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RULED HYPERSURFACES OF EUCLIDEAN SPACE

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In [**2**] J.R. Vanstone and one of the authors studied a minimal hypersurface of Euclidean space E^{n+1} which admits a foliation by Euclidean $(n-1)$–planes. Such a hypersurface was shown to be either totally geodesic or the product $M^2 \times E^{n-2}$ where M^2 is the standard helicoid in E^3. In this paper we are interested in this problem, not as a minimality one, but as a cylindricity problem and in the question of whether or not the mere existence of an $(n-1)$–plane through every point implies that the surface is foliated. Our basic assumption is that for a hypersurface M immersed in E^{n+1} we have the following condition.

CONDITION (*). *Through each point $x \in M$, there exists an entire $(n-1)$–plane contained in M.*

We shall show that for a surface M in E^3, this implies that the surface is ruled (i.e., foliated by lines), but note that the lines of the ruling need not be the lines hypothesized. For example consider a doubly ruled surface (hyperboloid of one sheet, hyperbolic paraboloid or plane) and for the lines of condition (*) make a random choice between the two rulings at each point. In general if the hypersurface M is not foliated by the given $(n-1)$–planes, we have points where these planes intersect. Our main result is to show that in a neighborhood of such a point $(n-2)$–dimensions break away and we have a product structure of an open set in E^{n-2} and a piece of a surface in E^3. If these intersections are dense, M is the product of E^{n-2} and a doubly ruled surface. Finally we show by example that M may be foliated by $(n-1)$–planes but not have a product structure with E^{n-2} as a factor; in particular the complement of the relative null distribution is not integrable.

Let $\overline{\nabla}$ denote the standard connection on E^{n+1} and ∇ the induced connection on M; the second fundamental form α of the immersion is

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then given by

$$\alpha(X, Y) = \overline{\nabla}_X Y - \nabla_X Y$$

where X and Y are tangent vector fields. For each $x \in M$ we define a subspace RN_x of the tangent space TM_x by

$$RN_x = \{X \in TM_x | \alpha(X, Y) = 0, \ \forall Y \in TM_x\}.$$

RN_x is called the *relative null space* at x, $\nu(x) = \dim RN_x$ is called the *relative nullity* at x and $\nu = \min_{x \in M}\{\nu(x)\}$ is called the *index of relative nullity.* Now let

$$G = \{x \in M | \nu(x) = \nu\};$$

by the upper semi–continuity of $\nu(x)$, G is open in M. Moreover RN_x for $x \in G$ defines a distribution RN on G. It is well known that RN is an integrable distribution on G with totally geodesic leaves [**1**].

LEMMA 1. $\nu \geq n - 2$.

PROOF. Let $e_1, \cdots, e_n$ be an orthonormal frame at $x \in M$ with $e_2, \cdots, e_n$ tangent to the $(n-1)$–plane through x. Then the Weingarten map has the following matrix representation with respect to this basis

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & 0 & \\ a_n & & \end{bmatrix}$$

which clearly has nullity $\geq n - 2$.

LEMMA 2. *Let ℓ_x be the $(n-1)$–plane through x and L_x the leaf of RN through $x \in G$. Then $L_x \subset \ell_x$.*

PROOF. Referring to the basis used in the proof of Lemma 1, let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be an eigenvector of the eigenvalue 0 of the Weingarten map. Then $\sum_{i=1}^n a_1 x_i = 0$ and $a_j x_1 = 0$ for $j = 2, \cdots, n$. Since

$\nu(x) = n-2$ for $x \in G$, at least one of $a_2, \cdots, a_n$ is non–zero and hence $x_1 = 0$ which means that X is tangent to ℓ_x. Thus since L_x and ℓ_x are linear subspaces of E^{n+1} the result follows.

LEMMA 3. *For any two $(n-1)$–planes ℓ_1 and ℓ_2 such that $\ell_1 \cap \ell_2 \neq \emptyset$, either $\ell_1 \cap \ell_2$ is a leaf of RN or no leaf of RN intersects with $\ell_1 \cap \ell_2$.*

PROOF. Suppose $x \in \ell_1 \cap \ell_2$ which is in a leaf L_x of RN. L_x lies in ℓ_1 and ℓ_2 by Lemma 2 giving the result.

