1} = 0$
Then there exists $\bar{t} > 0$ *such that* $(7)_t$ *has a negative solution* u_t *for all* $t \geq \bar{t}$*. Moreover,*

$$u_t \leq \frac{Ct}{\lambda_1 - \lambda}\varphi, \tag{22}$$

where C *is a positive constant independent of* t.

PROOF. We use an argument similar to the one in [**3**; Theorem 1]. Let us define

$$\tilde{f}(x, s) = \begin{cases} f(x, s) & \text{if } s < 0 \\ f(x, 0) & \text{if } s \geq 0. \end{cases}$$

We claim that the Dirichlet problem

$$(23)_t \qquad -\Delta u = \lambda u + \tilde{f}(x, u) + t\varphi + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a negative solution for $t > 0$ large. Such a solution is obviously a solution of the original problem $(7)_t$. Let $z_0 \in C^{2+\nu}(\overline{\Omega})$ be the solution of

$$-\Delta z_0 = \lambda z_0 + h \text{ in } \Omega, \quad z_0 = 0 \text{ on } \partial\Omega.$$

Now let us consider the Dirichlet problem

$$(24)_t \qquad -\Delta w = \lambda w + \tilde{f}(x, w + z_0 - t\beta\varphi) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega,$$

where $\beta = 1/(\lambda - \lambda_1)$. It follows from (F8) that

$$\lim_{|s| \to \infty} \tilde{f}(x, s)s^{-1} = 0$$

which implies that, given $\varepsilon > 0$, there is a constant C_ε such that $|\tilde{f}(x, s)| \leq \varepsilon|s| + C_\varepsilon$, for all $s \in R$ and all $x \in \Omega$. In view of this, it is then well known that problem $(24)_t$ has a solution w_t for each t and

$$||w_t||_{C^1} \leq C||\tilde{f}(x, w_t + z_0 - t\beta\varphi||_{C^0},$$

where the constant C does not depend on t and the norms considered here are maximum norms. Using the above estimate on $\tilde{f}$ we then obtain

$$||w_t||_{C^1} \leq \varepsilon C||w_t + z_0 - t\beta\varphi||_{C^0} + C.$$

Consequently we find, by taking $\varepsilon > 0$ small enough, that $||w_t||_{C^1} \leq C + C\varepsilon|t|$. So $||w_t/t||_{C^1} \to 0$. Then, using the fact that the eigenfunction φ is positive in Ω and its outer normal derivative at the boundary is strictly negative, we conclude that $w_t + z_0 - t\beta\varphi < 0$ in Ω for $t > 0$ and sufficiently large. Finally it is easy to check that $u_t = w_t + z_0 - t\beta\varphi$ is a solution of $(23)_t$ which is negative in Ω, and the estimate (22) is also clear.

Next we show that if one assumes (F3) instead of (F8), then u_t can be taken in such a way that u_t is a nondegenerate critical point of the Euler Lagrange functional associated with $(7)_t$ and its Morse index is k. That is a great deal of information about this critical point. Yet we use it here merely in a rather mild way. We prefer to work with a translation of $(7)_t$ as follows. Let $u = v + u_t$ in problem (7_t) for $t \geq \bar{t}$. Then v satisfies the equation

$$-\Delta v = \lambda v + f(x, v + u_t) - f(x, u_t) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{25}$$

Clearly, $v = 0$ is a solution of this equation, and we shall look for other solutions of $(25)_t$ as critical points of the functional

(26)

$$\Psi_t(v) = \frac{1}{2}\int |\nabla v|^2 - \frac{\lambda}{2}\int v^2 - \int F(x, v+u_t) + \int F(x, u_t) + \int f(x, u_t)v.$$

LEMMA 3. *Let us assume* (F1), (F3) *and* (F5). *Then there exists* $\hat{t} \geq \bar{t}$ *such that* $v = 0$ *is a nondegenerate critical point of* Ψ_t *with Morse index* k, *for all* $t \geq \hat{t}$.

REMARKS. (1) We are assuming that $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k < \lambda < \lambda_{k+1}$. So there are k eigenvalues of $(-\Delta, H_0^1(\Omega))$, counting multiplicities, which are less than λ. To each λ_j corresponds an eigenfunction $\varphi_j \in H_0^1(\Omega)$. Let V be the finite dimensional subspace of $H_0^1(\Omega)$ generated by $\varphi_1, \ldots, \varphi_k$, and let $W = V^\perp$. So $H_0^1(\Omega) = V \oplus W$.

