ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 4, Fall 1987

ON THE ISOMORPHISM PROBLEM FOR GROUP RINGS AND COMPLETED AUGMENTATION IDEALS

FRANK RÖHL

1. Introduction. Let G be a group, $\Delta_{\mathbf{Z}}G$ its integral and ΔG its modular augmentation ideal, i.e., over the field $\mathbf{F}_p$ of p elements. In this note, we consider the integral isomorphism problem–whether $\Delta_{\mathbf{Z}}G = \Delta_{\mathbf{Z}}H$ implies $G \tilde{\rightarrow} H$– for certain finite p–groups and emphasize the aspect of how much of our methods carry over to the modular case.

Having finite p–groups at our disposal, the most obvious approach to attack the isomorphism problem is to try some kind of induction. To put this idea to work, two ingredients turn out to be essential: One has to be able to lift automorphisms of $\Delta_{\mathbf{Z}}\hat{G}$, resp. $\Delta\hat{G}$, $\hat{G}$ a homomorphic image of G, to automorphisms of the augmentation ideal of a free group F with $F \twoheadrightarrow G$, and the lifting has to leave certain ideals invariant. Although it is not possible to solve the first problem in general, since the group ring of a free group does not contain enough units, one can do it for the completed group rings: Lemma 3.1 gives a solution, which can be easily generalized to other rings of coefficients.

However, to guarantee that the lifting is again an automorphism, we have to impose one further condition on the automorphisms under consideration: They have to induce the identity on $\Delta\hat{G}/\Delta^2\hat{G}$. Although this seems to be a severe restriction on the first glance, it is always satisfied for the start of the induction (and looks rather natural for these cases). The isomorphism $A \tilde{\rightarrow} \Delta_{\mathbf{Z}}A/\Delta_{\mathbf{Z}}^2 A$ for an abelian group A gives, in case $\Delta_{\mathbf{Z}}A = \Delta_{\mathbf{Z}}B$, an automorphism of $\mathbf{Z}A$ sending A onto B with the above property, and hence, even the Whitcomb isomorphism for metabelian torsion groups has it. Furthermore, the lifting, too, has this property (see (3.1)).

All these considerations lead to the following concept: A group G is $\mathbf{F}_p$–strongly characterized by its integral group ring, if $\Delta_{\mathbf{Z}}G = \Delta_{\mathbf{Z}}H$ implies the existence of an isomorphism $G \tilde{\rightarrow} H$, whose extension to an automorphism of $\Delta_{\mathbf{Z}}G$ induces the identity on $\Delta G/\Delta^2 G$. Although

This work is supported by a Feodor–Lynen Fellowship of the Alexander von Humboldt–Foundation.
Received by the editors on October 23, 1985.

we are not able to solve the second problem in general–invariance of certain ideals–there are a few cases in which it becomes trivial, thereby giving us our main result: (3.3) Let F be a finitely generated free group, R a normal subgroup contained in $F_2 \cdot F^p, p$ any prime such that F/R is a finite p–group. If F/R is $\mathbf{F}_p$–strongly characterized by its integral group ring, then so are $F/[R,R] \cdot R^p$ and $F/[F,R]R^p \cdot$ (F_2 denoting the second term of the lower central series and $[,]$ denoting group commutators).

Analyzing the proof of Whitcomb's theorem shows that finite metabelian p–groups are $\mathbf{F}_p$–strongly characterized by their integral group ring; hence our theorem applies to this situation.

Further on, we give in (3.6) an example of how the induction works.

All of these results would carry over to the modular case, if only there were ideal correspondence for $\Delta G = \Delta H$ as in the integral case (see for example [**6**; Chapter III.4]); and there seems to be some hope, that our results can be generalized to residually finite p–groups.

To be able to put this work into the proper context, it is recommended that the interested reader consult Sandling's excellent survey article *The isomorphism problem for group rings: a survey* in *Orders and their applications* (Springer LNM 1142, edited by I. Reiner and K.W. Roggenkamp).

Acknowledgements. The author would like to thank S.K. Sehgal for his helpful comments and the Department of Mathematics of the University of Alberta for its hospitality.

2. Preliminaries. In this section, we collect some more or less well–known facts on Δ–adic completions and some terminology.