Our final lemma is a simple one on $(n-1)$–planes in the n–dimensional space $\mathbf{R}^n$.

LEMMA 4. *Let ℓ_1 and ℓ_2 be $(n-1)$–planes in $\mathbf{R}^n$ such that $\ell_1 \cap \ell_2 \neq \emptyset$. For every point $x \in \ell_1 \cap \ell_2$ there exist neighborhoods U and V with $x \in U \subset V$ such that for every $y \in U$ and any $(n-1)$–plane ℓ_y through y, ℓ_y intersects at least one of ℓ_1 or ℓ_2 within V.*

PROOF. Choose V to be a section of a solid cylinder about $\ell_1 \cap \ell_2$ cut off by parallel $(n-1)$–planes perpendicular to $\ell_1 \cap \ell_2$ and consider the open rectangular prism about $\ell_1 \cap \ell_2$ inscribed to the cylinder. Now let U be a neighborhood of x contained in the prism (see figure 1). Clearly, an $(n-1)$–plane passing through a point in the open rectangular prism must intersect at least one of ℓ_1 and ℓ_2 within V. This completes the proof.

PROPOSITION. *Let M be a connected surface in E^3 satisfying condition $(*)$, then M is a ruled surface.*

PROOF. We first show that either M has an open cover $\{U_\alpha\}$ by coordinate neighborhoods such that the lines hypothesized induce a foliation on each U_α or M contains a triangle with simply connected interior and whose sides are three of the given lines. Let $\{V_\beta\}$ be an open cover of M and suppose that in some V_β two of the given lines, say ℓ_1 and ℓ_2 intersect at a point x. Choose a neighborhood W of x sufficiently small that for any $y \in W$, the line through y meets at least one of the ℓ_1, ℓ_2 in V_β. This can be seen as follows. Let TM_x

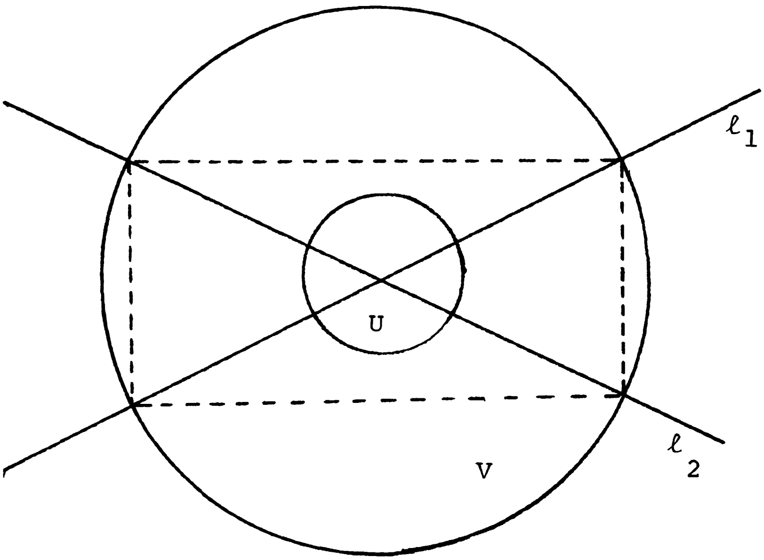

Figure 1.

be the tangent space of M at x. TM_x is realized as a 2–plane in $\mathbf{R}^3$ passing through x. Clearly, TM_x contains ℓ_1 and ℓ_2. Let P_x denote the orthogonal projection of $\mathbf{R}^3$ onto TM_x obtained by collapsing the normal direction of M at x. P_x restricted to M is a diffeomorphism in a small open neighborhood $W_\beta \subset V_\beta$ in M. Denote the restriction of P_x to W_β by the same letter P_x. In addition, W_β is chosen so small that any line in M passing through a point in W_β is projected under P_x onto a line in TM_x. $P_x(W_\beta)$ is an open neighborhood of x in TM_x. Let $U \subset V \subset P_x(W_\beta)$ be the open neighborhoods of x in TM_x constructed in the proof of Lemma 4. By Lemma 4, any line in TM_x passing through a point in U must intersect either ℓ_1 or ℓ_2 within $V \subset P_x(W_\beta)$. Now set $W = P_x^{-1}(U) \subset M$. Then any line passing through a point in W must intersect wither ℓ_1 or ℓ_2 within $P_x^{-1}(V) \subset W_\beta \subset V_\beta$. If for some y, the line through it meets both ℓ_1 and ℓ_2, we have a triangle of the desired sort and we assume in the rest of the argument that such a triangle does not exist. In particular note two things: 1) If for every

