ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 4, Fall 1987

A NOTE ON TWO–GENERATOR GROUPS

MARTIN J. EVANS

Following J.L. Brenner and James Wiegold [**1**], let $\Gamma_1^{(2)}$ stand for the collection of all finite non–abelian groups G with the property that every non–trivial element is in a two–element generating set of G in which one element is of order two.

In [**1**] it is shown that $\mathrm{PSL}\,(2, q) \in \Gamma_1^{(2)}$ except when $q = 2, 3$ or 9. This led the above mentioned authors to ask whether almost all finite simple groups in $\Gamma_1^{(2)}$ are projective special linear groups.

In this note we answer this question negatively by showing that $\Gamma_1^{(2)}$ contains the Suzuki groups $Sz(2^{2n+1}), (n \geq 1)$. However, in the opposite direction we prove that the groups $\mathrm{PSL}\,(2, p^m)$, with p an odd prime, $p^m \neq 3$ or 9, are the only simple Chevalley groups over a field of odd characteristic that are contained in $\Gamma_1^{(2)}$.

Throughout the proof of the following theorem we use standard facts concerning Suzuki groups. These can be found in [**3**].

THEOREM 1. *Let $G = Sz(q)$ be a Suzuki group, where $q = 2^{2n+1}$ and $n \geq 1$. Then $G \in \Gamma_1^{(2)}$.*

PROOF. Given $x \in G$, we shall say that y is a *mate* for x in G if $\langle x, y\rangle = G$. Let Q be a Sylow 2–subgroup of G and let $z \in G$ be an involution not contained in Q. It follows from [**3**; Proposition 13] that each non–trivial element of odd order in G is conjugate to an element of the form πz where π is an involution in Q. In particular there exist involutions $\pi_1, \pi_2 \in Q$ such that $\pi_1 z$ is of order $q - 1$ and $\pi_2 z$ is of order $q + r + 1$, where $r^2 = 2q$.

Let $x \in G$ be a non–trivial element of odd order. We aim to show that there exists an involution π_x such that $< x, \pi_x >= G$. Clearly it suffices to prove that some conjugate of x has a mate of order two in G. Therefore we may assume that $x = \pi z$ for some involution $\pi \in Q$. We distinguish two cases: (i) the order of x divides $q^2 + 1$; (ii) the order of

Received by the editors on April 24, 1986.

x divides $q-1$.

Case (i). Consider $H = <\pi_1\pi, \pi z>$. Now H contains $(\pi_1\pi)(\pi z) = \pi_1 z$ which is of order $q-1$. However, the maximal subgroups of G that contain elements of order $q-1$ are Frobenius groups of order $q^2(q-1)$ and dihedral groups of order 2(q-1). Since $\pi z \neq 1$ is of order dividing q^2+1 we deduce that $H = G$. Moreover, $\pi_1\pi$ is an involution, since each involution in Q is central in Q. Thus $x = \pi z$ has a mate of order two.

Case (ii). Consider $K = <\pi_2\pi, \pi z>$. Now K contains $(\pi_2\pi)(\pi z) = \pi_2 z$ which is of order $q+r+1$. A subgroup of order $q+r+1$ lies in a unique maximal subgroup of G, namely its normalizer which is of order $4(q+r+1)$. Since K contains $\pi z \neq 1$ which is of order dividing $q-1$, we deduce that $K = G$. Thus $\pi_2\pi$ is a mate of order two for x in G.

The above argument shows that each non–trivial element of G of odd order has a mate of order two. Furthermore, since all involutions in G are conjugate, it shows that every involution has a mate in G.

It only remains to show that each element of order four has a mate of order two in G. Now G has exactly two conjugacy classes of elements of order four, C_1 and C_2 say, and these are such that $x \in C_1$ if and only if $x^{-1} \in C_2$ [**3**; Proposition 18]. Therefore it suffices to show that some element of order four has a mate of order two. Such a pair of generators of G has been found by M. Suzuki [**3**, p. 140].

