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THE INTERMEDIATE VALUE THEOREM: PREIMAGES OF COMPACT SETS UNDER UNIFORMLY CONTINUOUS FUNCTIONS

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ABSTRACT. A strong constructive form of the intermediate value theorem is established. Let f be a uniformly continuous map from a connected, locally connected, compact metric space X to the real numbers $\mathbf{R}$ with $\alpha < \beta$ in the range of f. Except for countably many real numbers r, if $\alpha < r < \beta$, then the set $f^{-1}(r)$ is nonempty and compact. An application is a constructive proof of the Schoenflies theorem that the interior of a Jordan curve in the plane is homeomorphic to a disk.

0. Introduction. The intermediate value theorem is often cited as a theorem from classical mathematics which is not constructively valid. The standard Brouwerian counter example is as follows.

EXAMPLE 0.1. Let a be a real number and define the function f by $f(0) = -1, f(1/3) = 0 = f(2/3), f(1) = 1$, and linear inbetween. Solving the equation $f(x) = a$ for $-1 \leq a \leq 1$ is equivalent to determining whether $a \geq 0$ or $a \leq 0$.

Bishop [**2**] proves the following constructive form of the intermediate value theorem:

Let $f : [0,1] \to \mathbf{R}$ be uniformly continuous and let $\alpha < \beta$ be in the range of f. If $\alpha < r < \beta$ and $\varepsilon > 0$, then $f^{-1}(r-\varepsilon, r+\varepsilon)$ is nonempty.

The following stronger constructive version of the intermediate value theorem for $X = [0,1]$ is Problem 15 in [**2**, page 110].

(A) *Let $f : X \to \mathbf{R}$ be uniformly continuous with $\alpha < \beta$ in $fX = \{f(x) : x \in X\}$. For all but countably many real numbers r, if $\alpha < r < \beta$, then the set $f^{-1}(r)$ is nonempty and compact.*

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A proof of (A) for $X = [0,1]$ was attempted in [**3**]. The nature of this proof is such that, were it correct, it would show that (A) holds for an arbitrary compact, connected metric space X; we shall give a Brouwerian counterexample to this [Example 2.1]. Problem 16 in [**2**, page 110] asks the reader to prove:

(B) *Let X be a locally connected, compact metric space and let $f : X \to \mathbf{R}$ be uniformly continuous. Then for all but countably many real numbers r the set $f^{-1}(r)$ is compact.*

This would imply (A) for a connected, locally connected, compact metric space X, if one could assume that a classically nonempty set cannot be constructively empty. We shall construct a counterexample to this line of reasoning using Church's thesis [Example 5.1].

We shall prove a generalization of (B), replacing $\mathbf{R}$ by the unit circle T, and show that the map f^{-1} is continuous from above at all but countably many t in T [Theorem 4.7]. With this we arrive at (A) for a connected, locally connected, compact metric space X [Corollary 4.8]. We shall also give a condition on a compact subset A of $\mathbf{R}$ that ensures that $f^{-1}A$ is compact.

Using these results we establish the key fact needed for a constructive proof of the Schoenflies theorem–the interior of a Jordan curve in the plane is homeomorphic to a disk–along the lines of [**6**]. Specifically, we show in Corollary 4.9 that if J is a Jordan curve, and p is a point inside J, then for all but countably many rays M from p there is a point q on $J \cap M$ such that if x is on M, and strictly between p and q, then the segment $[p, x]$ is bounded away from J.

1. Connectivity. An *ε-approximation* E to a metric space X is a subset of X such that given any x in X, there is e in E such that $d(x, e) < \varepsilon$. A metric space is *totally bounded* if it has a finite ε-approximation for every $\varepsilon > 0$. We consider the empty set to be finite, contrary to the usage of Bishop and Cheng. A metric space X is *compact* if it is complete and totally bounded. If X is a totally bounded metric space, then the *diameter* of X is $\sup\{d(x,y) : x, y \in X\}$, and is denoted by $\operatorname{diam} X$. For A a subset of X, the *δ-neighborhood* $N_\delta(A)$ of A is $\{x : d(x, a) < \delta \text{ for some } a \in A\}$.