$y \in W$ the line through y passes through x, then M is already the plane; 2) if three lines pass through x, we can, using Lemma 4 for each pair of lines, choose W small enough to obtain a triangle. If now for every $y \in W$ the line through it meets one of the lines, say ℓ_1, we have the desired foliation. If some of the lines through points of W meet ℓ_1 and some meet ℓ_2, we have a quadrilateral in M with simply connected interior (choosing, e.g., Y sufficiently close to ℓ_1, if necessary (see figure 2)). Now for each point interior to the quadrilateral the line through it meets a pair of opposite sides (since we are assuming no triangles). Thus at least one pair of opposite sides has uncountably many lines connecting them. If this set of lines is dense in the interior of the quadrilateral we have a foliation there. If not, choose a curve joining a point of one side to a point of the other and lying in the complement of these lines; through each point of this curve there is a line connecting the other pair of opposite sides (and not intersecting) again giving a foliation. Now proceed similarly in the other quadrants determined by ℓ_1 and ℓ_2 to give a foliated neighborhood U_α of x. Finally doing this for each crossing point $x \in V_\beta$ if necessary, by the boundedness of V_β we may choose a finite cover of V_β and in turn desired cover of M.

Before continuing with the case of the foliated neighborhoods, we consider the case of the triangles. Let x be any point in the interior of such a triangle. By hypothesis there exists a line ℓ through x which is of course a geodesic in M and ℓ meets the triangle in at least two points (cf. Axiom of Pasch). Now a line meeting two points of a plane lies in the plane. Thus every point interior to the triangle lies in the plane π of the triangle.

Let S be the largest connected open set of M lying in π containing the interior of the triangle. We first claim that for a boundary point x of S, there exists an open line segment in M containing x and lying in π but containing no points of S near x. For since x is a boundary point of an open set in M and π, the tangent plane to M at x is π; thus the given line ℓ through x lies in π. Now suppose ℓ intersects S near x and consider a sequence of points $\{x_n\}$ in M not lying in π and converging to x. Also fix a neighborhood U of x in π such that neighborhood V of x in M is the graph of a function on U. Then for n sufficiently large the lines ℓ_n through x_n intersect ℓ by the fact that V is in $1-1$ correspondence with U. Now the projections of these lines onto U lie in the intersection of U with some half plane determined by a line m through

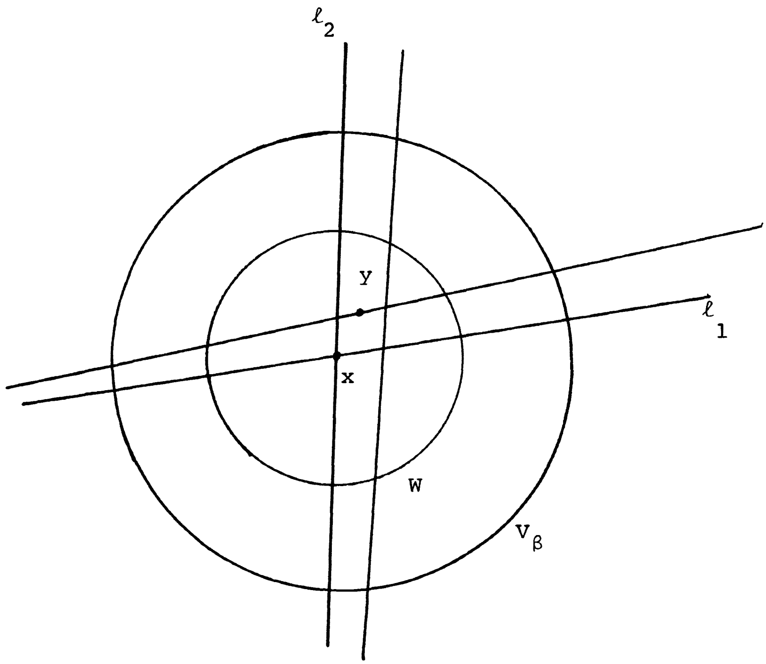

Figure 2.