Let G be a finite p–group and

$$1 \to R \to F \xrightarrow{\pi} G \to 1 \tag{2.1}$$

a minimal presentation of G (i.e., F is finitely generated free with $R \subset F_2 \cdot F^p$. Such presentations of finite p–groups G can always be found because G modulo its Frattini–subgroup is a finite–dimensional $\mathbf{F}_p$–vector space, each basis of which can be lifted to a generating system of G.) This gives rise to the following exact sequence

$$0 \to \Delta(F,R) \to \Delta F \xrightarrow{\pi} \Delta G \to 0;$$

$\Delta(F,R)$ denoting the ideal generated by all $\delta(r) := r - 1, r \in R$. (We always denote the map $\delta(g) := g - 1$ of a group into its augmentation ideal by δ without any distinction of the groups and rings of coefficients under consideration). Provide ΔF and ΔG with the Δ–adic and $\Delta(F,R)$ with the induced filtration, and denote completions with respect to the topologies defined by these filtrations by a "–". For their construction and their properties, we refer to [**1**; Chapter III] and [**2**; Chapter IX §3]. Then we know that the filtrations are exhaustive on each object, and π maps $\Delta^k F$ onto $\Delta^k G$. Hence, π has the Artin–Rees property, (see [**4**, p. 291]) and since $\Delta(F,R)$ carries the induced topology, the inclusion $\Delta(F,R) \subset \Delta F$, too, has the Artin–Rees property. By III.8 Thm. of [**4**, p. 291], we obtain for the completed sequence

LEMMA 2.2. $0 \to \overline{\Delta(F,R)} \to \overline{\Delta F} \xrightarrow{\pi} \overline{\Delta G} \to 0$ *is exact.*

Since G is a finite p–group, ΔG is a nilpotent $\mathbf{F}_p$–algebra, and the Δ–adic topology on ΔG is discrete so that ΔG coincides already with its Δ–adic completion.

For ΔF the situation is not quite that simple: By construction, one has

$$\overline{\Delta F} := \varprojlim \Delta F/\Delta^{i+1}F \subset \prod_{i\in N} \Delta F/\Delta^{i+1}F,$$

where each quotient is equipped with the discrete topology, and $\prod \Delta F/\Delta^{i+1}F$ is given the product topology. Thus, ΔF carries–when considered as the completion of ΔF–the induced topology. On the other hand, the "Δ–adic filtration" given by $\overline{\Delta F}^i$ defines a topology on $\overline{\Delta F}$, too, and it is natural to ask, in which way these topologies are related.

LEMMA 2.3. *Both topologies on $\overline{\Delta F}$ coincide; in particular, $\overline{\Delta F}$ is complete in the Δ–adic topology.*

PROOF. Since both topologies make $\overline{\Delta F}$ a topological ring, it is sufficient to show that some fundamental system of neighbourhoods of 0 in one of these topologies is a fundamental system in the other one. Since the $\Delta F/\Delta^{k+1}F$ are discrete, a fundamental system of neighbourhoods of 0 for the completion–topology is given by restricting the natural projections

$$pr_k : \prod_i \Delta F/\Delta^{i+1}F \to \Delta F/\Delta^{k+1}F$$

to $\overline{\Delta F}$ and taking kernels. This gives

$$\operatorname{Ker} pr_K|_{\overline{\Delta F}} = \overline{\Delta F}^{k+1},$$

which shows this assertion.

We will make use of this coincidence of topologies in the sequel without mentioning it explicitly.

As a first application of (2.3), we recall that

LEMMA 2.4. *$\overline{\Delta F}$ is a radical ring.*

PROOF. Since $\overline{\Delta F} \subset \prod \Delta F/\Delta^{i+1}F$, where $\prod \Delta F/\Delta^{i+1}F$–as a direct product of radical rings–is already a radical ring, all we have to do, is to show that the quasi–inverse $x^{(-1)}$ of an element $x \in \overline{\Delta F}$ (i.e., the inverse of x under the circle–composition $u \circ v := u + v + uv$) is contained in $\overline{\Delta F}$. But $x^{(-1)} = \sum_{i\geq 1}(-1)^i x^i$, and since the sequence $(x^i)_{i\in N}$ converges to 0 in $\overline{\Delta F}$, this is clear.