Bishop calls a space X *connected relative to a family F* of real valued functions if the closure of fX is convex for each f in F [**2**, Problem 12, page 73]. Thus we call a totally bounded space X *connected* if whenever

f is a uniformly continuous function from X to $\mathbf{R}$, and $f(a) < y < f(b)$, then for each $\varepsilon > 0$ there is x in X such that $|f(x) - y| < \varepsilon$.

A δ-chain from x to y in a metric space x is a sequence

$$x = x_1, x_2, \cdots, x_n = y$$

in X such that $d(x_i, x_{i+1}) < \delta$ for $i = 1, 2, \cdots, n-1$. A totally bounded metric space X is *stepwise connected* if for every x and y in X, and $\delta > 0$, there is a δ-chain from x to y.

The following theorem characterizes connected, totally bounded metric spaces.

THEOREM 1.1. *Let X be a totally bounded metric space. Then the following conditions are equivalent.*

(1) *X is connected,*

(2) *X is stepwise connected,*

(3) *X is not the union of two nonempty sets which are bounded apart.*

PROOF. We show that (1) implies (2). Let x and y be points of X, and $\delta > 0$. Let E be a finite $\delta/3$-approximation to X containing x and y. Choose ε such that $2\delta/3 < \varepsilon < \delta$ and if $u, v \in E$, then $d(u, v) \neq \delta$. Let

$$M = \{u \in E : \text{ there exists an } \varepsilon - \text{ chain in } E \text{ from } x \text{ to } u\}.$$

Now $x \in M$, and the two sets $K = N_{\delta/3}(M)$ and $L = N_{\delta/3}(E/M)$ satisfy $d(K, L) > \varepsilon - 2\delta/3$ with $X = K \cup L$. Define a uniformly continuous function $f : X \to \mathbf{R}$ by $fK = 0$ and $fL = 1$. If X is connected, then E/M is empty and we are done.

Clearly (2) implies (3). We now show that (3) implies (1). Let $f : X \to \mathbf{R}$ be uniformly continuous and suppose $f(a) < y < f(b)$. If $d(y, fX) > 0$, then $\{x \in X : f(x) < y\} \cup \{x \in X : f(x) > y\} = X$, a contradiction. Hence $d(y, fX) = 0$ so y is in the closure of fX.

Let C be a collection of totally bounded subsets of a metric space X. If the union of C is dense in X, we say that C is a *cover* of X. A cover is an *ε-cover* provided $\operatorname{diam} U < \varepsilon$ for all U in C.

A compact metric space is *locally connected* if for each $\varepsilon > 0$ there is a finite ε-cover consisting of connected, totally bounded subsets. This condition is slightly weaker than that in [**2**, Problem 16, page 110] where it is required that X be the union of the sets in the cover. Classically both are equivalent to locally connected as usually defined [**4**, Exercises 3-6, 3-9, page 115].

2. Cheng's solution to Bishop's problem 15. In [**3**] the proof of (A) for $X = [0,1]$ is based on two lemmas which are stated for an arbitrary compact metric space. The first lemma is technically incorrect in that the set $X(\mu,\nu)$ claimed to be compact may in fact be empty, contrary to Cheng's usage of the term *compact*; and the fact that $X(\mu,\nu)$ is nonempty, is essential in the application. To insure that $X(\mu,\nu)$ is nonempty, it is necessary that the space X be connected. In his proof of (A) for $X = [0,1]$, Cheng claims that the conclusion of his first lemma implies that the hypotheses of his second lemma are satisfied. If this claim were correct, then his two lemmas would imply that (A) holds for any connected, compact metric space X. We now give a Brouwerian counterexample to this.

EXAMPLE 2.1. Let $\{a_n\}$ be an increasing sequence in $\{0,1\}$, let $I = [0,1]$, and let $B_n = \{k/n : 0 \le k \le n\}$. With $B = \cup\{B_n : a_n = 0\}$, let X be the closure in the plane of $(B \times I) \cup (I \times \{0\})$. Then X is compact and connected. Let $f : X \to I$ be projection onto the first coordinate and observe that f is onto I. Let $\{s_n\}$ be a countable sequence of real numbers. By Cantor's theorem [**2**, Theorem 1, page 25] there is an irrational point r in I that is different from each point in S. If $f^{-1}(r)$ is compact, then let E be a finite $1/2$-approximation to $f^{-1}(r)$. Either all points in E have second coordinate less than $1/2$, or some point in E has second coordinate larger than 0. In the former case, the a_n cannot all be zero, while in the latter, the a_n must all be zero.

