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GREAT SPHERE FIBRATIONS OF MANIFOLDS

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1. Introduction. Let E be a smooth, closed n–manifold which is

a) smoothly fibred by k–spheres, and

b) smoothly embedded in S^N (the unit N–sphere in R^{N+1}) so that these k–sphere fibres appear as *great* k–spheres in S^N. We say simply that E is *fibred by great k–spheres.* In this paper we study great sphere fibrations of certain manifolds E and examine what restrictions the geometric constraint (b) above places on the class of topological k–sphere fibrations of E.

A good example to keep in mind is that of 3–sphere fibrations of the 7 sphere. There are infinitely many topologically inequivalent smooth 3–sphere fibrations of the 7–sphere [**5**]. By Proposition 2.1 below, each such fibration may be pictured as a fibration by great 3–spheres, provided we choose a suitable embedding of the 7–sphere into a large dimensional sphere S^N. If we insist that the 7–sphere appear as the unit sphere in $\mathbf{R}^8$, then Gluck, Warner, and Yang have recently shown that every smooth fibration of it by great 3–spheres is topologically equivalent to the Hopf fibration [**2**].

This illustrates the general expectation, namely, when we lower the dimension of the sphere in which we permit the total space to be embedded or place geometric constraints on the total space we correspondingly restrict the bundles whose fibres can thus be made into great k–spheres. It is in this way that geometric theory departs from the topological theory.

If $E \subseteq S^N$ is fibred by great k–spheres there is a hierarchy of three questions, in increasing order of difficulty, which guides our study:

1) Given two such fibrations, are they topologically equivalent?

2) If they are topologically equivalent, is it possible to deform one to the other through a one–parameter family of such fibrations?

3) What is the homotopy type of the space of all such fibrations?

In general, even question 1 remains unanswered for all but the simplest cases; however, all three questions were completely answered for great circle fibrations of the round of 3–sphere [**1**].

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The above referenced paper Gluck, Warner, and Yang shows how one is led to questions about great sphere fibrations of manifolds E, in particular where E is itself a round sphere, in working on the topological problem of the Blaschke conjecture.

Summary of Major Results. In §2 we prove a realization theorem which says that all reasonable k–sphere bundles can be pictured with great k–sphere fibres by embedding the total space into a large dimensional sphere.

PROPOSITION 2.1. *Let $\xi : S^k \to E \to B$ be a smooth k–sphere bundle with group $0(k+1)$ over a compact base space B. Then the total space E can be smoothly embedded into S^N for N sufficiently large so that each k–sphere fibre becomes a great k–sphere in S^N.*

The results in **[1]** and **[2]** deal with fibrations of the round sphere S^n by great k–spheres. The next simplest case to study seems to be $S^p \times S^q$ embedded in S^{p+q+1}, and the remainder of this paper is devoted to studying great k–sphere fibrations of such manifolds.

In §3 we prove some general statements about such fibrations and then we begin sampling the theory for small values of p, q, and k. Great circle fibrations of $S^1 \times S^3$ prove to be interesting and some elementary questions about them remain unanswered. Another sample: $S^6 \times S^{13}$ admits no fibrations by great k–spheres for any $k \geq 1$ (while it obviously admits fibrations by 1–spheres, 6–spheres and 13–spheres if we drop the restriction that the fibration be by great spheres in S^{20}).

By far the richest and most satisfying theory we develop is in §4 where we treat great 3–sphere fibrations of $S^3 \times S^3$. We completely answer the three questions posed above when we prove

THEOREM A. *The space of all oriented great 3–sphere fibrations of $S^3 \times S^3$ deformation retracts to the subspace of "Hopf fibrations" and has the homotopy type of a disjoint union of four copies of real projective 3–space, RP^3.*

In the course of proving this we also get:

1) There is a 2 to 1 correspondence between distance decreasing maps from S^3 to S^3 and great 3–sphere fibrations of $S^3 \times S^3$.

2) These fibrations are smooth if and only if the distance decreasing map is smooth and the norm of its differential is strictly less than 1.

3) Every such fibration has an orthogonal pair of fibres.

These are analogous to results obtained in [**1**] for great circle fibrations of S^3.

Using Theorem A we prove

THEOREM B. *Every smooth great 3–sphere fibration of $S^3 \times S^3$ can be extended to a smooth great 3–sphere fibration of S^7.*

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2. Realization theorem.

PROPOSITION 2.1. *Let $\xi : S^k \to E \xrightarrow{\pi} B$ be a smooth k–sphere bundle with group $0(k+1)$ over the compact base space B. Then the total space E can be smoothly embedded into S^N for N sufficiently large so that each k–sphere fibre becomes a great k–sphere in S^N.*

PROOF. 1. Since the group of the bundle is $0(k+1)$ there is an associated Euclidean $(k+1)$–plane bundle, $\xi' : \mathbf{R}^{k+1} \to E' \to B$, over B such that ξ is the unit sphere bundle of ξ'. By [**4**; Lemma 5.3] there exists an integer p and a bundle map $\tilde{f} : \xi' \to V^{k+1}\mathbf{R}^{p+1}$, where $V^{k+1}\mathbf{R}^{p+1}$ is the canonical $(k+1)$–plane bundle over the Grassmann manifold $G_{k+1}R^{p+1}$.

$$\begin{array}{ccc} \tilde{f} : E' & \longrightarrow & E(V^{k+1}\mathbf{R}^{p+1}) \\ \downarrow & & \downarrow \\ \bar{f} : B & \longrightarrow & G_{k+1}\mathbf{R}^{p+1} \end{array}$$

Now $E \subseteq E'$ and we may assume that $\tilde{f}|E : E \to S^p \subseteq \mathbf{R}^{p+1}$, so let $f = \tilde{f}|E$. Since $f(\pi^{-1}(b)) = \tilde{f}(\pi^{-1}(b)) \cap S^p$, it's clear that $f(K$–sphere fibre) is a great k–sphere in S^p. But while f is an embedding on each fibre, f is by no means an embedding of E in S^p.

2. If $n = 2 \cdot \dim B$ by the Whitney Embedding Theorem there exists an embedding $\psi : B \to S^n$. Consider the map

$$(\psi \circ \pi, f) : E \to S^n \times S^p.$$

For $a_1 \neq a_2$ in E, if $\pi(a_1) \neq \pi(a_2)$, then $\psi \circ \pi(a_1) \neq \psi \circ \pi(a_2)$. If $\pi(a_1) = \pi(a_2)$, then $f(a_1) \neq f(a_2)$ since f is injective on each fibre. Therefore $(\psi 0 \pi, f)$ is an injective map from a compact space E into a Hausdorff space $S^n \times S^p$, so it must be a homeomorphism onto its image.

