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# SOME CONNECTIONS BETWEEEN PETTIS INTEGRATION AND OPERATOR THEORY

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**1. Introduction.** Suppose that $X$ is a Banach space with continuous dual $X^*, (\Omega, \Sigma, \mu)$ is a finite measure space, $f : \Omega \to X$ is a scalarly measurable function so that $x^*f \in L^1(\mu)$ for each $x^* \in X^*$, and $T_f : X^* \to L^1(\mu)$ is the operator defined by $T_f(x^*) = x^*f$. In recent papers Huff [**9**] and Bator [**2**] effectively used properties of the operator $T_f$ to study the Pettis integral. In this paper we extend this study to operators between general Banach spaces. In particular, we present a characterization of $(w^*, w)$–continuous linear transformations, as well as new expositions of the Riddle, Saab, and Uhl result on universal Pettis integrability [**16**] and Odell's characterization (in terms of completely continuous operators) of spaces which contain $\ell^1$ [**18**].

Throughout the paper, $X$ and $Y$ will denote real Banach spaces. We write $X \approx Y$ to denote that $X$ and $Y$ are isomorphic (= linearly homeomorphic), and we denote the unit ball of $X$ by $B_X$. By an operator $T$ from $X$ to $Y$ we shall mean a continuous linear transformation $T : X \to Y$; the adjoint of $T$ will be denoted by $T^*$. An operator $T : X^* \to Y$ is said to be $(w^*, w)$–continuous provided that $(T(x^*_\alpha))$ converges to $T(x^*)$ in the weak topology of $Y$ whenever $(x^*_\alpha)$ is a net which converges to $x^*$ in the weak* topology of $X^*$. We denote weak (weak*) convergence by $\xrightarrow{w}(\xrightarrow{w^*})$. If $F$ is a finite subset of $X$ and $\varepsilon > 0$, set

$$K(F, \varepsilon) = (x^* \in B_{X^*} : |x^*(x)| \le \varepsilon \text{ for } x \in F).$$

**2. $(w^*, w)$–Continuity.** If $(\Omega, \Sigma, \mu)$ is as above and $f : \Omega \to X$ is a function, then we say that $f$ is scalarly measurable with respect to $\mu$ if $x^*f$ is $\mu$–measurable for $x^* \in X^*$, and we say that $f$ belongs to weak $L^1(\mu, X)$ if $x^*f \in L^1(\mu)$ for all $x^* \in X^*$. If $f \in$ weak–$L^1(\mu, X)$, then we define the operator $T_f : X^* \to L^1(\mu)$ by $T_f(x^*) = x^*f$ ([**4**], p. 52), and we say that $f$ is a $\mu$–Pettis integrable if $T^*_f$ maps $L^\infty(\mu)$ into the canonical image of $X$ in $X^{**}$. In [**9**] Huff gave a simple proof

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of the Geitz–Talagrand Pettis integrability criteria [8], [19] in terms of the $(w^*, w)$–continuity of $T_f$ and the action of $T_f$ on the sets $K(F, \varepsilon)$. In the following theorem we show that Huff's ideas may be used to characterize $(w^*, w)$–continuity in general.

THEOREM 2.1. *Suppose that* $T : X^* \to Y$ *is a continuous linear transformation. Then the following are equivalent:*
(i) $T^*(Y^*) \subseteq X$;
(ii) $T$ *is* $(w^*, w)$*–continuous;*
(iii) $T$ *is weakly compact,* $T(K(F, \varepsilon))$ *is closed for all* $(F, \varepsilon)$, *and*

$$\cap_{(F,\varepsilon)} T(K(F, \varepsilon)) = \{0\}.$$

PROOF. (i)⇒(ii). Suppose that $T^*(Y^*) \subseteq X, x^*_\alpha \xrightarrow{w^*} x^*$ in $X^*$, and $y^* \in Y^*$. Then $\langle T(x^*_\alpha), y^* \rangle = \langle x^*_\alpha, T^*(y^*) \rangle \to \langle x, Ty) \rangle = \langle T(x^*), y^* \rangle$, and it follows that $T$ is $(w^*, w)$–continuous.