3. Admissibility. Let Y be a metric space and S a subset of Y. We distinguish subsets A of Y having no boundary points in S. A subset A of Y will be called *S-admissible* if for each $s \in S$, there is $\delta(s) > 0$ such that $N_{\delta(s)}(s) \subset A$ or $d(s,A) > \delta(s)$. A point p is *S-admissible* if $\{p\}$ is *S-admissible.* If p is distinct from each point of S, then p is S-admissible.

A metric space A is *two-complete* if every Cauchy sequence in A

with at most two values converges to an element of A. The set $\{1/n : n = 1, 2 \ldots\}$ is an example of a set which is two-complete but not complete. Classically every metric space is two-complete, so adding an hypothesis that a space is two-complete does not decrease the classical strength of a theorem. Two-completeness allows us to find S-admissible points in S-admissible sets.

THEOREM 3.1. *Let Y be a complete metric space such that for each y in Y the set of points distinct from y is dense. Let S be a countable subset of Y, and A a two-complete, S-admissible subset of Y. Then the S-admissible points of A are dense in A.*

PROOF. Let $S = \{s_1, s_2, \ldots\}$. Let $a_1 \in A$ and let $\varepsilon > 0$. For each $n \geq 1$, let $S_n = \{s_1, s_2, \cdots, s_n\}$, and choose $\delta_n \leq \inf\{\delta(s_i) : i = 1, 2, \ldots, n\}$ such that $0 < 2\delta_{n+1} < \delta_n < \varepsilon$. Define a sequence $\{\lambda_n\}$ in $\{0, 1\}$ with the property that

$$\lambda_n = 0 \text{ implies there is } m < n \text{ with } \lambda_m = 1, \text{ or } d(a_1, S_n) > 0.5\delta_n,$$
$$\lambda_n = 1 \text{ implies } d(a_1, S_n) < \delta_n \text{ and } \lambda_m = 0 \text{ for } m < n.$$

We now define a sequence $\{a_n\}$ as follows.

If $\lambda_n = 0$, then $a_{n+1} = a_n$.

If $\lambda_n = 1$, then $d(a_1, S_n) < \delta_n$, so $a_1 \in \{y \in Y : d(y, s_n) < \delta_n\}$ which is a subset of A, as A is S-admissible and $\delta_n \leq \delta(s_n)$. Choose, by the Baire category theorem [**2**, Corollary, page 88], a point a_{n+1} of A within $0.5\delta_n$ of a_1, that is distinct from each point of S, and therefore S-admissible.

Since A is two-complete the Cauchy sequence $\{a_n\}$ converges to an element r of A. Clearly $d(a_1, r) < \varepsilon$. We shall show for $n > 0$ that $d(r, s_n) > 0$. If there is $m < n$ such that $\lambda_m = 1$, we are done as r is S-admissible. If $\lambda_m = 0$ for all $m \leq n$, then for $m > n$ we have $d(a_m, a_{m+1}) < 0.5\delta_m$. Therefore $d(a_1, r) < 0.5\delta_{n+1}$. So $d(r, S_n) \geq d(a_1, S_n) - d(r, a_1) \geq 0.5(\delta_n - \delta_{n+1}) > 0$.

The hypothesis that A is two-complete is needed to construct even a single S-admissible point of A, as the following example shows.

EXAMPLE 3.2. Let $\{\lambda_n\}$ be an ascending sequence in $\{0,1\}$. Set

$$Y = [-1,1]$$
$$A = \cup_n[-n^{-1}\lambda_n, n^{-1}\lambda_n],$$
$$S = \{s_n : s_n = n^{-1}(1-\lambda_n) \text{ for some positive integer } n\}.$$

Then A is S-admissible, taking $\delta(s_n) < n^{-1} - (n+1)^{-1}$. However, given an S-admissible point a in A, we could find a positive integer m such that $a \in [-m^{-1}\lambda_m, m^{-1}\lambda_m]$. If $\lambda_m = 0$, then $a = 0$; so $0 \notin S$ and $\lambda_n = 0$ for all n.