LEMMA 2.5. *Let I be an ideal in ΔG and J its full preimage under $\pi : \Delta F \to \Delta G$. Then $\overline{\pi}^{-1}(I) = \overline{J}$.*

PROOF. Since ΔG is already complete, we can form $\overline{\pi}^{-1}(I)$. We now have $I = \pi(J) \subset \overline{\pi}(\overline{J}) \subset \overline{\overline{\pi}(J)} = \overline{\pi(J)} = I$; hence, $\overline{\pi}^{-1}(I) = \overline{J} + \operatorname{Ker}\overline{\pi}$. Since $\operatorname{Ker}\pi = \Delta(F,R) \subset J$ implies $\operatorname{Ker}\overline{\pi} = \overline{\Delta(F,R)} \subset \overline{J}$, the assertion follows.

LEMMA 2.6. *If H is an arbitrary group and N a normal subgroup of H, then $\delta : H \to \Delta H$ induces*

a) $N/[N,N]N^p \xrightarrow{\sim} \Delta(H,N)/\Delta H \cdot \Delta(H,N)$

b) $N/[H,N]N^p \xrightarrow{\sim} \Delta(H,N)/\Delta H \Delta(H,N) + \Delta(H,N)\Delta H.$

(2.6a) is well–known (see for example [**6**; Prop III.1.15, p. 76]) and (2.6b) is an easy corollary: One obviously has $\delta([H,N]), \delta(N^p) \subset$

$\Delta H \cdot \Delta(H,N) + \Delta(H,N) \cdot \Delta H$; the latter follows from $\delta(n^p) = \delta(n)^p$ in an $\mathbf{F}_p$–algebra. Thus we may assume $[H,N] = 1 = N^p$, hence $\Delta H \Delta(H,N) = \Delta(H,N)\Delta H$. Since

$$\begin{aligned} N &\to \Delta(H,N)/\Delta H \Delta(H,N) \\ n &\mapsto \delta(n) + \Delta H \Delta(H,N) \end{aligned}$$

is by (2.6a) surjective with kernel $[N,N]N^p = 1$, the result follows.

3. Lifting isomorphisms. For the next Lemma, let the situation be as in (2.1), resp. (2.2). We then have

LEMMA 3.1. *If γ is an algebra endomorphism of ΔG, then there exists an algebra endomorphism Γ of $\overline{\Delta F}$ with*

$$\begin{array}{ccc} \overline{\Delta F} & \xrightarrow{\Gamma} & \overline{\Delta F} \\ {\scriptstyle\pi}\downarrow & /// & \downarrow{\scriptstyle\pi} \\ \Delta G & \xrightarrow[\gamma]{} & \Delta G \end{array}$$

and if furthermore $gr_1\gamma = Id$ (i.e., γ induces the identity on $\Delta G/\Delta^2 G$), then $gr_1\Gamma = Id$.

PROOF. By (2.2) it is sufficient to show the assertion for

$$\begin{array}{ccc} \overline{\Delta F} & & \overline{\Delta F} \\ \downarrow & & \downarrow \\ \overline{\Delta F}/\overline{\Delta(F,R)} & \xrightarrow[\gamma]{} & \overline{\Delta F}/\overline{\Delta(F,R)}, \end{array}$$

where the vertical maps are the natural homomorphisms. Let $F = < f_i | i \in I >$ be freely generated by the f_i. Then

$$\delta f_i + \overline{\Delta(F,R)} \overset{\gamma}{\mapsto} x_i + \overline{\Delta(F,R)}, \ i \in I,$$

where x_i denotes an arbitrary but fixed preimage in $\overline{\Delta F}$ of $\gamma(\delta f_i + \overline{\Delta(F,R)})$.

Define

$$\Gamma\delta(f_i) := x_i \text{ for } i \in I.$$

By (2.4), the circle monoid $(\overline{\Delta F}, \circ)$ of $\overline{\Delta F}$ forms already a group so that Γ extends to a group homomorphism $\Gamma : \delta F \to (\overline{\Delta F}, \circ)$, which by the universal property of the augmentation ideal can be extended to an $\mathbf{F}_p$–algebra homomorphism

$$\Gamma : \Delta F \to \overline{\Delta F}.$$

As a homomorphism of algebras, Γ is continuous with respect to the Δ–adic topologies, and since ΔF is dense in $\overline{\Delta F}$, we can extend Γ still further to obtain an algebra endomorphism

$$\Gamma : \overline{\Delta F} \to \overline{\Delta F}.$$

We are now going to show $\Gamma(\overline{\Delta(F,R)}) \subset \overline{\Delta(F,R)}$, thereby obtaining an algebra endomorphism

$$\overline{\Delta F}/\overline{\Delta(F,R)} \to \overline{\Delta F}/\overline{\Delta(F,R)},$$

which, by construction, coincides with γ.