(ii)⇒(i). Suppose that $T$ is $(w^*, w)$–continuous, and let $y^* \in Y^*$. But then $T^*(y^*)$ is a $w^*$–continuous member of $X^{**}$, i.e., $T^*(y^*) \in X$.

(ii)⇒(iii). Since $B_{X^*}$ is $w^*$–compact, it is clear that $T(B_{X^*})$ is weakly compact, i.e., $T$ is a weakly compact operator. Further, since $K(F, \varepsilon)$ is $w^*$–compact for each $(F, \varepsilon)$, it follows immediately that $T(K(F, \varepsilon))$ is weakly compact and therefore closed. Now suppose that $y \in \cap_{(F,\varepsilon)} T(K(F, \varepsilon))$. Let $x^*_{(F,\varepsilon)} \in K(F, \varepsilon)$ so that $T(x^*_{(F,\varepsilon)}) = y$ for each $(F, \varepsilon)$. But $(x^*_{(F,\varepsilon)})$ forms a net in the obvious ordering, and certainly $(x^*_{F,\varepsilon)}) \xrightarrow{w^*} 0$. Therefore $T(x^*_{(F,\varepsilon)}) \xrightarrow{w} 0, y = 0$, and (ii) implies (iii).

(iii)⇒(i). Suppose $x^*_\alpha \xrightarrow{w^*} x^*$ in $\frac{1}{2} B_{X^*}$. Then $x^*_\alpha - x^* \in B_{X^*}$ for each $\alpha$, and $(x^*_\alpha - x^*)_\alpha$ is eventually in $K(F, \varepsilon)$ for each pair $(F, \varepsilon)$. Now $(T(x^*_\alpha - x^*))_\alpha \subseteq T(B_{X^*})$, a relatively weakly compact subset of $Y$. Suppose that $y$ is a weak cluster point of $(T(x^*_\alpha - x^*))_\alpha$. Therefore $y \in w$–closure $(T(K(F, \varepsilon)))$. But $T(K(F, \varepsilon))$ is convex and norm closed. Consequently, $y \in T(K(F, \varepsilon))$ for each pair $(F, \varepsilon)$. Thus $y = 0$, and $T(x^*_\alpha) \xrightarrow{w} T(x^*)$.

Now suppose that $y^* \in Y^*$, and consider $y^*T$. The preceding paragraph shows that $\ker(y^*T) \cap 1/2 B_{X^*}$ is $w^*$–closed. By scaling, it follows that $\ker(y^*T) \cap \alpha B_{X^*}$ is $w^*$–closed for each $\alpha > 0$. Therefore by the Krein–Smulian theorem ([6], p. 429), $y^*T$ is $w^*$–continuous, and it follows that $T^*(Y^*) \subseteq X$.

We remark that in [**9**] Huff shows that

(1) if $f \in$ weak–$L^1(\mu, X)$, then $T_f(K(F,\varepsilon))$ is closed for each pair $(F,\varepsilon)$, and

(2) if $\operatorname{cor}_E(f) \neq \emptyset$ for $E \in \Sigma$ and $\mu(E) > 0$, then

$$\cap_{(F,\varepsilon)} T_f(K(F,\varepsilon)) = \{0\}.$$

The reader may consult Geitz [**8**] for a definition of core.

In [**2**] Bator showed that if $X$ is a Banach space and $(\Omega, \Sigma, \mu)$ is a perfect measure space, then $X^*$ has the $\mu$–PIP if and only if $||\tilde{T}_f^*(\chi_\Omega)|| = ||T_f^*(\chi_\Omega)||$ for each scalarly measurable function $f : \Omega \to B_{X^*}$, where $\tilde{T}_f$ is the restriction of $T_f$ to $X$. In the next few theorems we consider an operator $T : X^{**} \to Y$ and its restriction $\tilde{T} : X \to Y$ in connection with the conclusions of Theorem 2.1 and the equality of the norms cited above.