4. Solution to Bishop's problem 16. Let X be a compact connected, locally connected metric space and $f : X \to T$ be uniformly continuous. We are concerned with when $f^{-1}A$ is compact for a given compact subset A of T. Note that this problem includes the same problem with the unit circle T replaced by $\mathbf{R}$. It is hard to find finite approximations to $f^{-1}A$ because there may be a point s in T at which f^{-1} is discontinuous, but we can neither bound s away from A nor establish that s is in A. We can avoid this difficulty if we can cover X with small connected sets U such that the endpoints of fU are not in the boundary of A. This motivates the following definitions.

Let T denote the unit circle. If A is a totally bounded subset of T of diameter less than 1, then A is contained in a unique arc *Arc* A of diameter less than 1, with endpoints *Ends* A in the closure of A.

Let X be a compact metric space and let $f : X \to T$ be uniformly continuous. Let A be a totally bounded subset of T. A cover C *f-admits* A at ε if for each U in C:

(1) U is connected, totally bounded, $\operatorname{diam} fU < 1$, and $\operatorname{diam} U < \varepsilon$.

(2) A is (Ends fU)-admissible.

The set A is *f-admissible* if for each $\varepsilon > 0$ there is a finite cover that f-admits A at ε. Thus if A if f-admissible, then X is locally connected. A point t is *f-admissible* if $\{t\}$ is f-admissible. Refering to Example 0.1 we note that 0 is not f-admissible, while every $a \neq 0$ is f-admissible.

LEMMA 4.1. *Let X be a compact, locally connected, metric space. Let $f : X \to T$ be uniformly continuous. Let $U \subset X$ be connected and $\operatorname{diam} fU < 1$. If t is f-admissible, and t is in the interior of $\operatorname{Arc} fU$, then for all $\varepsilon > 0$ there is $x \in X$ with $f(x) = t$ and $d(x,U) < \varepsilon$.*

PROOF. We shall construct a sequence of connected subsets U_n of X such that $U_o = U$, and for $n \geq 1$ we have:

(0) U_n is totally bounded,

(1) $\operatorname{diam} U_n < 2^{-n}\varepsilon$ and $\operatorname{diam} fU_n < 1$,

(2) there is $x_n \in U_{n-1}$ such that $d(x_n, U_n) < 2^{-n}\varepsilon - \operatorname{diam} U_n$,

(3) t is in the interior of Arc fU_n.

Assume that we have constructed U_{n-1}. As t is f-admissible, there is a $2^{-n}\varepsilon$-cover C of X consisting of totally bounded, connected sets whose images under f have diameters less than 1, and the distance δ from t to Ends $fC = \cup\{$ Ends $fU : U \in C\}$ is positive. As U_{n-1} is connected and t is in the interior of Arc fU_{n-1}, there exists $x_n \in U_{n-1}$ with $d(f(x_n), t) < \delta/2$. Let $\xi < 2^{-n}\varepsilon$ be so that diam $V < \xi$ if V is in C. As C is a cover, there exists U_n in C with $d(x_n, U_n) < 2^{-n}\varepsilon - \xi$, and $d(f(x_n), fU_n) < \delta/2$, the latter by continuity of f at x_n. Conditions (0), (1), and (2) are clear. As $d(t, \text{Ends } fU_n) \geq \delta$ and $d(f(x_n), t) < \delta/2$, we have (3).

The sequence $\{x_n\}$ is Cauchy as $d(x_n, x_{n+1}) < 2^{-n}\varepsilon$. If $x = \lim x_n$, then $d(x, U) \leq \varepsilon$ as $x_1 \in U$. As $\lim \operatorname{diam} U_n = 0$, and $x_n \in U_{n-1'}$ and $t \in \operatorname{Arc} fU_n$ and f is uniformly continuous, it follows that $f(x) = t$.