By continuity of Γ, it is sufficient to show $\Gamma\delta(R) \subset \overline{\Delta(F,R)}$. Let

$$f_{i_1}^{k_{i_1}} \cdots f_{i_r}^{k_{i_r}} \in R, \text{ i.e., } f_{i_1}^{k_{i_1}} \cdots f_{i_r}^{k_{i_r}} \cdot R = R.$$

Since for radical rings A with a two–sided ideal I, the relation $x + I = x \circ I$ defines an isomorphism $(A/I, \circ) \tilde{\to} (A, \circ)/(I, \circ)$ (see for example [**5**; 1.1 Lemma p. 301]), we obtain

$$(\delta f_{i_1}^{(k_{i_1})}) \circ \cdots \circ (\delta f_{i_r})^{(k_{i_r})} \circ \overline{\Delta(F,R)} = \overline{\Delta(F,R)}.$$

(To avoid confusion with ordinary powers, we have denoted powers in the circle composition by round brackets.) Since, as an algebra homomorphism, γ preserves the circle composition, we obtain

$$\begin{aligned}
\overline{\Delta(F,r)} &= \gamma(\overline{\Delta(F,R)})) \\
&= (\gamma(\delta f_{i_1} \circ \overline{\Delta(F,R)}))^{(k_{i_1})} \circ \cdots \circ (\gamma(\delta f_{i_r} \circ \overline{\Delta(F,R)})^{k_{i_r}} \\
&= (x_{i_1} \circ \overline{\Delta(F,R)})^{(k_{i_1})} \circ \cdots \circ (x_{i_r} \circ \overline{\Delta(F,R)})^{(k_{i_r})} \\
&= x_{i_1}^{(k_{i_1})} \circ \cdots \circ x_{i_r}^{(k_{i_r})} \circ \overline{\Delta(F,R)};
\end{aligned}$$

thus $\Gamma\delta(f_{i_1}^{k_{i_1}}) = x_{i_1}^{(k_{i_1})} \circ \cdots \circ x_{i_r}^{(k_{i_r})} \in \overline{\Delta(F,R)}$.

Now let $gr_1\gamma = \mathrm{Id}$. Because of the natural isomorphism

$$(\overline{\Delta F}/\overline{\Delta(F,R)})(\overline{\Delta F}^2 + \overline{\Delta(FR)}/\overline{\Delta(F,R)}) \xrightarrow{\sim} \overline{\Delta F}/\overline{\Delta F}^2 + \overline{\Delta(F,R)}$$

and by minimality of (2.1), which implies $\Delta(F,R) \subset \Delta^2 F$ and thus $\overline{\Delta(F,R)} \subset \overline{\Delta F}^2$, the diagram (3.2) (with Γ) induces

$$\begin{array}{ccc} \overline{\Delta F}/\overline{\Delta F}^2 & \xrightarrow{gr_1\Gamma} & \overline{\Delta F}/\overline{\Delta F}^2 \\ {\scriptstyle Id}\downarrow & /// & \downarrow{\scriptstyle Id} \\ \overline{\Delta F}/\overline{\Delta F}^2 & \xrightarrow[gr_1\gamma = Id]{} & \overline{\Delta F}/\overline{\Delta F}^2 \end{array}$$