Throughout the remainder of this note our notation is that of [**2**] and is more or less standard. We shall also use some basic facts about Chevalley groups which can again be found in [**2**].

THEOREM 2. *Let* $G = \mathfrak{L}(K)$ *be a simple Chevalley group over a finite field* K *of odd characteristic* p. *If* $G \in \Gamma_1^{(2)}$, *then* $G \simeq \mathrm{PSL}\,(2, K)$.

PROOF. Let ϕ denote a system of roots of $\mathfrak{L}$, so that G is generated by $\{x_r(k) | r \in \phi, k, \in K\}$ and let X_r denote the root subgroup generated by $\{x_r(k) | k \in K\}$. Furthermore, let U be the subgroup generated by $\{x_r(k) | r \in \phi^+, k \in K\}$ where ϕ^+ is a positive system of roots with respect to some fundamental system of ϕ. Now U is nilpotent and, letting s denote a root of greatest height, X_s is a central subgroup of U [**2**; Theorem 5.3.3].

From now on suppose that $G \in \Gamma_1^{(2)}$. Since $x_s(1)$ is of order p and $G \in \Gamma_1^{(2)}$, there exists an involution $y \in G$ such that $G =$

$\langle x_s(1), y\rangle$. Notice that $G = \langle x_s(1), y^{-1}x_s(1)y\rangle$, since $x_s(1)$ and y normalize $\langle x_s(1), y^{-1}x_s(1)y\rangle$ and $G = \langle x_s(1), y\rangle$ is simple. We shall only require the fact that $G = \langle X_s, X_s^y\rangle$.

Using the Bruhat decomposition of G we can write $y = u_1h_1n_tu_2h_2$ where $u_1, u_2 \in U, h_1, h_2 \in H$ and $n_t \in N$. Here H and N denote the diagonal and monomial subgroups of G respectively. For the definitions and basic properties of H and N see [**2**; Chapter 7]. Now X_s is central in U and $H \leq N_G(X_s)$, so

$$G = \langle X_s, X_s^{n_t u_2 h_2}\rangle = \langle X_s^{h_2^{-1}u_2^{-1}}, X_s^{n_t}\rangle = \langle X_s, X_s^{n_t}\rangle.$$

Let w_t denote the image of n_t in the Weyl group W under the natural homomrphism from N to $N/H \simeq W$. Now $n_t^{-1}X_sn_t = X_{w_t(s)}$ [**2** Lemma 7.2.1] so $G = \langle X_s, X_{w_t(s)}\rangle$.

If $r_1, r_2 \in \phi$ are linearly independent then there exists $w \in W$ such that $w(r_1), w(r_2) \in \phi^+$ [**2**; Proposition 2.1.8, Lemma 2.1.5]. It follows that $\langle X_{r_1}, X_{r_2}\rangle$ is conjugate to a subgroup of U and is therefore nilpotent. Since $G = \langle X_s, X_{w_t(s)}\rangle$ is simple, we deduce that $w_t(s) = -s$, so $G = \langle X_s, X_{-s}\rangle$. Now there exists a homomorphism from $\mathrm{SL}(2, K)$ onto $\langle X_s, X_{-s}\rangle$ [**2**; Theorem 6.3.1] and as $\langle X_s, X_{-s}\rangle = G$ is simple, it follows that $G \simeq \mathrm{PSL}(2, k)$ as required.

References

1. J.L. Brenner and James Wiegold, *Two–generator groups,* I Michigan Math. J. **22** (1975), 53-64.

2. Roger W. Carter, *Simple Groups of Lie Type* Interscience Publishers, New York, 1972.

3. Michio Suzuki, *On a class of doubly transitive groups,* Ann. of Math. (2) **75**, (1962) 105-145.

Department of Mathematics, University of Alabama, Univerity, AL 35486