PROPOSITION 2.2. *Let $T : X^{**} \to Y$ be an operator, and let $\tilde{T} = T|_X$*

(i) *If $T$ is weakly compact, then $\tilde{T}^{**}$ is $(w^*, w)$–continuous.*

(ii) *The operator $T$ is $(w^*, w)$–continuous iff $T = \tilde{T}^{**}$.*

PROOF. (i) Suppose that $T : X^{**} \to Y$ is weakly compact. Then $\tilde{T}$ is weakly compact, $\tilde{T}^{**} : X^{**} \to Y \to Y^{**}$, and consequently $\tilde{T}^{**}$ is $(w^*, w)$–continuous.

(ii) Suppose that $T = \tilde{T}^{**}$. Therefore $\tilde{T}^{**} : X^{**} \to Y$, and $\tilde{T}^{**}$ is weakly compact by a theorem of Gantmacher [**6**], p. 482. Thus $T$ is weakly compact and the $(w^*, w)$–continuity of $\tilde{T}^{**}$ follows from (i). Conversely, suppose that $T$ is $(w^*, w)$–continuous. Let $x^{**} \in B_{X^{**}}$, and choose a net $(x_\alpha)$ from $B_X$ so that $x_\alpha \xrightarrow{w^*} x^{**}$ (Goldstine's Theorem). Then $T(x_\alpha) = \tilde{T}(x_\alpha) = \tilde{T}^{**}(x_\alpha)$ for each $\alpha$. Further, $T(x_\alpha) \xrightarrow{w} T(x^{**})$ by hypothesis, and $\tilde{T}^{**}(x_\alpha) \xrightarrow{w} \tilde{T}^{**}(x^{**})$ by (i). Therefore $\tilde{T}^{**}(x^{**}) = T(x^{**})$.

Before proceeding to our next characterization of $(w^*, w)$-continuity, we establish the following simple lemma.

LEMMA 2.3. *If $F$ is a finite subset of $X^*$ and $\varepsilon > 0$, then $K(F,\varepsilon) \cap X$ is $w^*$–dense in $K(F,\varepsilon)$.*

PROOF. Suppose that $x^{**} \in B_{X^{**}}$, and $|\langle x^{**}, x_i^* \rangle| < \in$ for each $x_i^* \in F$. Let $(x_\alpha)$ be net in $B_X$ so that $(x_\alpha) \xrightarrow{w^*} x^{**}$. Then eventually $|x_i^*(x_\alpha)| < \varepsilon$ for each $x_i^* \in F$, i.e., there is an $\alpha_0$ so that $x_\alpha \in K(F,\varepsilon) \cap X$ for $\alpha > \alpha_0$. Thus $x^{**} \in w^*$–closure $(K(F,\varepsilon) \cap X)$. But if $x^{**}$ is an arbitrary non–zero member of $K(F,\varepsilon)$, then the preceding argument shows that $(1 - 1/n)x^{**} \in w^*$–closure $(K(F,\varepsilon) \cap X)$ for each $n \in N$. Therefore $K(F,\varepsilon) \cap X$ is $w^*$–dense in $K(F,\varepsilon)$.

PROPOSITION 2.4. *Suppose that $T : X^{**} \to Y$ is weakly compact. Then $T$ is $(w^*, w)$–continuous if and only if $\tilde{T}(K(F,\varepsilon) \cap X)$ is weakly dense in $T(K(F,\varepsilon))$ for each finite set $F \subseteq X^*$ and each $\varepsilon > 0$.*

PROOF. Suppose $T$ is $(w^*, w)$–continuous and that $x^{**} \in K(F,\varepsilon)$. Use Lemma 2.3, and let $(x_\alpha)$ be a net from $K(F,\varepsilon) \cap X$ so that $x_\alpha \xrightarrow{w^*} x^{**}$. Then $T(x_\alpha) \xrightarrow{w} T(x^{**})$, and $\tilde{T}(x_\alpha) = T(x_\alpha)$ for each $\alpha$. (Since $\tilde{T}(K(F,\varepsilon) \cap X)$ is convex, of course it follows immediately that $\tilde{T}(K(F,\varepsilon) \cap X)$ is norm dense in $T(K(F,\varepsilon))$.)