3. At $a \in \pi^{-1}(x) \subseteq E$, the tangent space T_aE decomposes into a direct sum, $T_aE = T_a\pi^{-1}(x) \oplus T_a\hat{B}$ where $T_a\hat{B}$ is transverse to the fibre $\pi^{-1}(x)$, and consequently maps bijectively via $d\pi$ on T_xB. Since $df_a|T_a\pi^{-1}(x)$ is injective and $d(\psi \circ \pi)_a T_a\hat{B}$ is injective, it follows that $(\psi o f, \pi)$ is an immersion.

4. Together, the results of 2 and 3 imply $(\psi \circ \pi, f) : E \to S^n \times S^p$ is a smooth embedding. The following lemma completes the proof of Theorem 2.1.

LEMMA 2.2. *For any pair of positive integers n and p, there exists an N and a smooth embedding of $S^n \times S^p$ into S^N, taking $\{a\} \times S^p$ linearly onto a great p–sphere in S^N for each $a \in S^n$.*

PROOF. Let $N = (p+1)(n+1)+p$ and $S^N \subseteq \mathbf{R}^{N+1}$. We show there is a smooth embedding $\phi : S^n \to V_{p+1}\mathbf{R}^{N+1}$, the Stiefel manifold of orthonormal $(p+1)$–frames in $\mathbf{R}^{N+1}$, such that, for $a \neq b$ in S^n, the $(p+1)$–plane { span $\phi(a)$}, intersects the $(p+1)$–plane { span $\phi(b)$} only at the origin.

Define $\phi : S^n \to \mathbf{R}^{N+1} \times \mathbf{R}^{N+1} \times \cdots \times \mathbf{R}^{N+1}$ $(p+1$ copies)

$$\phi_1(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{2}}(1, 0, \dots, 0, x_1, \dots, x_{n+1}, 0, \dots, 0)$$

where x_1 is in the $p + 2^{\text{nd}}$ entry

$$\phi_2(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{2}}(0, 1, 0, \ldots, 0, \ldots, 0, x_1, \ldots, x_{n+1}, 0, \ldots, 0)$$

where x_1 is in the $p + n + 3^{\mathrm{rd}}$ entry

$$\vdots$$

$$\phi_{p+1}(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, 0, \ldots, 0, x_1, \ldots, x_{n+1})$$

where 1 is in the $p + 1^{\mathrm{st}}$ entry and x_1 is in the $p + pn + p + 2^{\mathrm{nd}}$ entry.

Let $\phi = (\phi_1, \phi_2, \ldots, \phi_{p+1})$. It is simple to verify that the map $\tau : S^n \times S^p \to S^N$ given by

$$\tau(a, (\alpha_1, \ldots, \alpha_{p+1})) = \sum_{i=1}^{p+1} \alpha_i \phi_i(a)$$

satisfies the requirements of the lemma.

Finally, $\tau o(\psi o \pi, f) : E \to S^N$ is an embedding that satisfies the requirements of Proposition 2.1.

DEFINITION 2.3. The bad cone of a point $Q \in G_k\mathbf{R}^n$ is the set of points $P \in G_k\mathbf{R}^n$ that correspond to k–planes in $\mathbf{R}^n$ which intersect the k–plane Q in more than the origin; bad cone of $Q = \{P \in G_k\mathbf{R}^n : \exists v \in P \cap Q \text{ with } v \neq 0\}$.

COROLLARY 2.4. *Given the hypothesis of Proposition* 2.1, *the base space B of a smooth k–sphere bundle has a smooth embedding in $G_{k+1}\mathbf{R}^{r+k+1}$, for r sufficiently large, such that B is transverse to the bad cone through each of its points.*

PROOF. From Proposition 2.1 we have produced a smooth great k–sphere fibration of a submanifold, $\tau o(\psi o \pi, f)(E)$ of S^N. Although we don't have a fibration of all of S^N by great k–spheres, the identical proof of [**2**; Theorem 4.1] carries through to produce the result.

If P is a k–plane in $\mathbf{R}^n, k < n$, and $P^\perp$ denotes the orthogonal $(n-k)$–plane and $\phi : U \to \operatorname{Hom}(P, P^\perp)$ is an embedding of an open subset U of a manifold X, then since ϕ is injective, for $x \neq y$ in $U, \phi(x) \neq \phi(y)$. So, for $x \neq y$, there exists a non–zero $v \in P$ with $\phi(x)(v) \neq \phi(y)(v)$.

DEFINITION 2.5. Given the situation just described, we say ϕ is a *strongly injective embedding* if, for $x \neq y$ in $U, \phi(x)(v) \neq \phi(y)(v)$ for *all* non–zero $v \in P$. And ϕ is a *smooth strongly injective embedding* if, in addition, for all non–zero $v \in P$, the map $\phi_v : U \to P^\perp$ given by $\phi_v(x) = \phi(x)(v)$ is an immersion.

If B is an embedded submanifold of $G_{k+1}\mathbf{R}^{N+1}$ such that B represents the base space of a great k–sphere fibration of some submanifold $E = (\bigcup_{Q\in B} Q) \bigcap S^N \subseteq S^N$, then for all $Q \in B$ and coordinate maps $\phi_Q : U_Q \subseteq G_{k+1}\mathbf{R}^{N+1} \to \operatorname{Hom}(Q, Q^\perp), \phi_Q|U_Q \cap B$ is a strongly injective embedding. In addition, if B is the base space of a smooth great k–sphere fibration of E, then B is transverse to the bad cone through each of its points and $\phi_Q|U_Q \cap B$ is a smooth strongly injective embedding.

3. Great k–sphere fibrations of $S^m \times S^n$. Throughout this part, by abuse of notation, we let $S^m \times S^n$ denote the submanifold of S^{m+n+1} given by

$$S^m_{\frac{1}{\sqrt{2}}} \times S^n_{\frac{1}{\sqrt{2}}} = \left\{(x_1, \ldots, x_{m+n+2}) : \sum_{i=1}^{m+1} x_i^2 = \frac{1}{2}, \sum_{i=m+2}^{m+n+2} x_i^2 = \frac{1}{2}\right\}.$$

It should be clear from the context and thus cause no confusion when we write $S^m \times S^n$ whether we mean a product of unit spheres or a product of spheres of radius $\frac{1}{\sqrt{2}}$.

Let $p_1 : \mathbf{R}^{m+n+2} \to \mathbf{R}^{m+1}$ be the projection on the first $m+1$ coordinates and $p_2 : \mathbf{R}^{m+n+2} \to \mathbf{R}^{n+1}$ be the projection on the last $n+1$ coordinates.