LEMMA 4.2. *Let X be a compact, locally connected, metric space, and $\varepsilon > 0$. Let $f : X \to T$ be uniformly continuous. Let C be a finite $\varepsilon/3$-cover of X by connected sets with $\operatorname{diam} fU < 1$ for all $U \in C$. Let $I \subset T$ be a nonempty arc bounded away from Ends fC. If $t \in I$ is f-admissible, then $f^{-1}I$ has a finite ε-approximation contained in $f^{-1}(t)$.*

PROOF. Since I is bounded away from Ends fC and C is finite, either $f^{-1}(I) = \phi$ or there is $U \in C$ with $I \subset \operatorname{Arc} fU$. Let $S = \{U \in C : I \subset \operatorname{Arc} fU\}$. Let $t \in I$ be f-admissible. For each $U \in S$ construct by Lemma 4.1 a point x_U in $f^{-1}(t)$ with $d(x_U, U) < \varepsilon/3$.

We show that $\{x_U : U \in S\}$ is an ε-approximation to $f^{-1}I$: Let $y \in f^{-1}I$. Since f is continuous and C is a cover, we can choose U in C so that $d(y, U) < \varepsilon/3$ and $d(f(y), fU) < d(I, \text{Ends } fU)$. Since $f(y)$ and t are in I, it follows that U is in S. Then $d(x_U, y) < \varepsilon$.

If A and B are totally bounded subsets of a metric space X, then the Hausdorff metric $\rho(A, B)$ is the supremum of $\sup_{a \in A} d(a, B)$ and

$\sup_{b \in B} d(b, A)$. Thus $\rho(A, B) < \varepsilon$ if and only if $d(a, B) < \varepsilon$ for each a in A and $d(b, A) < \varepsilon$ for each b in B. Note that $\rho(\phi, \phi) = 0$ but $\rho(\phi, B) = \infty$ if B is nonempty. The next theorem shows that f^{-1} is a pointwise continuous function from the set of f-admissible points to the compact subsets of X in the Hausdorff metric.

THEOREM 4.3. *Let X be a compact, locally connected, metric space, let $f : X \to T$ be uniformly continuous and let r be f-admissible. Then $f^{-1}(r)$ is compact. Moreover for each $\eta > 0$ there is $\delta > 0$ such that if s and t are f-admissible points in $N_\delta(r)$, then $\rho(f^{-1}(s), f^{-1}(t)) < \eta$.*

PROOF. Let $\varepsilon > 0$ and let C be a finite cover f-admitting r at $\varepsilon/3$. By Lemma 4.2, $f^{-1}(r)$ has a finite ε-approximation. Thus $f^{-1}(r)$ is totally bounded, and since it is closed in X, it is compact.

Now let C be a finite cover f-admitting r at $\eta/3$, let $\delta = \frac{1}{2}d(r, \operatorname{Ends} fC)$, and let s and t be f-admissible points of $N_\delta(r)$. Suppose that $f(x) = s$. Choose $U \in C$ so $d(s, fU) < \delta/2$ and $d(x, U) < \eta/3$. Then s and t are in the interior of $\operatorname{Arc} fU$. Apply Lemma 4.1 to get a point y in X such that $f(y) = t$ and $d(y, U) < \eta/3$. As $\operatorname{diam} U < \eta/3$, we have $d(x, y) < \eta$. Thus $\rho(f^{-1}(s), f^{-1}(t)) < \eta$.

The next theorem shows that f^{-1} is continuous from above at each admissible two-complete set.

THEOREM 4.4. *Let X be a compact, locally connected, metric space and let $f : X \to T$ be uniformly continuous. Let $A \subset T$ be f-admissible and two-complete. Then for all $\varepsilon > 0$, there is $\delta > 0$ so that $f^{-1}A$ contains a finite ε-approximation to $f^{-1}N_\delta(A)$.*

PROOF. Let $\varepsilon > 0$ and let C be a finite cover of X that f-admits A at $\varepsilon/3$. Choose $\delta > 0$ so that for all s in $\operatorname{Ends} fC$ either $N_\delta(s) \subset A$ or $d(s, A) > 2\delta$. Write $C = M \cup N$ disjoint so that for each $U \in C$:

If $U \in M$, then there is a in A which is either in $\operatorname{Ends} fU$ or in the interior of $\operatorname{Arc} fU$.