Conversely, suppose that $\tilde{T}(K(F,\varepsilon) \cap X)$ is weakly dense in $T(K(F,\varepsilon))$ for each pair $(F,\varepsilon)$. Since $\tilde{T}^{**}$ is $(w^*, w)$–continuous, $\tilde{T}^{**}(K\ (F,\varepsilon) \cap X)$ is norm dense in $T(K(F,\varepsilon))$ for each pair $(F,\varepsilon)$. Since $\tilde{T}^{**}$ is $(w^*, w)$–continuous, $\tilde{T}^{**}(K(F,\varepsilon))$ is closed by Theorem 2.1, and thus $T(K(F,\varepsilon)) \subseteq \tilde{T}^{**}(K(F,\varepsilon))$. Therefore, again by Theorem 2.1, $\cap_{(F,\varepsilon)} T(K(F,\varepsilon)) = \{0\}$.

Suppose now that $(x_\alpha^{**}) \xrightarrow{w^*} x^{**}$ in $\frac{1}{2}B_{X^{**}}$, and fix $(F,\varepsilon)$. Then $(x_\alpha^{**} - x^{**})$ is eventually in $K(F,\varepsilon)$, and the preceding observation ensures that 0 is the only weak cluster point of $(T(x_\alpha^{**}) - T(x^{**}))$. Therefore $T(x_\alpha^{**}) \xrightarrow{w} T(x^{**})$, and the concluding portion of the argument of 2.1 shows that $T$ is $(w^*, w)$–continuous.

The following definition sets the stage for a slight pertubation of Proposition 2.4 which will be useful in a subsequent result.

DEFINITION. If $X$ is a Banach space, $F$ is a finite subset of $X$, and $\varepsilon > 0$, set $0(F,\varepsilon) = \{x^* \in B_{X^*} : |x^*(x)| < \varepsilon \text{ for each } x \in F\}$.

COROLLARY 2.5. *Suppose that $T : X^{**} \to Y$ is weakly compact. Then $T$ is $(w^*, w)$–continuous if $\tilde{T}(0(F, E) \cap X)$ is dense in $T(0(F, \varepsilon))$ for each pair $(F, \varepsilon)$.*

PROOF. If $\tilde{T}(0(F, \varepsilon) \cap X)$ is dense in $T(0(F, \varepsilon))$, then $\tilde{T}(K(F, \varepsilon) \cap X)$ is dense in $T(K(F, \varepsilon))$, and the result follows from 2.4.

Suppose now that $f : \Omega \to B_{X^*}$ is scalarly measurable. Then it follows easily that $T_f : X^{**} \to L^1(\mu)$ is $(w^*, w)$–sequentially continuous. In fact, if $x_n^{**} \xrightarrow{w^*} x^{**}$, then $x_n^{**} f \to x^{**} f$ pointwise. Consequently the bounded convergence theorem guarantees that

$$\int_{\Omega} |x_n^{**} f - x^{**} f| d\mu \to 0.$$

Thus $T_f$ is $(w^*$, norm)–sequentially continuous in this case. (Further, the boundedness of $f$ also ensures that $T_f$ is weakly compact since $T_f(B_{X^{**}})$ is norm bounded and uniformly integrable.) If $(w^*, w)$–sequential compactness guaranteed $(w^*, w)$-continuity, then $f$ would be Pettis integrable by Proposition 1 of Huff [**9**] or Theorem 2.1. The following theorem presents a general operator theoretic version of this fact in a setting which guarantees $(w^*, w)$–continuity. The proof makes use of a deep theorem of Odell and Rosenthal [**11**].