LEMMA 3.1. *A great k–sphere of S^{m+n+1} lying inside $S^m \times S^n$ gives an isometry from a great k–sphere in S^m onto a great k–sphere in S^n.*

PROOF. First observe that $S^k \subseteq S^m \times S^n$ implies $k \leq \min(m,n)$. For suppose $m = \min(m,n)$ and $k > m$. Let P denote the $(k+1)$–plane spanned by S^k. Since $k+1+n+1 > m+n+2, P$ and $p_2(\mathbf{R}^{m+n+2}) = \mathbf{R}^{n+1}$ must intersect at least along a line. Hence there exists $\nu \in S^k \cap p_2(\mathbf{R}^{m+n+2})$, but this v cannot be in $S^m \times S^n$ since $||p_2(v)|| = ||v|| = 1$. Hence we must have $k \leq \min(m,n)$.

This also shows $p_1|P$ and $p_2|P$ both have rank $= k+1$ and kernel $= \{0\}$. Therefore P represents the graph of an isomorphism from $p_1(P)$ to $p_2(P)$ which is clearly an isometry.

LEMMA 3.2. *If m or n is even then $S^m \times S^n$ cannot be fibred by great circles of S^{m+n+1}.*

PROOF. Say n is even. Fix $x \in p_1(S^m \times S^n)$. If there were a fibration of $S^m \times S^n$ by great circles, then, for each $y \in p_2(S^m \times S^n)$, the great circle fibre through $(x,y) \in S^m \times S^n$ would project to a great circle $S'(y)$ on $p_2(S^m \times S^n)$ through y. Each such circle $S'(y)$ has a well defined tangent line, at y varying continuously with y on $p_2(S^m \times S^n) = S^n_{\frac{1}{\sqrt{2}}}$. But n is even, so this is impossible.

COROLLARY 3.3. *$S^1 \times S^{2n}$ has no great k–sphere fibrations for any n or $k \geq 1$.*

If n is even and $m \leq n$ then we can apply Corollary 3.3 to conclude that there is no great m sphere fibration of $S^m \times S^n$. For suppose P is the $(n+3)$–plane in R^{m+n+2},

$$P = \{(x_1, \ldots, x_{m+n+1}) : x_1 = x_2 = \cdots = x_{m-1} = 0\}.$$

Note that $P \cap S^{m+n+1} = S^{n+2}$ and $P \cap (S^m \times S^n) = S^1 \times S^n$. If there were a great m–sphere fibration of $S^m \times S^n$, then each fibre projects bijectively on $p_1(S^m \times S^n)$, intersecting the fibration with P would cut each fibre down to a great circle in S^{n+2} and give a great circle fibration of $S^1 \times S^n$. By Corollary 3.3 this is impossible.

In particular we conclude $S^{2n} \times S^{2n}$ cannot be fibred by great $2n$–spheres.

What about great $(2n-1)$–sphere fibrations of $S^{2n-1} \times S^{2n-1}$? Clearly, any such fibration is trivial (let $S_i^{2n-1} = p_i(S^{2n-1} \times S^{2n-1})$, then fix $q_o \in S_2^{2n-1}$ and define $\psi : S_1^{2n-1} \times S_2^{2n-1} \to S^{2n-1} \times S^{2n-1}$ by $\psi(r,q) = (r, L_{r,q_o} q)$, where L_{r,q_o} is the isometry from S_1^{2n-1} to S_2^{2n-1} determined by the fibre through $(r, q_o))$. If S^{2n-1} denotes the base space of a great $(2n-1)$–sphere fibration of $S^{2n-1} \times S^{2n-1} \subseteq S^{4n-1} \subseteq \mathbf{R}^{4n}$ then from §2 we conclude that we have a strongly injective embedding $\phi : S^{2n-1} \to \mathrm{Hom}\,(p_1(\mathbf{R}^{4n}), p_2(\mathbf{R}^{4n}))$. Since every fibre in fact gives an isometry, if we identify the i^{th} coordinate in $p_1(R^{4n})$ with the $2n + i^{\mathrm{th}}$ coordinate of $p_2(R^{4n})$ we can assume we have a strongly injective embedding $\phi : S^{2n-1} \to 0(2n)$, the orthogonal group on R^{2n}. Such a strongly injective embedding ϕ induces a map $\hat{\phi} : S^{2n-1} \times S^{2n-1} \to S^{2n-1}, \hat{\phi}(a,b) = \hat{\phi}(a)(b)$. Clearly $\hat{\phi}$ has bidegree$(1,1)$, so by a theorem of Adams and Atiyah, [**3**; Chapter 14] we conclude $n = 1,3,7$. Such strongly injective embeddings certainly exists in these dimensions, namely the unit spheres in $\mathbf{R}^2, \mathbf{R}^4$ or $\mathbf{R}^8$ considered as the complex numbers, quaternions, or Cayley numbers respectively. Hence we have proven.

COROLLARY 3.4. *$S^n \times S^n$ can be fibred by great n–spheres if and only if $n = 1, 3$ or 7.*

Since $S^{4n+1}, n \geq 1$, does not admit a continuous field of tangent k–planes $2 \leq k \leq 4n-1$, [**6**; Section 27.18], the idea in the proof of Lemma 3.2 generalizes to

LEMMA 3.5. *$S^m \times S^{4n+1}, n \geq 1$, admits no great k–sphere fibration for $2 \leq k \leq 4n-1$.*

PROOF. To be specific, suppose we had a great 2–sphere fibration of $S^m \times S^{4n+1}$. Fix $x \in p_1(S^m \times S^{4n+1})$. Then, for each $y \in p_2(S^m \times S^{4n+1})$, let $T_{x,y}S^2$ be the tangent space to the great 2–sphere fibre at $(x,y) \in S^m \times S^{4n+1}$. Since the 2–sphere projects onto an embedded great 2–sphere in $p_2(S^m \times S^{4n+1})$, $dp_2(T_{x,y}S^2)$ is a 2–plane in $T_y p_2(S^m \times S^{4n+1})$. In this way we get a continuous field of tangent 2–planes on S^{4n+1}, but this is impossible.

Using these general facts we now turn our attention to some specific low dimensional cases.

1. Great circle fibrations of $S^1 \times S^1 \subseteq S^3$. From the discussion after Corollary 3.3, any such fibration is trivial with base space S^1 and it gives a strongly injective embedding of S^1 in $0(2)$. Now $0(2) \simeq S^1 \bigcup_{\text{disj}} S^1$, so, module reparametrization, there are only two possible embeddings of S^1 in $0(2)$ with image either $S0(2)$ or $0(2) - S0(2)$. Its easy to see that either such embedding is a strongly injective embedding. So the space of great circle fibrations of $S^1 \times S^1$ is just 2 points, one point corresponding to a fibration by (1,1) curves (homotopy type of typical fibre in $\pi_1(S^1 \times S^1)$) with typical fibre of the form

$$\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i(\theta+\alpha)}) : 0 \leq \theta \leq 2\pi\}, 0 \leq \alpha \leq 2\pi,$$

and the other point corresponding to a fibration by $(1,-1)$ curves, with typical fibre $\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i(-\theta+\alpha)}) : 0 \leq \theta \leq 2\pi\}, 0 \leq \alpha \leq 2\pi$.