x such that $m \cap U \subset M$ (see figure 3); for if not the line through some nearby point $y \in S$ would intersect these projections contradicting the local $1-1$ correspondence. Moreover in this case where we have the line ℓ through x meeting S we see that part of the boundary of S is an open line segment containing x. We now show this in general. First note that what we proved above shows that the set S is convex (see, e.g., [**4**], p. 53). Let m be the given line through x. If for all $y \in S \cap U$ the given line through y does not meet m, we clearly again have the result. So suppose that for some $y \in S \cap U$ the line ℓ through y meets m, say at $w \neq x$. Join y to x by a segment in S. Again considering projections we see that for z in the triangle bounded by ℓ, m and this segment, $z \in S$. Therefore the boundary of S contains a line segment with x as an endpoint. Now consider a boundary point x' near x but not in this segment; it has a line segment m' associated to it as m is to x. Since m' does not meet $S \cap U$, m' meets m either at x or at a point on the side of x opposite the segment. In the latter case, the argument just given gives the interval about x. The former case gives us vertical angles at x with one angle as the boundary of S. Again some $y' \in S \cap U$ has a line ℓ' meeting m and m' (otherwise the lines through points of $S \cap U$ would "enlarge" S to the other side of m' or m). Now for z near x on the side of m containing $S \cap U$, the projection of the line through z meets at least two of m, m' and ℓ' giving $z \in S$ and the desired interval.

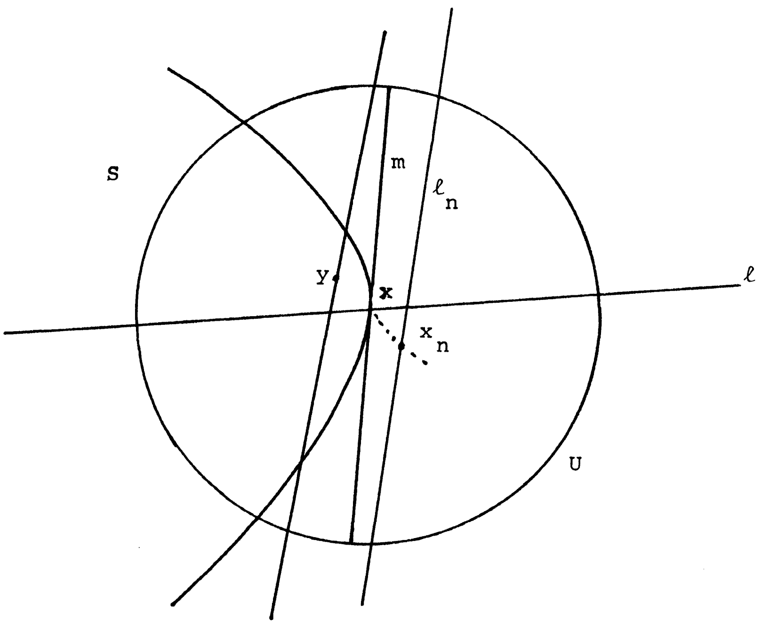

Figure 3.

Now since the boundary of S contains an open line segment about each of its points it must be either a line in π or a pair of parallel lines lying in π. But S contains a triangle whose sides are given lines and moreover at most one of these is parallel to the boundary of S. Considering two sides ℓ_1 and ℓ_2 transversal to a boundary line m (figure

4), we "enlarge" S. Arguing as above, the line ℓ through a point x near an intersection point, e.g., $\ell_1 \cap m$, must meet ℓ_1 (or already lie in π) and its projection must be parallel to m. Therefore ℓ meets ℓ_2 and hence lies in π. Thus S was not the largest connected open set of M lying in π and containing the interior of the triangle, a contradiction unless S (and therefore M) is the entire plane π which is a ruled surface.