(2) In view of the hypotheses, the functional Ψ_t is well defined in H_0^1 and it is of class C^2. Their first and second derivatives are represented

as follows:

$$\langle \Psi_t'(u), v\rangle = \int \nabla u \nabla v - \lambda \int uv - \int f(x, u+u_t)v + \int f(x, u_t)v,$$

$$\langle \Psi_t''(u)v_1, v_2\rangle = \int \nabla v_1 \nabla v_2 - \lambda \int v_1 v_2 - \int f_s'(x, u+u_t)v_1 v_2,$$

for all u, v, v_1, v_2 in $H_0^1(\Omega)$.

PROOF OF LEMMA 3. It suffices to prove that there exists $\hat{t} \geq \bar{t}$ such that, for each $t \geq \hat{t}$, there is a positive constant $c(t)$ depending only on t such that

$$\langle \Psi_t''(0)v, v\rangle \leq -c(t)\|v\|_{H^1}^2, \quad v \in V, \tag{27}$$

and

$$\langle \Psi_t''(0)w, w\rangle \geq c(t)\|w\|_{H^1}^2, \quad w \in W. \tag{28}$$

Instead of proving (27) and (28) it suffices to show that

$$\langle \Psi_t''(0)v, v\rangle < 0, \quad \text{for all } v \in V \text{ with } \|v\|_{H^1} = 1, \tag{29}$$

and

$$\langle \Psi_t''(0)w, w\rangle > 0, \text{ for all } w \in W \text{ with } \|w\|_{H^1} = 1. \tag{30}$$

Indeed (29) clearly implies (27) in view of the compactness of the unit ball in V. To see that (30) implies (28), let us assume by contradiction that there exists a sequence $w_n \in W$ with $\|w_n\|_{H^1} = 1, w_n \to w_0$ weakly in H_0^1 and strongly in L^2, such that $\langle \Psi_t''(0)w_n, w_n\rangle \downarrow 0$, that is

$$\lim_{n\to\infty} \left\{ \int |\nabla w_n|^2 - \lambda \int w_n^2 - \int f_s'(x, u_t)w_n^2 \right\} = 0. \tag{31}$$

On one hand (31) gives

$$1 - \lambda \int w_0^2 - \int f_s'(x, u_t)w_0^2 = 0$$

which implies $w_0 \neq 0$. On the other hand, (31) also yields the inequality

$$\int |\nabla w_0|^2 - \lambda \int w_0^2 - \int f_s'(x, u_t) w_0^2 \leq 0$$

which is a contradiction to (30) with $w = w_0/||w_0||_{H^1}$. Now let us prove (30). The relation (29) is proved likewise. Suppose, by contradiction, that there is a sequence $t_n \to +\infty$ and $w_n \in W$ with $||w_n||_{H^1} = 1$ and $\langle \Psi_{t_n}''(0) w_n, w_n \rangle \leq 0$. Then

$$\int |\nabla w_n|^2 - \lambda \int w_n^2 \leq \int f_s'(x, u_{t_n}) w_n^2. \tag{32}$$

The left side of (32) can be estimated using $\lambda_{k+1} \int w_n^2 \leq \int |\nabla w_n|^2$, and we are led to

$$1 - \frac{\lambda}{\lambda_{k+1}} \leq \int f_s'(x, u_{t_n}) w_n^2. \tag{33}$$

The sequence w_n can be chosen in such a way that $w_n \to w_0$ weakly in H_0^1, strongly in L^2, a.e. in Ω and $|w_n(x)| \leq k(x)$ where $k(x)$ is some $L^2(\Omega)$ function. It follows from the estimate (22) in Lemma 2 that the integrand in (33) converges a.e. to 0. Since u_{t_n} is a negative function, this same integrand is bounded uniformly by an L^2 function. So applying the Lebesgue Dominated Theorem to the expression (33) we are led to a contradiction.

4. Existence of a second solution. In this section we show that the functional Ψ_t defined in (26) possesses a nontrivial critical point for $t \geq \hat{t}$. In this way one obtains a second solution of $(7)_t$ in addition to the solution u_t found in §3. And this will complete the proof of the Theorem. So in this section we assume all the hypotheses of the Theorem. The proof below uses the Generalized Mountain Pass Theorem, see [**8**] or [**9**; Theorem 4.1], and the lemmas in the sequel are the verification of the hypotheses of the theorem applied to our functional Ψ_t.

LEMMA 4. *The functional ψ_t satisfies the* PS *condition.*

PROOF. The function $g(x, s) = \lambda s + f(x, s + u_t(x)) - f(x, u_t(x))$ satisfies all conditions of Lemma 1, as consequences of (F1), (F2), (F4), (F5) and (F7). The θ in (G4) should be slightly larger than the one in (F4).

From now on let us fix $t \geq \hat{t}$.