2. By Corollary 3.3, $S^1 \times S^2$ admits no great sphere fibrations.

3. By Lemma 3.2 and the discussion after Corollary 3.3, $S^2 \times S^2$ admits no great sphere fibrations.

4. Great circle fibrations of $S^1 \times S^3 \subseteq S^5$. Given a great circle fibration of S^3, we get a great circle fibration of $S^1 \times S^3$ by lifting each great circle on S^3 to a "diagonal" on $S^1 \times S^3$. Assume the fibration of S^3 is oriented, and, for $y \in S^3$, let $y^\perp$ denote that element of S^3 gotten by rotating $\frac{\pi}{2}$ radians in the oriented direction along the fibre through y. For each $y \in S^3$,

$$S(y) = \{\frac{1}{\sqrt{2}}\cos\theta(1, 0, q) + \frac{1}{\sqrt{2}}\sin\theta(0, 1, q^\perp) : 0 \leq \theta \leq 2\pi\}$$

is a great circle of S^5 lying entirely in $S^1 \times S^3$. Its easy to see that the family of all such great circles fibres $S^1 \times S^3$.

As a consequence of our work in §4 below we will see that a great circle fibration can be obtained in a natural way from any distance decreasing map from S^3 to S^2. Some of these maps will give fibrations of $S^1 \times S^3$ that do not correspond to great circle fibrations of the S^3 factor as above. It is not clear whether every great circle fibration of $S^1 \times S^3$ arises from a distance decreasing map of S^3 to S^2 so this remains an open question.

5. Great sphere fibrations of $S^2 \times S^3$. The only possibility is for a

fibration by great 2–spheres. In §4 below we completely catalogue all great 3–sphere fibrations of $S^3 \times S^3$. Suppose we have a great 3–sphere fibration of $S^3 \times S^3 \subseteq S^7$. Let $S^6 \subseteq S^7$ be given by $S^6 = S^7 \cap \ell_1^{\perp}$. Note that $S^2 \times S^3 = (S^3 \times S^3) \cap \ell_1^{\perp}$. Exactly as in the discussion following Corollary 3.3 we conclude that any great 3–sphere fibration of $S^3 \times S^3$, desuspends, via intersection with $\ell_1^{\perp}$ to a great 2–sphere fibration of $S^2 \times S^3$.

Conversely, since any isometry from S^2 to S^3 extends in one of two distinct ways to an isometry from S^3 to S^3 it is not hard to show that any great 2–sphere fibration of $S^2 \times S^3$ extends in one of 2 distinct ways to a great 3–sphere fibration of $S^3 \times S^3$. Furthermore this extension process is inverse to the desuspension process above; in summary we have the conclusion: There is a 2–to–1 correspondence between great 3–sphere fibrations of $S^3 \times S^3$ and great 2–sphere fibrations of $S^2 \times S^3$. Therefore all the results we obtain in §4 pertaining to great 3–sphere fibrations of $S^3 \times S^3$ can, with minor modification, be applied to great 2–sphere fibrations of $S^2 \times S^3$.

6. Great circle fibrations of $S^3 \times S^3 \subseteq S^7$. Let $F_i : S^1 \to S^3_{\frac{1}{\sqrt{2}}} \xrightarrow{q_i} S^2$ be any oriented great circle fibrations of $p_i(S^3 \times S^3), i = 1, 2$. Given a great circle fibre on the first factor and one on the second, their product is an $S^1 \times S^1 \subseteq S^3 \times S^3$. By (1), this $S^1 \times S^1$ admits a unique fibration by $(1,1)$ great circles (since F_1 and F_2 are oriented, the notion of $(1,1)$ makes sense globally). In this way we can associate to any pair F_1 and F_2 a great circle fibration of $S^3 \times S^3$.

As in (4) above, there remain unanswered questions here also. Are all great circle fibrations of $S^3 \times S^3$ obtained by a product of two such fibrations of the factors?

7. Great 2–sphere fibrations of $S^3 \times S^3$. None of the above results address the case of fibrations of $S^3 \times S^3$ by great 2–spheres. From the Gysin sequence we can settle this case by proving that in fact $S^3 \times S^3$ does not even admit a topological fibration by 2–spheres.

Suppose we had such a fibration, $S^2 \to S^3 \times S^3 \to M^4$. Since $S^3 \times S^3$ is simply connected and S^2 is path connected, M^4 must be connected and simply connected, hence $H^o(M, Z) \simeq Z$ and the fibration is orientable. The Gysin sequence of our hypothetical fibration gives:

$$\begin{array}{ccccccc} \longrightarrow H^5(M^4) & \longrightarrow & H^5(S^3 \times S^3) & \longrightarrow & H^3(M^4) & \longrightarrow & H^6(M^4), \\ \| & & \| & & & & \| \\ 0 & & 0 & & & & 0 \end{array}$$

so by exactness we must have $H^3(M^4) = 0$. But another segment of the Gysin sequence gives:

$$\begin{array}{ccccccc} \longrightarrow H^3(M^4) & \longrightarrow & H^2(S^3 \times S^3) & \longrightarrow & H^0(M^4) & \longrightarrow & H^3(M^4) \longrightarrow \\ & & \| & & \| & & \\ & & 0 & & Z & & \end{array}$$

so the conclusion $H^3(M^4) = 0$ destroys the exactness of this segment. Hence no such fibration exists.

8. Examples of $S^m \times S^n$ which admit no great k–sphere fibrations, $k > 0$.

$S^1 \times S^2$	$S^2 \times S^2$	$S^2 \times S^5$	$S^4 \times S^9$
$S^1 \times S^4$	$S^2 \times S^4$	$S^2 \times S^9$	$S^4 \times S^{13}$
$S^1 \times S^6$	$S^2 \times S^6$	$S^2 \times S^{13}$	$S^6 \times S^9$
$S^1 \times S^8$	$S^2 \times S^8$	$S^2 \times S^{17}$	$S^6 \times S^{13}$

4. Great 3–sphere fibrations of $S^3 \times S^3$. From §3 we know that every great 3–sphere fibration of $S^3 \times S^3$ is trivial and it gives a strongly injective embedding of S^3 in $0(4)$. Conversely, given a strongly injective embedding $\phi : S^3 \to 0(4)$, the family of 3–spheres $x \mapsto \{\frac{1}{\sqrt{2}}(b, \phi(x)(b)) : b \in S^3\}$ for each $x \in S^3$, gives a great 3–sphere fibration of $S^3 \times S^3$.