REMARKS. 1) $gr_1\Gamma = \mathrm{Id}$ (resp. $gr_1\gamma = \mathrm{Id}$) implies that Γ (resp. γ) is an automorphism: $gr_1\Gamma = \mathrm{Id}$ forces already $gr\ \Gamma \in \mathrm{End}\ \sum_{i\geq 1} \oplus \overline{\Delta F}^i/\overline{\Delta F}^{i+1}$ to be the identity. Hence, $\mathrm{Ker}\,\Gamma \subset \overline{\Delta F}^n$ for all $n \in \mathbf{N}$, and since $\overline{\Delta F}$ is Hausdorff, Γ is injective. And Γ is surjective, for $\Gamma = \mathrm{Id} + \varphi$, where $\varphi : \overline{\Delta F} \to \overline{\Delta F}^2$ is linear with the further property $\varphi(xy) = \varphi(x)y + x\varphi(y) + \varphi(x)\varphi(y)$, which shows inductively $\varphi(\overline{\Delta F}^i) \subset \overline{\Delta F}^{i+1}$. Hence, for a given $x \in \overline{\Delta F}, \sum_{i\geq 0}(-1)^i\varphi^i(x)$ exists and

$$\Gamma(\sum(-1)^i\varphi^i(x)) = (\mathrm{Id} + \varphi)(\sum_{i\geq 0}(-1)^i\varphi^i(x)) = x.$$

The same applies to γ.

2) The reasoning in the last part of the proof giving $gr_1\Gamma = \mathrm{Id}$ holds also for the following situation: Let $\tau : \Delta G \to \Delta H$ be induced by a surjective homomorphism $G \to H$ and I an ideal contained in $\ker \tau$. Then τ induces $\hat{\tau} : \Delta G/I \to \Delta H$. If

$$\begin{array}{ccc} \Delta G/I & \xrightarrow{\Gamma} & \Delta G/I \\ {\scriptstyle \hat{\tau}}\downarrow & /// & \downarrow{\scriptstyle \hat{\tau}} \\ \Delta H & \xrightarrow[\gamma]{} & \Delta H \end{array}$$

with $\mathrm{Ker}\,\tau \subset \Delta^2 G$ and $gr_1\gamma = \mathrm{Id}$, then $gr_1\Gamma = \mathrm{Id}$ (because, once again, $\hat{\tau}$ induces an isomorphism $\Delta G/\Delta^2 G \to \Delta H/\Delta^2 H$).

Now, we have the tools together to prove

THEOREM 3.3. *Let F be a finitely generated free group and $R \subset F_2 \cdot F^p$ a normal subgroup such that F/R is a finite p–group. If F/R is $\mathbf{F}_p$–strongly characterized by its integral group ring, then so are*

$$F/[F,R]R^p \quad \textit{and} \quad F/[R,R]R^p.$$

PROOF. For notational convenience let us call $G := F/R$ and $G_0 := F/[F,R]R^p$ resp. $G_0 := F/[R,R]R^p$ (we will handle both cases simultanously). Then it is well–known that "G a finite p–group" implies the same for G_0. Now let H_0 be a group such that $\Delta_{\mathbf{Z}} G_0 = \Delta_{\mathbf{Z}} H_0$, and let $R_0 := R/[F,R]R^p$, resp. $R_0 := R/[R,R]R^p$. Then there exists a normal subgroup N_0 of H_0 such that $\Delta_{\mathbf{Z}}(G_0, R_0) = \Delta_{\mathbf{Z}}(H_0, N_0)$ and, moreover, if I is an ideal in $\Delta_{\mathbf{Z}} G_0$ with $I \cap \delta G_0 = 0$, then $I \cap \delta H_0 = 0$ and vice versa. By means of the natural (!) isomorphism

$$\Delta_{\mathbf{Z}} G_0 / \Delta_{\mathbf{Z}}(G_0, R_0) \xrightarrow{\sim} \Delta_{\mathbf{Z}}(G_0/R_0) \xrightarrow{\sim} \Delta_{\mathbf{Z}} G,$$

we may identify $\Delta_{\mathbf{Z}} G$ and $\Delta_{\mathbf{Z}}(H_0/N_0)$. Passing to the modular situation, this gives

$$\begin{aligned} \Delta(G_0, R_0) &= \Delta(H_0, N_0), \\ \Delta G &= \Delta(H_0/N_0), \end{aligned} \tag{3.4}$$

and the "intersection zero property", too, carries over. Since G is $\mathbf{F}_p$–strongly characterized by its integral group ring, there exists an isomorphism $\gamma : G \xrightarrow{\sim} H_0/N_0$, which–when extended to an automorphism of ΔG–induces the identity on $\Delta G/\Delta^2 G$. Thus, we obtain, by (3.1),