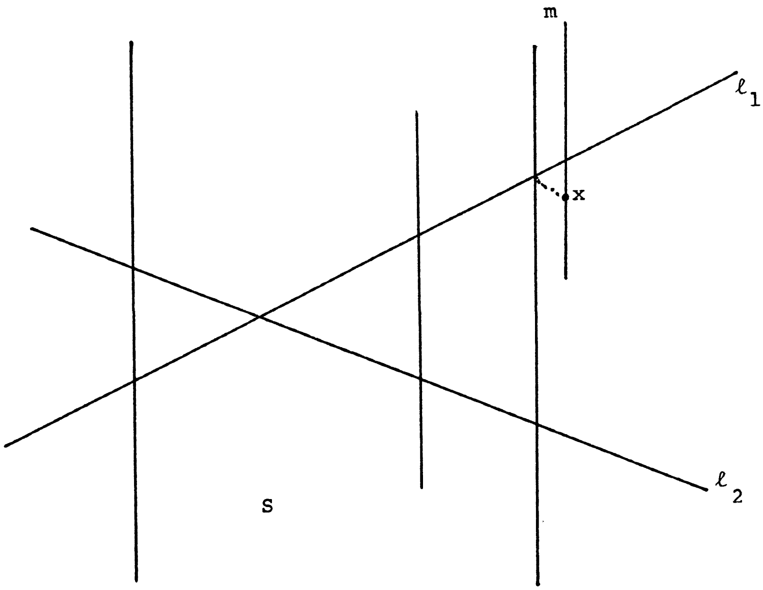

Figure 4.

Finally returning to the case of the foliated neighborhoods, we assume that M has an open cover $\{U_\alpha\}$ such that each U_α is foliated by line segments which belong to entire lines lying in M. If now in the overlaps of these neighborhoods the foliations agree, M is a ruled surface. If in the overlap of two neighborhoods the foliations do not agree, then this overlap is a piece of a plane or a quadric (the only doubly ruled surfaces are quadrics except for the plane (see, e.g., [**3**] p. 227, p. 345)). Moreover since the entire lines lie in M we see that M contains a closed "thick X" and conclude that M is a plane or a ruled quadric.

THEOREM. *Let M be a hypersurface in E^{n+1} satisfying condition $(*)$. Let x be a point in M through which there are at least two $(n-1)$–planes. Then there exists an open neighborhood U of x in M which is the Riemannian product of a ruled surface S and E^{n-2}.*

PROOF. Let ℓ_1 and ℓ_2 be two such $(n-1)$–planes through x and TM_x the tangent space to M at x to be regarded as an n–plane in E^{n+1}. Clearly then $\ell_1 \cup \ell_2 \subset TM_x$. Denote by p the projection of E^{n+1} onto TM_x. Let V be a neighborhood of x in M on which p is a diffeomorphism. Then for every $y \in V$, the $(n-1)$–plane ℓ_y through y is also mapped diffeomorphically as an $(n-1)$–plane into TM_x. Therefore $p(\ell_y) \cap p(\ell_z)$ is an $(n-2)$–plane unless they are parallel. Now choose a neighborhood U of x in M so small that for every $y \in U, p(\ell_y)$ intersects either ℓ_1 or ℓ_2 (in TM_x) in $p(V) \subset TM_x$ (cf. Lemma 4).

Suppose that for an arbitrary $y \in U, \ell_y$ intersects ℓ_2. The idea of the proof is to show ultimately that $\ell_y \cap \ell_2$ is parallel to $\ell_1 \cap \ell_2$. Assume $\ell_y \cap \ell_2 || \ell_1 \cap \ell_2$. Since $\ell_y \cap \ell_2$ and $\ell_1 \cap \ell_2$ lie in ℓ_2 and have dimension $n-3$, $(\ell_y \cap \ell_2) \cap (\ell_1 \cap \ell_2) \neq \emptyset$ and has dimension $n-2$. In particular then $\dim(\ell_y \cap \ell_1) = n-3$. Now if $\ell_1 \cap \ell_2 = \mathrm{L}_x$, a leaf of RN, $(\ell_y \cap \ell_2) \cap (\ell_1 \cap \ell_2)$ contains a point in a leaf of RN and hence by Lemma 3, $\ell_2 \cap \ell_y = L_x$. Thus each ℓ_y for $y \in U$ is a union of $(n-2)$–planes parallel to $\ell_1 \cap \ell_2$ in E^{n+1} giving the desired product structure. On the other hand if $\ell_1 \cap \ell_2$ intersects with no leaf on RN, then the relative nullity at any point in $\ell_1 \cap \ell_2 \geq n-1$. By the argument of Lemma 1 we see that in fact the relative nullity at any point in $\ell_1 \cap \ell_2 = n$, i.e., the second fundamental form vanishes along $\ell_1 \cap \ell_2$. We now distinguish two cases.