So our approach to studying great 3–sphere fibrations of $S^3 \times S^3$ will be to analyze the equivalent problem of strongly injective embeddings of S^3 in $0(4)$. It is well known that $S0(4) \simeq S^3 \times RP^3$, hence $S0(4)$ has $S^3 \times S^3$ as a double cover. In all that follows we identify R^4 with the quaternions in the usual manner and S^3 will represent the quaternions of norm 1. With these identifications, we take for the double cover projection $h : S^3 \times S^3 \to S0(4)$ the map determined by

$h(u,v)(x) = uxv$ (quaternion multiplication).

Since S^3 is simply connected, any map $\phi : S^3 \to S0(4)$ lifts to a map $\hat{\phi} : S^3 \to S^3 \times S^3$ such that the diagram

$$\begin{tikzcd} & S^3 \times S^3 \arrow[d, "\hat{\phi}"', "h"] \\ S^3 \arrow[ur, phantom, "\hat{\phi}"] \arrow[r, "\phi"] & S0(4) \end{tikzcd}$$

commutes $\hat{\phi}$ is unique up to choice of base point lying over, say $\phi(1)$.

Now we address the question: What criteria are there to guarantee that a 3–sphere embedded in $S^3 \times S^3$ projects via h to the image of a strongly injective embedding?

Let $d : S^3 \times S^3 \to [0,\pi]$ denote distance on $S^3; d(x,y)$ equals the minimum value of the length of a great circle arc joining x and y. Since multiplication by a norm 1 quaternion is an isometry of S^3, we get immediately,

$$d(x,y) = d(xq,yq) = d(qx,qy), \text{ for all } x,y,q \in S^3.$$

LEMMA 4.1. *A necessary condition for $h(x,y)(w) = h(u,v)(w)$ for some $w \in S^3$ is that $d(x,u) = d(y,v)$.*

PROOF. $xwy = uwv \Leftrightarrow xwyv^{-1} = uw$ so $d(x,u) = d(xw,uw) = d(xw,xwyv^{-1}) = d(1,yv^{-1}) = d(v,y) = d(y,v)$.

LEMMA 4.2. *$h(x,y) \in S0(4)$ has a fixed point ($+1$ eigenvalue) if and only if $d(x,1) = d(y,1)$.*

PROOF. $(\Rightarrow)$.$h(x,y)(w) = w = h(1,1)(w)$, so, by Lemma 4.1, $d(x,1) = d(y,1)$.

$(\Leftarrow)$. If $x = \pm 1$, then $d(x,1) = d(y,1)$ implies $y = \pm 1$ and both $h(1,1)$ and $h(-1,-1)$ have fixed points, so the conclusion follows for $x = \pm 1$.

For $x \neq \pm 1$, suppose $d(x,1) = d(y,1) = c$ with $0 < c < \pi$. For any $z \in S^3, z \neq \pm 1$, let $\hat{z}$ denote the quaternion on the great circle through 1 and z with $d(1,\hat{z}) = \frac{\pi}{2}$ and $d(z,\hat{z}) < \frac{\pi}{2}$.

We recall that $SO(3) \simeq RP^3$ and one way to make this identification is to conjugate by norm 1 quanternions and view this action restricted to the unit 2–sphere in S^3 of quaternions with real part 0. So there exists $w \in S^3$ such that $w^{-1}\widehat{x^{-1}}w = \hat{y}$.

Now, conjugation by w is an isometry of S^3 fixing ± 1 so it takes the great circle through 1 and $\widehat{x^{-1}}$ to the great circle through 1 and $\hat{y}$. Since $d(1, x^{-1}) = d(1, y)$ we conclude $y = w^{-1}x^{-1}w$, but this implies $w = xwy = h(x, y)(w)$ and consequently $h(x, y)$ has a fixed point.

THEOREM 4.3. *Given $x, y, u, v \in S^3, h(x, y)(w) = h(u, v)(w)$ for some $w \in S^3$ if and only if $d(x, u) = d(y, v)$.*

PROOF. ($\Rightarrow$). Lemma 4.1.

($\Leftarrow$). $d(x, u) = d(y, v)$ implies $d(1, x^{-1}u) = d(x, u) = d(y, v) = d(1, vy^{-1})$. So, by Lemma 4.2, there exists $w \in S^3$ with $w = h(x^{-1}u, vy^{-1})(w) = x^{-1}uwvy^{-1}$. This says that $h(x, y)(w) = xwy = uwv = h(u, v)(w)$.

This is the key result we were after and it will allow us to completely characterize strongly injective embeddings of S^3 in $S0(4)$.

THEOREM 4.4. *A submanifold of $S^3 \times S^3$ corresponds to the image of a strongly injective embedding of S^3 in $S0(4)$ if and only if it is the graph of a smooth distance decreasing map ψ from either S^3 factor to the other.*

PROOF. Using Theorem 4.3, the proof is identical to [**1**; Theorem A].

THEOREM 4.5. *$\phi : S^3 \to S0(4)$ is a smooth strongly injective embedding if and only if the corresponding distance decreasing map ψ is differentiable with $|d\psi| < 1$.*

PROOF. ($\Rightarrow$). By Theorem 4.4, ϕ corresponds to a smooth distance decreasing map ψ from one factor of S^3 to the other. Therefore we have $|d\psi| \leq 1$.

Suppose $|d\psi| = 1$ at some point $(u,v) \in S^3 \times S^3, v = \psi(u)$. Left and right multiplication in the Lie group S^3 are both diffeomorphisms of norm 1, so replacing h by $\tilde{h}(x,y) = h(u^{-1}x, yv^{-1})$ we may and shall assume $(u,v) = (1,1)$.

$|d\psi| = 1$ implies there is a parametrized curve $\sigma : (-\varepsilon, \varepsilon) \to S^3, \sigma(0) = 1, \sigma'(0) = V$ with $|V| = 1$ such that $|(\psi \circ \sigma)'(0)| = |V| = 1$. We assume σ traverses a portion of a great circle, and by a conjugation action applied to the first factor which rotates the purely imaginary 2–sphere we can suppose $\sigma(t) = \cos t + i \sin t$.