LEMMA 5. *There are positive constants ρ and δ such that*

$$\Psi_t(w) \geq \delta \quad \textit{for } w \in W \textit{ with } ||w||_{H^1} = \rho. \tag{34}$$

PROOF. Since Ψ_t is of class C^2 it follows that, given $\varepsilon > 0$, there is $\rho > 0$ such that $|||\Psi''(u) - \Psi''(0)||| \leq \varepsilon$ if $||u|| \leq \rho$, where $|||\cdot|||$ denotes the norm in the space $L(H_0^1, H_0^1)$ of linear bounded operators in H_0^1. Now take $\varepsilon > 0$ such that $\varepsilon < c(t)/2$. We claim that the corresponding ρ does the job. Indeed, given any $\overline{w} \in W$ with $||\overline{w}||_{H^1} = 1$, define the function $\theta : R \to R$ by

$$\theta(s) = \Psi_t(s\overline{w}).$$

So $\theta'(s) = \langle \Psi_t'(s\overline{w}), \overline{w}\rangle, \theta''(s) = \langle \Psi_t(s\overline{w})\overline{w}, \overline{w}\rangle$. Using Taylor's formula, $\theta(s) = \theta(0) + \theta'(0)s + \frac{1}{2}\theta''(\xi)s^2$, where ξ is a number between 0 and s, we have

$$\Psi_t(sw) = \frac{1}{2}\langle \Psi_t''(\xi\overline{w})\overline{w}, \overline{w}\rangle s^2.$$

Now take $s = \rho$ and write

$$\Psi_t(\rho\overline{w}) = \frac{1}{2}\langle \Psi_t''(0)\rho\overline{w}, \rho\overline{w}\rangle + \frac{1}{2}\langle (\Psi_t''(\xi\overline{w}) - \Psi_t''(0))\rho\overline{w}, \rho\overline{w}\rangle.$$

Using Lemma 3 we then estimate

$$\Psi_t(\rho\overline{w}) \geq \frac{1}{2}c(t)||\rho\overline{w}||_{H^1}^2 - \frac{1}{2}\varepsilon||\rho\overline{w}||_{H^1}^2 \geq \frac{1}{4}c(t)||\rho\overline{w}||_{H^1}^2.$$

So the lemma is proved with $\delta = \frac{1}{4}c(t)p^2$ and the above ρ.

LEMMA 6. *Given $\varepsilon > 0$ there exists an element $e \in W$ with $||e||_{H^1} = \varepsilon$ such that the set*

$$\{x \in \Omega : v(x) + e(x) > 1\}$$

has positive measure for each $v \in V$ with $||v||_{H^1} \leq 1$.

PROOF. Suppose by contradiction that, for each $e \in W$ with $||e||_{H^1} = \varepsilon$, there is a $v_e \in V$ with $||v_e||_{H^1} \leq 1$ and such that

$$v_e(x) + e(x) - 1 \leq 0 \text{ a.e. in } \Omega.$$

This implies that

$$e(x) \leq 1 - v_e(x) \leq \text{k a.e. in } \Omega.$$

Similarly, we get $-e(x) \leq$ const. a.e. in Ω. So $e \in L^\infty(\Omega)$, for all $e \in W$. This would give that $H_0^1(\Omega) \subset L^\infty(\Omega)$, which is false for all $N \geq 2$.

Now we claim that there is a constant $\eta > 0$ such that

$$\inf_{\substack{v \in V \\ ||v||_{H^1} = 1}} \int [(v + e - 1)^+]^2 \geq \eta > 0 \tag{35}$$

Indeed, otherwise there would exist a sequence $v_n \in V$ with $||v_n||_{H^1} = 1$ such that $[v_n + e - 1]^+ \to 0$ in L^2. Since V is finite dimensional we may assume that $v_n \to v_0$ strongly in both H_0^1 and L^2 and $||v_0||_{H^1} = 1$. Hence $[v_0 + e - 1]^+ = 0$ which contradicts the statement of Lemma 6.

LEMMA 7. *There exist $R > 0$ and $\varepsilon > 0$ such that*

$$\Psi_t(u) \leq 0 \textit{ for all } u \in \partial Q$$

where $Q = (\overline{B}_r \cap V) \oplus \{re : 0 \leq r \leq R\}$, where $\overline{B}_R$ is a closed ball in H_0^1 centered at 0 and e, with $||e||_{H^1} = \varepsilon$, is the function found in Lemma 6.