I. For every neighborhood of x, there exists a point w in the neighborhood such that $\nu(w) = n-2$.

II. There exists a neighborhood of x such that $\nu(y) \geq n-1$ for every y in the neighborhood.

I. We may assume the $(n-1)$–plane ℓ_w meets ℓ_1 in the neighborhood V (cf. Lemma 4). We also suppose that $\ell_w \neq \ell_1$ or ℓ_2, since G is an open set. Then $\ell_1 \cap \ell_w$ is an $(n-2)$–plane. As above if $\ell_1 \cap \ell_w$ is not parallel to $\ell_1 \cap \ell_2, \ell_2 \cap \ell_w \neq \emptyset$. By Lemma 3 therefore the leaves of RN in ℓ_w must intersect with ℓ_1 and/or ℓ_2. This is possible only if $\ell_w = \ell_1$

or ℓ_2, a contradiction. Therefore $\ell_1 \cap \ell_w || \ell_2 \cap \ell_w || L_w$. Now let y be any point in U. Then ℓ_y intersects ℓ_1 and/or ℓ_2, say $\ell_y \cap \ell_1 \neq \emptyset$ and $\ell_y \cap \ell_1 || \ell_1 \cap \ell_2$. Moreover the leaf L_w of RN through w meets ℓ_y in a point z (see figure 5). $\nu(z) = n - 2$ and the above argument again gives a contradiction. Thus through each point $y \in U$, there exists an $(n-2)$–plane parallel to $\ell_1 \cap \ell_2$ giving the desired product structure.

II. We now shrink V, if necessary, so that the relative nullity $\geq n-1$ on V and construct $U \subset V$ via Lemma 4 as before. We shall show that the relative nullity is actually equal to n on U. Let $y \in U$ with $\nu(y) = n - 1$. Now ℓ_y meets at least one of ℓ_1 or ℓ_2, say ℓ_1, join y to a point in $\ell_y \cap \ell_1$ by a line segment lying in $\ell_y \cap U$. As before the relative nullity along $\ell_y \cap \ell_1$ is n. Let z be the first point in the line segment such that $\nu(z) = n$, i.e., the relative nullity along the segment $[y, z)$ is $n - 1$. Since the subset of V where the relative nullity is $n - 1$ is open, there exists a neighborhood of the line segment $[y, z)$ in which the relative nullity equals $n - 1$. Thus this neighborhood is foliated by the leaves of this $(n-1)$–dimensional relative null distribution and each leaf is an open subset of an $(n-1)$–plane. An argument similar to the one given to prove the completeness of the relative null distribution [**1**] then shows that the rank of the second fundamental form cannot decrease at z. Thus $\nu(z) = n - 1$, a contradiction. Therefore $\nu(y) = n$ and hence U is totally geodesic in E^{n+1}, i.e., U is an open subset of E^n.

COROLLARY. *If through every point of M (or every point of a dense subset of M) there exists two $(n-1)$–planes in M, then M is the product of E^{n-2} and a doubly ruled surface in E^3.*

COROLLARY. *If M is a hypersurface of E^{n+1} satisfying condition $(*)$, then M is foliatable by $(n-1)$–planes.*

We close with an example of hypersurface of E^4 which is foliated by planes but which is not a Riemannian product of E^1, the leaves of RN, and a surface. In particular the orthogonal distribution $RN^{\perp}$ is not integrable. We give the position vector $\underline{\mathrm{X}}$ in E^4 as a function of three parameters u, v, w.