Now $\psi \circ \sigma(t) = \psi_1(t) + \psi_2(t)i + \psi_3(t)j + \psi_4(t)k$, with $\psi_1(0) = 1, \psi_i(0) = 0, 2 \le i \le 4; \psi_1'(0) = 0, \psi_2'(0)^2 + \psi_3'(0)^2 + \psi_4'(0)^2 = 1$. The matrix for $h(\sigma(t), \psi \circ \sigma(t)) \in S0(4)$ is given by:

$$H(t) = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \begin{bmatrix} \psi_1(t) & -\psi_2(t) & -\psi_3(t) & -\psi_4(t) \\ \psi_2(t) & \psi_1(t) & \psi_4(t) & -\psi_3(t) \\ \psi_3(t) & -\psi_4(t) & \psi_1(t) & \psi_2(t) \\ \psi_4(t) & \psi_3(t) & \psi_2(t) & \psi_1(t) \end{bmatrix}$$

$$\begin{aligned} \frac{d}{dt}|_{t=0} H(t) &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot I + I \cdot \begin{bmatrix} 0 & -\psi_2'(0) & -\psi_3'(0) & -\psi_4'(0) \\ \psi_2'(0) & 0 & \psi_4'(0) & -\psi_3'(0) \\ \psi_3'(0) & -\psi_4'(0) & 0 & \psi_2'(0) \\ \psi_4'(0) & \psi_3'(0) & -\psi_2'(0) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1-\psi_2'(0) & -\psi_3'(0) & -\psi_4'(0) \\ 1+\psi_2'(0) & 0 & \psi_4'(0) & -\psi_3'(0) \\ \psi_3'(0) & -\psi_4'(0) & & -1+\psi_2'(0) \\ \psi_4'(0) & \psi_3'(0) & 1-\psi_2'(0) & 0 \end{bmatrix} \end{aligned}$$

It is a straightforward but tedius calculation to compute:

$$\text{(4.5.1)} \qquad \det\left[\frac{d}{dt}|_{t=0} H(t)\right] = (-1 + \psi_2'(0)^2 + \psi_3'(0)^2 + \psi_4'(0)^2)^2.$$

So $|\psi \circ \sigma'(0)| = |\sigma'(0)| = 1$ implies $\det[\frac{d}{dt}|_{t=0} H(t)] = 0$. Let $w \in S^3$ be a vector such that $[\frac{d}{dt}|_{t=0} H(t)](w) = 0$. Assuming, WLOG, that $\phi : S^3 \to S^3 \times S^3$ is given by $\phi = (\mathrm{id}, \psi)$ we get (recalling $\phi_w : S^3 \to S^3$ is given by $\phi_w(x) = \phi(x)(w)$)

$$\text{(4.5.2)} \qquad \begin{aligned} (d\phi_w)_1(V) &= \frac{d}{dt}|_{t=0} \phi_w \circ \sigma(t) = \frac{d}{dt}|_{t=0} h(\sigma(t), \psi \circ \sigma(t))(w) \\ &= \left[\frac{d}{dt}|_{t=0} H(t)\right](w) = 0, \end{aligned}$$

hence ϕ_w is not a diffeomorphism at $1 \in S^3$. Therefore ϕ is not a smooth strongly injective embedding.

($\Leftarrow$). We already know that, given a smooth distance decreasing map $\psi : S^3 \to S^3$, we get a strongly injective embedding $\phi : S^3 \to S0(4)$, $\phi(x) = h(x, \psi(x))$. It remains to show that $|d\psi| < 1$ implies ϕ is a smooth strongly injective embedding.

Suppose ϕ is not a smooth injective embedding, then, for some $w \in S^3, \phi_w$ is not a diffeomorphism. Hence there is a point $p \in S^3$ and a unit tangent vector V at p such that $(d\phi_w)(V) = 0$. By an argument completely analogous to that at the beginning of the "if" part of the proof, we may assume $p = \psi(p) = 1$ and V is the unit tangent vector to the curve $\sigma(t) = \cos t + i \sin t$ at $t = 0$.

So we can apply equation 4.5.2, this time knowing $(d\phi_w)_1(V) = 0$, to conclude $[\frac{d}{dt}|_{t=0}H(t)](w) = 0$. By 4.5.1 we must have

$$|(\psi \circ \sigma)'(0)| = \psi_2'(0)^2 + \psi_3'(0)^2 + \psi_4'(0)^2 = 1 \text{ and } |d\psi| = 1.$$

THEOREM 4.6. *Any great 3–sphere fibration of $S^3 \times S^3$ must contain some orthogonal pair of fibres.*

PROOF. Corresponding to any great 3–sphere fibration of $S^3 \times S^3$ is a strongly injective embedding $\phi : S^3 \to 0(4)$, and we assume image ϕ lies in $S0(4)$. Corresponding to ϕ is a distance decreasing map $\psi : S^3 \to S^3$, mapping one factor of the double cover of $S0(4)$ to the other, say WLOG, the first factor to the second.

ψ distance decreasing implies $-\psi(x) \notin \operatorname{Im}\psi$ for any $x \in S^3$, so ψ is not surjective. By the Borsuk–Ulam Theorem, there exist $\pm u \in S^3$ such that $\psi(u) = \psi(-u)$. Let $P_{\pm u}$ denote the fibres over $\pm u$,

$$P_{\pm u} = \{\frac{1}{\sqrt{2}}(v, \pm uv\psi(u)) : v \in S^3\}.$$

For $V = (v, uv\psi(u)) \in P_u$ and $W = (w, -uw\psi(u)) \in P_{-u}$ we compute $V \cdot W$. Note that, for quaternions $a = a_1 + a_2 i + a_3 j + a_4 k$ and $b = b_1 + b_2 i + b_3 j + b_4 k$, Re $(a\bar{b}) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = a \cdot b$, hence

$$\begin{aligned} V \cdot W &= \Re(v\overline{w}) + \Re(uv\psi(u)(\overline{-uw\psi(u)})) \\ &= \Re(v\overline{w} - uv\psi(u)\overline{\psi(u)}\overline{w}\overline{u}) = \Re(\overline{v}w - uv\overline{w}\overline{u}) \\ &\Re(v\overline{w} - uvwu^{-1}) = \Re(v\overline{w}) - \Re(v\overline{w}) = 0. \end{aligned}$$

Therefore we have shown $P_u = P_{-u}^{\perp}$.

From these three theorems we see a strong analogy with great circle fibrations of S^3 and the work of Gluck and Warner [**1**]. To keep this analogy going we would like to distinguish a certain "nice" subspace of the space of all great 3–sphere fibrations of $S^3 \times S^3$ and call them Hopf fibrations.