If $U \in N$, then $d(fU, A) > 2\delta$. We can do this as follows. Let $u \in C$. Either there is a in $A \cap \operatorname{Ends} fU$, in which case put $U \in M$, or $d(\operatorname{Ends} fU, A) > 2\delta$. In the latter case, either $\sigma = \operatorname{diam} \operatorname{Ends} fU < \delta$, in which case put $U \in N$, or $\sigma > 0$. In the latter case choose a finite $\sigma/2$-approximation E to A. As $d(\operatorname{Ends} fU, A) > \delta$, we can decide whether any element of E is in Arc fU; if no element of E is in Arc fU, then put $U \in N$, otherwise put $U \in M$.

Suppose $U \in M$. If there is $a \in A$ in the interior of Arc fU, then by Theorem 3.1, we may assume that a is f-admissible. Choose by Lemma 4.1 a point x_U in $f^{-1}A$ such that $d(x_U, U) < \varepsilon/3$. On the other hand, if there is a in $A \cap \text{Ends}\, fU$, then $N_\delta(a) \subset A$, so there is a point x_U in U with $f(x_U)$ in A.

We now show that $\{x_U : U \in M\}$ is a finite ε-approximation in $f^{-1}A$ to $f^{-1}N_\delta(A)$: Let $y \in f^{-1}N_\delta(A)$ and choose U in C such that $d(y, U) < \varepsilon/3$ and $d(f(y), fU) < \delta$. Since $f(y)$ is in $N_\delta(A)$ we have U in M. Thus $d(x_U, y) < \varepsilon$.

COROLLARY 4.5. *Let X be a compact metric space and let $f : X \to T$ be uniformly continuous. Let $A \subset T$ be f-admissible and compact. Then $f^{-1}A$ is compact.*

While Theorem 4.4 shows that f^{-1} is continuous from above at each admissible two-complete set, it need not be continuous from below as the next example shows:

EXAMPLE 4.6. Let f be the piecewise linear function from $[0, 1]$ to $\mathbf{R}$ defined by $f(0) = -1, f(1/3) = 0 = f(2/3)$, and $f(1) = 1$. Let $A = [-1/2, 1/2]$ and $B_\delta = [-1/2, -\delta] \cup [\delta, 1/2]$. Then $\lim_{\delta \to 0} \rho(B_\delta, A) = 0$, but $\rho(f^{-1}B_\delta, f^{-1}A) \geq 1/6$.

THEOREM 4.7. *Let X be a compact, connected, locally connected metric space and $f : X \to T$ be uniformly continuous. Then there is a countable subset S of T so that if t is S-admissible, then $f^{-1}(t)$ is compact. Furthermore the restriction of f^{-1} to the S-admissible points of T is pointwise continuous. Finally, if t is f-admissible and $f^{-1}(t)$ is empty, then there exists $\delta > 0$ such that $f^{-1}N_\delta(t)$ is empty.*

PROOF. Let $\varepsilon > 0$ be such that $\text{diam}\, U < \varepsilon$ implies that $\text{diam}\, fU < 1$. For each positive integer n, let C_n be a finite ε/n-cover of X, consisting of connected totally bounded subsets of X. Let $S_n = \text{Ends}\, fC_n$ and $S = \cup S_n$. Let A be the set of S-admissible points of X. Note that each point in A is f-admissible. By Theorem 4.3, $f^{-1}(t)$ is compact for each $t \in A$ and f^{-1} restricted to A is continuous. Let $t \in A$ be such that $f^{-1}(t)$ is empty. Then by Theorem 4.4 with $\varepsilon = 1$, there is $\delta > 0$ such that $f^{-1}(t)$ contains a 1-approximation to $f^{-1}N_\delta(t)$. Thus

$f^{-1}N_\delta(t)$ is empty.

Example 5.1 will show that it is necessary that t be S-admissible in the last sentence of the Theorem 4.7. We now show that Problem 15 holds with $[0,1]$ replaced by any compact, connected locally connected space X.