PROOF. The set ∂Q is composed of three sets: $\Gamma_1 = \overline{B}_R \cap V, \Gamma_2 = \{u \in H_0^1 : u = v + se$, with $v \in V, ||v||_{H^1} = R, 0 \leq s \leq R\}$, and $\Gamma_3 = \{u \in H_0^1 : u = v + Re$, with $v \in V, ||v||_{H^1} \leq R\}$. The proof of the lemma is divided in three parts, each one showing that $\Psi_t \leq 0$ in each of the three sets above.

(i) Here we show even more. Namely $\Psi_t \leq 0$ on V. Indeed, for $v \in V$, one uses the estimate $\int |\nabla v|^2 \leq \lambda_k \int v^2$ and Taylor's formula $F(x, v+u_t) - F(x, u_t) + f(x, u_T)v = \frac{1}{2} f_s'(x, \xi(x))v^2$ where $\xi(x)$ is some number between $u_t(x)$ and $u_t(x) + v(x)$ to obtain the estimate

$$\Psi_t(v) \leq \frac{1}{2}(\lambda_k - \lambda) \int v^2 - \frac{1}{2} \int f_s'(x, \xi(x))v^2.$$

Next, using (F6), we obtain

$$\Psi_t(v) \leq \frac{1}{2}(\lambda_k - \lambda + \mu) \int v^2 \leq 0.$$

(ii) Here we take $u \in \Gamma_2$, and show that $\Psi_t(u) \leq 0$ for any choice of R. The number $\varepsilon > 0$ will be chosen at this step. We could estimate the functional as in (i):

$$\Psi_t(u) \leq \frac{1}{2}\Big((1 - \frac{\lambda}{\lambda_k}\Big) \int |\nabla u|^2 + \frac{s^2}{2}\Big(\int |\nabla e|^2 - \lambda \int e^2\Big) + \frac{\mu}{2} \int |v + se|^2$$

or

$$\Psi_t(u) \leq \frac{R^2}{2}\left(\left(1 - \frac{\lambda - \mu}{\lambda_k}\right) + \int |\nabla e|^2 - (\lambda - \mu) \int e^2\right). \tag{36}$$

So $\Psi_t(u) \leq 0$ if one chooses $\varepsilon > 0$ such that

$$\varepsilon^2 + \frac{\lambda - \mu}{\lambda_{k+1}}\varepsilon^2 \leq \frac{\lambda - \mu}{\lambda_k} - 1. \tag{37}$$

Indeed, since $||e||_{H^1} = \varepsilon$, we obtain

$$\int |\nabla e|^2 - (\lambda - \mu) \int e^2 \leq \left(1 + \frac{\lambda - \mu}{\lambda_{k+1}}\right) \int |\nabla e|^2 \leq \frac{\lambda - \mu}{\lambda_k} - 1,$$

which, applied in (36), gives that $\Psi_t(u) < 0$.

(iii) Now we show how to select $R > 0$ in such a way that $\Psi_t(u) \leq 0$ for $u \in \Gamma_3$. Using the fact that $\int |\nabla v|^2 \leq \lambda_k \int v^2 < \lambda \int v^2$ we obtain the estimate

$$\begin{aligned}\Psi_t(u) \leq &\frac{R^2}{2}\left(\int |\nabla e|^2 - \lambda \int e^2\right) \\ &- \int (F(x, v + Re + u_t) - F(x, u_t) - f(x, u_t)(v + Re))\end{aligned}$$

which could be estimated further by

$$\Psi_t(u) \leq C + CR + CR^2 - \int_{v+Re+u_t>0} F(x, v + Re + u_t),$$

where the constants C do not depend on R; observe that $|F(s)| \leq C|s|^{1+\alpha} + C$ if $s < 0$. Now, by the superlinearity of f at $+\infty$, given any $K > 0$ there exist $s_K > 0$ such that $f(x,s) \geq 2Ks$ for $s > s_K$. So there is a constant C_K such that $F(x,s) \geq C_K + Ks^2$ for $s > 0$. Then

$$\Psi_t(u) \leq C + CR + CR^2 - C_K - KR^2 \int \left(\left(\frac{v}{R} + e + \frac{u_t}{R}\right)^+\right)^2.$$

Now we pick $R > 0$ such that $u_t(x)/R \geq -1$ for all $x \in \Omega$; this can be done because of the continuity of u_t. So

$$\left(\frac{v}{R} + e + \frac{u_t}{R}\right)^+ \geq \left(\frac{v}{R} + e - 1\right)^+,$$

and then

$$\Psi_t(y) \leq C + CR + CR^2 - C_K K R^2 \int \left(\left(\frac{v}{R} + e - 1\right)^+\right)^2.$$

Applying (35) we get finally that

$$\Psi_t(u) \leq C + CR + CR^2 - C_K - K\eta R^2.$$

So the result is achieved if K is chosen in such a way that $K\eta > C$.

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