If we took the Hopf fibration of S^7 by great 3–spheres as the graphs of left quaternion multiplication and restrict to graphs of norm one quaternion multiplication, then clearly these great 3–spheres fibre $S^3 \times S^3$. This fibration corresponds to the distance decreasing map $\psi : S^3 \to 1 \in S^3$ (first factor to the second). Certainly this fibration of $S^3 \times S^3$ should be called a Hopf fibration. As in [**1**] any orthogonal transformation of this fibration, which still fibres $S^3 \times S^3$, should also be called a Hopf fibration. Those orthogonal transformations which fix $S^3 \times S^3$ are of the form $\left[\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right]$ where $A_i \in 0(4)$.

Restricting to the special orthogonal group, so we stay in the class of strongly injective embeddings of S^3 in $S0(4)$, suppose first that $A_i \in S0(4)$, for $i = 1$ and 2. In this case we can represent the action of such a transformation as

$$(a, b)| \to (xay, ubv) \text{ for } (a, b) \in S^3 \times S^3 \subseteq S^7,$$

where x, y, u and v are in S^3.

If $P_r = \{\frac{1}{\sqrt{2}}(c, rc) : c \in S^3\}$ is a great 3–sphere of our Hopf fibration then under this transformation,

$$\begin{aligned} P_r \mapsto & \{\frac{1}{\sqrt{2}}(xcy, urcv) : c \in S^3\} \\ = & \{\frac{1}{\sqrt{2}}(c', (urx^{-1})c'(y^{-1}v) : c' \in S^3\}. \end{aligned}$$

So the distance decreasing map corresponding to this new fibration is $\psi'(w) = y^{-1}v =$ constant.

Suppose now $A_i \in 0(4) - S0(4)$ for $i = 1$ and 2. We get

$$(a_1, a_2) \mapsto (A_1 a_1, A_2 a_2) = (A_1' \bar{a}_1, A_2' \bar{a}_2),$$

where $\bar{a}_i$ is the conjugate of a_i and $A_i' \in S0(4)$ for $i = 1$ and 2. So nothing new happens in this case and we conclude: Any special orthogaonal transformation of the Hopf fibration of $S^3 \times S^3$, which still fibres $S^3 \times S^3$, has a corresponding distance decreasing map of the form $\psi(S^3) =$ constant. Clearly all of this could have been applied to the "other" Hopf fibration of S^7 given by graphs of right multiplication so the map $\psi(S^3) =$ constant can be a map from wither S^3 factor to the other.

DEFINITION 4.7. A Hopf fibration of $S^3 \times S^3$ is any great 3–sphere fibration of $S^3 \times S^3$, which induces a strongly injective embedding $\phi : S^3 \to S^3 \times S^3$ such that $\operatorname{Im} \phi = (S^3, pt)$ or (pt, S^3).

THEOREM A. *The space of all oriented great 3–sphere fibrations of $S^3 \times S^3 \subseteq S^7$ deformation retracts to the subspace of Hopf fibrations and hence has the homotopy type of disjoint union of four copies of RP^3.*

PROOF. Let DDM (S^n) be the space of distance decreasing maps of S^n to itself. We give DDM (S^n) the compact open or C° topology. Two maps f and g from S^n to itself are within ε of each other provided $f(x)$ and $g(x)$ are within ε of each other for all $x \in S^n$. For $f \in$ DDM (S^n), f distance decreasing implies $-f(x) \notin \operatorname{Im} f$ for all $x \notin S^n$ so f is not surjective. By the Borsuk–Ulam Theorem, there are $\pm u \in S^n$ with $f(u) = f(-u) = u'$. For all $x \in S^n$, $\min d(x, \pm u) \leq \pi/2$, so distance decreasing implies $d(f(x), u') < \pi/2$ hence Image $f \subseteq$open hemisphere of S^n.

LEMMA 4.8. *There is a continuous map $c : DDM(S^n) \to S^n$ such that, for each $f \in DDM(S^n)$, the image $f(S^n)$ lies in the open hemisphere centered at $c(f)$*

PROOF. $\operatorname{Im} f$ certainly varies continuously with f by the choice of topology on DDM (S^n). Let $B(f)$ be the closed ball of smallest radius which contains the closed set $\operatorname{Im} f$.

1) $B(f)$ is uniquely determined by f.

This follows from the fact that on the unit n–sphere the intersection of two closed balls, each of radius $< \pi/2$ is contained in some closed ball of smaller radius.

Let $c(f)$ denote the center of the ball $B(f)$ and $r(f)$ its radius.

2) $r(f)$ varies continuously with f.

If f is perturbed by less than ε to g, then $r(g) < r(f) + \varepsilon$ and by symmetry $r(f) < r(g) + \varepsilon$.

3) Let B be a ball of radius $r < \pi$ on S^n with center p, then, for any ball C inside B of radius $\alpha, \alpha > r - \beta$, with center $q, d(p,q) < \beta$.

Now, for $\frac{\pi}{2} > \varepsilon > 0$ given, let U_ε be the open ball about f, $U_\varepsilon = \{g \in \mathrm{DDM}(S^n)$: distance from f to g is less than $\{\varepsilon/2\}$. Let B be the ball of radius $r(f) + \varepsilon/2$ with center $c(f)$. Then Image $g \subseteq B$ for all $g \in U_\varepsilon$, hence $B(g) \subseteq B$ for all $g \in U_\varepsilon$. Now (2) implies $r(f) - \varepsilon/2 < r(g)$, so $(r(f) + \varepsilon/2) - \varepsilon < r(g)$ and, by (3), we conclude $d(c(f), c(g)) < \varepsilon$. Since we can find such a U_ε for all $\varepsilon > 0, c$ is continuous.

For any $f \in \mathrm{DDM}(S^n)$, radial contraction of $\mathrm{Im} f$ to $c(f)$ homotopes f, through distance decreasing maps to a constant map, $f_o : S^n \to c(f) \in S^n$. By Lemma 4.8 $c(f)$ depends continuously on f so this process is a deformation retraction. Hence $\mathrm{DDM}(S^n)$ has the homotopy type of S^n.

Finally set $n = 3$ and observe that the two constant maps $\psi : S^3 \to p$ and $\psi_2 : S^3 \to -p$ both from the first factor to the second or vice versa determine the same great 3–sphere fibration of $S^3 \times S^3$. So the family of great 3–sphere fibrations of $S^3 \times S^3$ determined by strongly injective embeddings $\phi : S^3 \to S0(4)$ has the homotopy type of a disjoint union of two copies of RP^3. We get two more copies of RP^3 by considering strongly injective embeddings $\phi : S^3 \to 0(4) - S0(4)$.

Except for the last paragraph and part (3) of the Lemma, the proof of Theorem A is essentially identical to [**1**; Theorem D].

With this theorem we have answered completely, for great 3–sphere fibrations of $S^3 \times S^3$, the three questions posed in §1. Now we ask: Can every great 3–sphere fibration of $S^3 \times S^3$ appear as a portion of a great 3–sphere fibration of S^7?