COROLLARY 4.8. *Let X be a connected, locally connected, compact metric space and let $f : X \to \mathbf{R}$ be uniformly continuous. Suppose that $\alpha < \beta$ are in fX. Then for all but countably many t with $\alpha < t < \beta$ the set $f^{-1}(t)$ is non-empty and compact.*

PROOF. Since X is compact, we can assume f maps X to T and $\operatorname{diam} fX < 1$. Note that since X is connected the closure of fX contains $\operatorname{Arc}\{\alpha, \beta\}$. Let S be as in Theorem 4.7. If t is S-admissible it follows that $f^{-1}(t)$ is compact, and hence either empty or non-empty. Moreover, if t is also in $\operatorname{Arc}\{\alpha, \beta\}$ then $f^{-1}(t)$ cannot be empty for otherwise there would exist $\delta > 0$ such that $f^{-1}N_\delta(t)$ is empty. But the closure of fX contains $\operatorname{Arc}\{\alpha, \beta\}$. Thus $f^{-1}(t)$ must be nonempty.

The following corollary is what is needed to give a constructive proof of the Schoenflies Theorem [**6**, Chapter 9, Theorem 2].

COROLLARLY 4.9. *If J is a Jordan curve and p is a point inside J, then for all but countably many rays M from p there is a point q on $J \cap M$ such that if x is on M and strictly between p and q then the segment $[p, x]$ is bounded away from J.*

PROOF. Let $\eta = d(p, J)$ and let f be radial projection of J into the unit circle T with center p. Let S be as in Theorem 4.7. If $t \in T$ is S-admissible, and M is the ray from p through t, then $f^{-1}(t) = J \cap M$ is compact. Thus $J \cap M$ is empty or nonempty. Were $J \cap M$ empty, then by Theorem 4.7 there would be $\delta > 0$ so that $f^{-1}N_\delta(t)$ is empty. Some neighborhood of the ray M, connecting p to infinity, would then be contained in

$$w_\delta = N_\eta(p) \cup \{(1-r)p + r(t+\theta) : r \geq 0, |\theta| < \delta\}$$

which misses J. But this cannot occur for p inside J [**1**]. Thus $M \cap J$ is non-empty. The set $\{r \geq 0 : (1-r)p + rt \in J\}$ is isometric to $M \cap J$, so is compact; let r_0 denote its infimum. Let $q = (1 - r_0)p + r_0 t$. Let

$x = (1-r)p + rt$ for $0 \leq r < r_0$ and $2\varepsilon = r - r_0$. By Theorem 4.4 choose $\delta > 0$ so that $f^{-1}(t)$ contains an ε-approximation to $f^{-1}N_\delta(t)$. Then some neighborhood of $[p, x]$ is contained in $W_\delta \cap N_{r+\varepsilon}(p)$ which misses J.

5. Church's thesis. It is plausible that Bishop's Problem 15 can be easily derived from his Problem 16. The reasoning goes as follows: Let X be a compact, connected, locally connected space, f a uniformly continuous function from X to $\mathbf{R}$ and $f(a) < t < f(b)$. Classically the set $f^{-1}(t)$ is nonempty, so even constructively it should be contradictory for $f^{-1}(t)$ to be empty. Thus if by Problem 16 we conclude that $f^{-1}(t)$ is compact, then we can rule out that $f^{-1}(t)$ is empty and incorrectly conclude that $f^{-1}(t)$ is nonempty. However the classical impossibility that $f^{-1}(t)$ is empty, cannot be established constructively. In fact if we assume Church's thesis that the computable functions are precisely the recursive functions, together with the standard constructive thesis that all functions are computable, then we can construct an example where $f^{-1}(t)$ is in fact empty. Of course Church's thesis in this context is classically false, as not every function is recursive, but it is unlikely to be constructively refutable.

EXAMPLE 5.1. Using Church's thesis we can construct a positive function g on the unit interval with infimum 0 [**5**], [**7**]. Let X be the union of the graphs of g and $-g$; let f be projection onto the y-axis. It is easy to show that X is compact (because g is positive), locally connected (because X is the union of two Jordan arcs), and connected (because $\inf g = 0$). For each positive δ the set $f^{-1}N_\delta(0)$ is non-empty but $f^{-1}(0)$ is empty.

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