$$\begin{array}{ccc} \overline{\Delta F} & \xrightarrow{\ \Gamma\ } & \overline{\Delta F} \\ {\scriptstyle \overline{\pi}}\downarrow & /// & \downarrow{\scriptstyle \overline{\pi}} \\ \Delta G & \xrightarrow[\ \gamma\ ]{} & \Delta G \end{array}$$

with $gr_1 \Gamma = \mathrm{Id}$. In particular, $\Gamma(\operatorname{Ker} \overline{\pi}) \subset \operatorname{Ker} \overline{\pi}$, i.e., $\Gamma(\overline{\Delta(F,R)}) \subset \overline{\Delta(F,R)}$. If $\pi_0 : F \to F/[F,R]R^p$ denotes the natural homomorphism and $\overline{\pi}_0$ its extension to $\overline{\Delta F} \to \Delta(F/[F,R]R^p)$, then the full preimage $\overline{J}$ of $I := \Delta G_0(G_0,R_0) + \Delta(G_0,R_0)\Delta G_0$ under $\overline{\pi}_0$ is, according to (2.5),

$$\overline{J} = \overline{\Delta(F,[F,R]R^p) + \Delta F\Delta(F,R) + \Delta(F,R)\Delta F},$$

and since $\Delta(F,[F,R]R^p) \subset \Delta F \cdot \Delta(F,R) + \Delta(F,R)\Delta F$ by (2.6b), one has

$$\overline{J} = \overline{\Delta F\Delta(F,R) + \Delta(F,R)\Delta F}.$$

Now, $\overline{\Delta(F,R)}$ was already seen to be invariant under Γ, so we obtain

$$\Gamma(\Delta F\Delta(F,R) + \Delta(F,R)\Delta F) \subset \overline{\Delta F} \cdot \overline{\Delta(F,R)} + \overline{\Delta(F,R)} \cdot \overline{\Delta F} \subset \overline{J}.$$

Since Γ is continuous with respect to the Δ–adic topology, this implies that $\overline{J}$ is invariant under Γ, so that Γ induces an endomorphism γ_0 of $\Delta G_0/I$. By $R \subset F_2 \cdot F^p$, one has $I \subset \Delta^2 G_0$ and by the remarks following (3.1), γ_0 is already an automorphism. (To deal with the case $F/[R,R]R^p$, one has to replace $\Delta G_0\Delta(G_0,R_0) + \Delta(G_0,R_0)\Delta G_0$ by $\Delta G_0 \cdot \Delta(G_0,R_0)$, and the corresponding preimage $\overline{J}$ turns out to be $\overline{J} = \overline{\Delta F \cdot \Delta(F,R)}$ by (2.5a).) Since $\Gamma(\overline{\Delta(F,R)}) \subset \overline{\Delta(F,R)}$ and thus $\gamma_0(\Delta(G_0,R_0)/I) \subset \Delta(G_0,R_0)/I, \gamma_0$ induces γ on ΔG. Altogether, we have established

(3.5)

$$\begin{array}{ccc}
\overline{\Delta F} & \xrightarrow{\Gamma} & \overline{\Delta F} \\
\downarrow & /// & \downarrow \\
\Delta G_0/I & \xrightarrow[\gamma_0]{} & \Delta G_0/I \\
\downarrow & /// & \downarrow \\
\Delta G & \xrightarrow[\gamma_0]{} & \Delta G,
\end{array}$$

where the vertical maps are given by

$$\delta(f) \mapsto \delta(f \cdot [F,R]R^p) + I \mapsto \delta(fR),$$

respectively

$$\delta(f) \mapsto \delta(f \cdot [R,R]R^p) + I \mapsto \delta(fR).$$

By (2.6), $\hat{\delta}$ with $G_0 \to \Delta G_0/I$ with $g \mapsto \delta(g) + I$ is injective; and since $\Delta(G_0, R_0) = \Delta(H_0, R_0)$ and (3.4), the same applies to H_0; and

$$\begin{array}{cc} R_0 \xrightarrow{\sim} \Delta(G_0, R_0)/I = & \Delta(H_0, N_0)/I \xleftarrow{\sim} N_0 \\ r \mapsto \hat{\delta}(r) = \delta(r) + I & \delta(n) + I = \hat{\delta}(n) \leftarrow\!\shortmid n. \end{array}$$