The result will follow easily from Theorem A and the following two simple observations:

1) Every great 3–sphere fibration of $S^3 \times S^3$ can be "fattened up." Suppose $\psi : S^3 \to S^3$ is a distance decreasing map giving a great 3–sphere fibration of $S^3 \times S^3$. It is a simple matter to check

that the family of 4–planes, $\{P_{ta} : a \in S^3, 0 < t < \infty\}$, where $P_{ta} = \{(u,\ tau\psi(a)) : u \in H\}$, gives a fibration of $S^7 - (P_o \cup P_\infty)$, where $P_o = \text{span}\,(\ell_1, \ell_2, \ell_3, \ell_4) \subseteq \mathbf{R}^8$ and $P_\infty = \text{span}\,(\ell_5, \ell_6, \ell_7, , \ell_8) \subseteq \mathbf{R}^8$. This "fattened up" fibration is smooth if and only if the original fibration is smooth. Including the 4–planes P_o and P_∞ we get a topological fibration of S^7.

2) If $\psi : S^3 \to$ constant $\in S^3$, then in the above "fattening up" process we can include P_o and P_∞ and still retain differentiability.

THEOREM B. *Every (smooth) great 3–sphere fibration of $S^3 \times S^3$ can be extended to a (smooth) great 3–sphere fibration of S^7.*

PROOF. Given a great 3–sphere fibration of $S^3 \times S^3$, let ψ as usual be the corresponding distance decreasing map. Let $\psi_t : S^3 \to S^3, t \in I$, be the homotopy of ψ, through distance decreasing maps, to a constant map provided by Theorem A with $\psi_1 = \psi$ and $\psi_o(x) = c(\psi) = c$, a constant for all $x \in S^3$.

Let $n : [0, \infty] \to [0, 1]$ be a C^∞ function such that $n(1) = 1$ and support $n \subseteq (\frac{1}{2}, 2)$. We suppose $\mathbf{R}^8 = H \times H$ and consider the family of 4–planes:

$$\begin{aligned}&\{P_o\} \bigcup \{P_\infty\} \bigcup \{P_{tv} : 0 < t < \infty\},\\ &\qquad\qquad \text{where } P_{tv} = \{(u, tvu\psi_{n(t)}(v)) : u \in H\}.\end{aligned}$$

Note that, for $t < \frac{1}{2}$ and $t > 2, \psi_{n(t)}$ is a constant map so the fibration is C^∞ is a neighborhood of P_o and P_∞. With this it is easy to verify that the above family of 4–planes cut out a C^∞ fibration of S^7 by great 3–spheres and the fibration is smooth if and only if the original fibration of $S^3 \times S^3$ is smooth.

As a final application of Theorem 4.3 we obtain a further insight into great circle fibrations of $S^1 \times S^3 \subseteq S^5$.

Let $F : S^3 \to S^3$ be a smooth map with $F(q) \in q^\perp$ for all $q \in S^3$, i.e., F is a smooth, unit tangent vector field on S^3. Let $H_F : [0, 2\pi] \times S^3 \to S^3$ be given by

$$H(\theta, q) = \cos\theta \cdot q + \sin\theta \cdot F(q).$$

One readily verifies that a necessary and sufficient condition for the family of great circles in $S^1 \times S^3 \subseteq S^5$, $S^1(q) = \{\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, H(\theta, q)) :$

$0 \le \theta \le 2\pi\}$ for all $q \in S^3$, to fibre $S^1 \times S^3$, is that $H(\theta,\ ) : S^3 \to S^3$ is injective for all $0 \le \theta < 2\pi$.

Recall that there is a bijective correspondence between smooth unit tangent vector fields on S^3 and smooth maps from S^3 to S^2 where $S^2 = S^3 \cap 1^\perp$. Given F as above we get this correspondence by defining $f : S^3 \to S^2$ by $f(p) = p^{-1}F(p)$. In terms of f the map H_F is given by:

$$H_F(\theta, p) = \cos\theta \cdot p + \sin\theta \cdot (pf(p)) = p(\cos\theta + \sin\theta \cdot f(p)).$$

For each p and θ we get an element of $S0(4)$:

$$(p, \theta) \mapsto h(p, \cos\theta + \sin\theta f(p)) \in S0(4)$$

(recall $h : S^3 \times S^3 \to S0(4)$ is the double cover projection). For brevity we denote this element of $S0(4)$ by $h(p, \theta)$.

If $f : S^3 \to S^2 \subseteq S^3$ is distance decreasing, then since $d(\cos\theta + \sin\theta f(p), \cos\theta + \sin f(q)) \le d(f(p), f(q))$ for all θ and all p and q, we conclude that, for all θ and all $p \ne q$,

$$d(p, q) > d(f(p), f(q)) \ge d(\cos\theta + \sin\theta f(p), \cos\theta + \sin\theta f(q)).$$

Therefore, from Theorem 4.3, we get

$$H_F(\theta, p) = h(p, \theta)(1) \ne h(q, \theta)(1) = H_F(\theta, q)$$

for $p \ne q$ and all θ, i.e., $H_F(\theta,\ )$ is injective for all θ.

So we conclude: For every distance decreasing map $f : S^3 \to S^2$ we get a great circle fibration of $S^1 \times S^3 \subseteq S^5$. It remains an interesting open question whether the converse also holds.

This conclusion is sufficient for us to exhibit a great circle fibration of $S^1 \times S^3$ which does not project a great circle fibration of the S^3 factor. If $f : S^3 \to \{i\} \subseteq S^2$ is the constant map, then one can readily check that the flow along the related map H_F determines a great circle fibration of S^3 (the Hopf fibration). Now let $\tilde{f}$ be a perturbation of f in a neighborhood of 1 such that it is still distance decreasing and $\tilde{f}(x) = i$ for all $x \in S^3$ with $d(x, 1) \ge \frac{\pi}{4}$ and $\tilde{f}(1) \ne i$. Since $\tilde{f}$ is distance decreasing, $H_{\tilde{f}}(\theta,\ ) : S^3 \to S^3$ is injective for all θ; however, if $\tilde{S}(1) = \{H_{\tilde{F}}(\theta, 1) : 0 \le \theta \le 2\pi\}, \tilde{S}(i) = \{H_{\tilde{F}}(\theta, i) : 0 \le \theta < 2\pi\}$, then $1 \in \tilde{S}(1) \cap \tilde{S}(i)$, but $\tilde{S}(1) \ne \tilde{S}(i)$ since $i \in \tilde{S}(i)$ but $i \notin \tilde{S}(1)$. Therefore

$H_{\tilde{F}}$ does not determine a great circle fibration of S^3 although it lifts to a great circle fibration of $S^1 \times S^3$.

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