$$\underline{\mathrm{X}} = (u \cos w, u \sin w + \cos w, w + v \sin w, w).$$

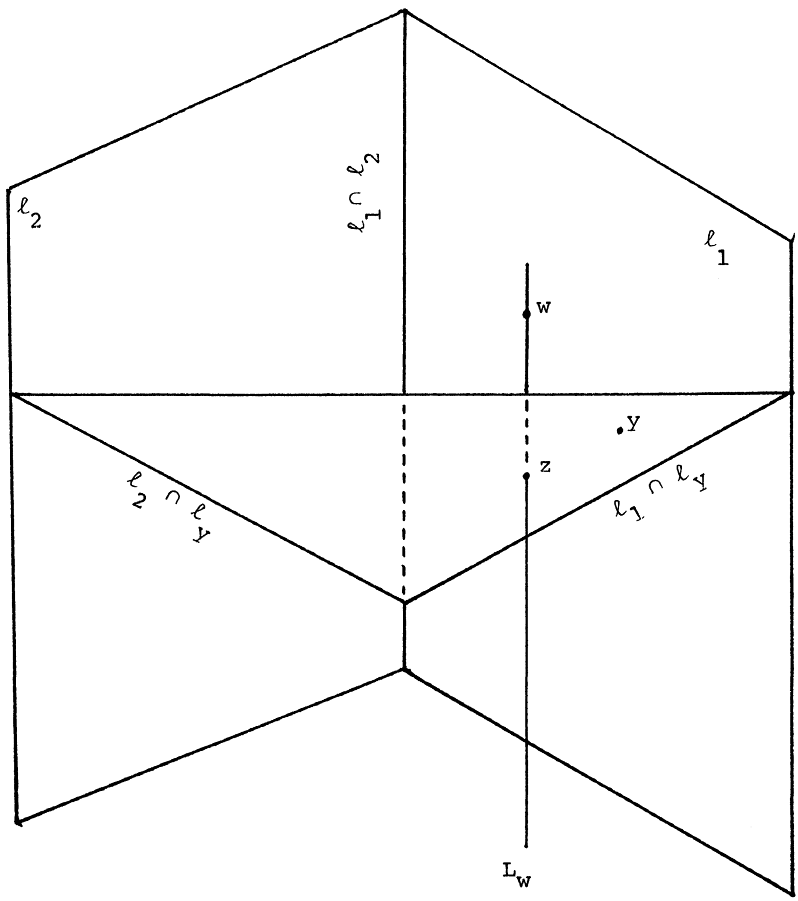

Figure 5.

The matrix of the differential of the immersion is

$$\begin{bmatrix} \cos w & \sin w & 0 & 0 \\ 0 & \cos w & \sin w & 0 \\ -u\sin w & u\cos w - v\sin w & 1 + v\cos w & 1 \end{bmatrix}$$

The vector

$$\underline{\mathrm{N}} = (-\sin^2 w, \cos w \sin w, -\cos^2 w, \cos^2 w - u \sin w + v \cos w)$$

is normal to the hypersurface. We compute the components h_{ij} of the second fundamental form except for h_{33}. The matrix is

$$\frac{1}{|\underline{\mathrm{N}}|}\begin{bmatrix} 0 & 0 & \sin w \\ 0 & 0 & -\cos w \\ \sin w & -\cos w & /// \end{bmatrix}.$$

RN is spanned by $\cos w(\partial/\partial u) + \sin w(\partial/\partial v)$. The matrix g_{ij} of the first fundamental form is easily computed. We then see that $RN^{\perp}$ is spanned by

$$\underline{\mathrm{Y}} = \frac{\sin w \cos w(u \cos w - v \sin w) + \sin^2 w}{\cos w(1 + \sin^2 w)} \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

and

$$\underline{\mathrm{Z}} = \frac{\partial}{\partial u} - \frac{\cos w(1 + \sin^2 w)}{\sin w(1 + \cos^2 w)} \frac{\partial}{\partial v}.$$

A direct computation of the Lie bracket $[\underline{\mathrm{Y}}, \underline{\mathrm{Z}}]$ shows that $[\underline{\mathrm{Y}}, \underline{\mathrm{Z}}]$ is not orthogonal to RN.

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