So δG, resp. δH, considered as subsets of ΔG has $\hat{\delta} G_0$, resp. $\hat{\delta} H_0$, resp. $\hat{\delta} H_0$, as full preimage under $\Delta G_0/I \to \Delta G$. By $\gamma\delta G = \delta H$ and commutativity of the bottom part of diagram (3.5), we thus obtain

$$\gamma_0 \hat{\delta}(G_0) \subset \hat{\delta}(H_0),$$

and since γ_0 is an automorphism, equality holds. This defines an isomorphism

$$\sigma : G_0 \xrightarrow{\sim} H_0,$$

and since $\gamma_0(\Delta(G_0, R_0)/I) = \Delta(G_0, R_0)/I$, we obtain

$$\sigma(R_0) = N_0.$$

Extending σ to an automorphism of ΔG_0 thus gives

$$\sigma(\Delta(G_0, R_0)) = \Delta(H_0, N_0) = \Delta(G_0, R_0);$$

hence, $\sigma(I) = I$, and σ induces

$$\hat{\sigma} : \Delta G_0/I \to \Delta G_0/I.$$

Because $\hat{\sigma}$ coincides on $\hat{\delta} G_0$ with $\gamma_0, \hat{\sigma}$ and γ_0 are equal, and we have

$$\begin{array}{ccc} \Delta G_0/I & \xrightarrow{\hat{\sigma}} & \Delta G_0/I \\ \downarrow & /// & \downarrow \\ \Delta G & \xrightarrow[\gamma]{} & \Delta G \end{array}$$

The remarks following (3.1) now imply $gr_1\sigma = gr_1\gamma = \mathrm{Id}$.

REMARK. The only reason to start in (3.3) with integrally isomorphic augmentation ideals is to carry over ideal correspondence as in (3.4) to the modular case. Once one could establish ideal correspondence for the modular situation $\Delta G_0 = \Delta H_0$ (or find a suitable substitute), (3.3) would hold for modular group algebras.

For an arbitrary group H define $\lambda_1(H) := H$ and $\lambda_{n+1}(H) := [H, \lambda_n(H)]\lambda_n(H)^p$, p a prime. Then the series $\lambda_1(H) \supset \cdots \lambda_n(H) \supset \cdots$ forms a descending central series of H, and it turns out that

$$\lambda_n(H) = H_1^{p^{n-1}} \cdot H_2^{p^{n-2}} \cdot \cdots \cdot H_n$$

(see [**3**, p. 242, 243]). If F is now a finitely generated free group, then $F/F_2F^p = F/\lambda_2(F)$ is - as an abelian group - $\mathbf{F}_p$-strongly characterized by its integral group ring, and (3.3) thus, inductively, gives

COROLLARY 3.6. *$F/\lambda_n(F)$ is $\mathbf{F}_p$–strongly characterized by its integral group ring for all $n \in N$.*

Addendum. With the (trivial) generalization of (3.1) to the situation where γ is an algebra endomorphism of $\Delta G/\Delta^n G, n$ arbitrary, it is possible to show, along the same lines as in (3.6): If F is a finitely generated free group and R a normal subgroup lying between two successive terms of the series of modular dimension subgroups of F with respect to the prime p–say, $M_{n+1,p}(F) \subset R \subset M_{n,p}(F)$–then F/R is characterized by $\mathbf{F}_p(F/R)$. This generalizes Sehgal's result that groups G with $M_{3,p}(G) = 1$ are characterized by F_pG (see [**6**; III.6.25, p. 117]).

REFERENCES

1. N.Bourbaki, *Elements of mathematics, General topology,* Part 1, Addison Wesley, Reading, MA, 1966.

2. ———, *Elements of mathematics, General topology,* Part 2, Addison Wesley, Reading, MA, 1966.

3. B. Huppert and N. Blackburn, *Finite groups* II, Springer Grundlehren **242,** Heidelberg, 1982.

4. C. Năstăsescu and F. van Oystayen, *Graded ring theory,* Northern Holland, Amsterdam, The Netherlands, 1982.

5. F. Röhl and J. Ullrich, *Über Komplettierungen und Lokalisierungen von Zirkelgruppen nilpotenter Ringe,* Arch. Math. **37** (1981), 300-305.

6. S.K. Sehgal, *Topics in group rings,* Marcel Dekker, New York, N.Y., 1978.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487