PROPOSITION 2.6. *If $X$ is separable, $T : X^{**} \to Y$ is weakly compact and $(w^*, w)$–sequentially continuous, and $X$ does not contain an isomorphic copy of $\ell^1$, then $T$ is $(w^*, w)$–continuous.*

PROOF. By Corollary 2.5, it suffices to show that $\tilde{T}(0(F, E) \cap X)$ is dense in $T(0(F, \varepsilon))$ for each pair $(F, \varepsilon)$. Let $x^{**} \in 0(F, \varepsilon)$. Then, by Odell and Rosenthall [**11**], we can (and do) select a sequence $(x_n)$ from $B_X$ so that $x_n \xrightarrow{w^*} x^{**}$. Since $0(F < \varepsilon)$ is $w^*$–open, it follows that $(x_n)$ is eventually in $0(F, \varepsilon) \cap X$. Therefore $\tilde{T}(x_n) \xrightarrow{w} T(x^{**})$ since $T$ is $(w^*, w)$–sequentially continuous. Thus $T(x^{**}) \in w$–closure $(\tilde{T}(0(F, \varepsilon) \cap X)$, i.e., $\tilde{T}(0(F, \varepsilon) \cap X)$ is dense in $T(0(F, \varepsilon))$.

In the next theorem we use some of the previous results on $(w^*, w)$–continuity together with results of Bourgain, Fremlin, and Talagrand [**3**] to present an alternate proof of a theorem of Riddle, Saab, and Uhl [**16**]. We recall that if $K$ is a compact Hausdorff space, then $f : K \to X$ is said to be universally measurable if $f$ is scalarly $\mu$–measurable for all Radon measures $\mu$ on $K$.

THEOREM 2.7. *Suppose that $X$ is a separable Banach space, $H$ is a compact Hausdorff space, and $f : H \to B_{X^*}$ is universally measurable. Then $f$ is universally Pettis integrable, i.e., $f$ is $\mu$–Pettis integrable for each Radon measure $\mu$ on $K$.*

PROOF. Suppose that $f$ and $H$ are as in the hypothesis, let $\varepsilon_1 > 0$, and let $\mu$ be a Radon measure on $K$. Using the separability of $X$, let $H_1$ be a compact subset of $H$ so that $\mu(H \backslash H_1) < \varepsilon_1$ and $xf$ is continuous on $H_1$ for each $x \in X$. Let $\tilde{f} = f|_{H_1}$.

Now suppose that $F$ is a finite subset of $X^*, \varepsilon > 0$, and let $(x_n)$ be a norm dense sequence from $K(F, \varepsilon) \cap X$. If $x^{**}$ is any cluster point of $(x_n)$, then $x^{**}\tilde{f}$ is a cluster point of $(x_n\tilde{f})_{n=1}^{\infty}$ in the topology of pointwise convergence. Therefore $A = \{x_n\tilde{f} : n \in \mathbf{N}\}$ is relatively countably compact in $M_\mu(H_1)$ in the topology of pointwise convergence. (Here $M_\mu(H_1)$ is the set of all $\mu$–measurable real valued functions on $H_1$.) Thus, by Theorem 4D of [**3**], the closure of $A$ in the product topology on $\mathbf{R}^{H_1}$ is angelic. (See [**3**], p. 857, for a formal definition of angelic.) Consequently, if $x^{**} \in K(F, \varepsilon)$, there is a sequence $(x_n)$ in $K(F, \varepsilon) \cap X$ so that $x_n\tilde{f} \to x^{**}\tilde{f}$ pointwise. Hence $x_n\tilde{f} \to x^{**}\tilde{f}$ in $L^1(\mu|_{H_1})$. Since $\varepsilon_1$ was arbitrary, it follows that the closure of $\{xf : x \in K(F, \varepsilon) \cap X\}$ in the $L^1(\mu)$–norm contains $\{x^{**}f : x^{**} \in K(F, \varepsilon)\}$. Therefore $T_f : X^{**} \to L^1(\mu)$ is $(w^*, w)$–continuous by 2.4; hence $f$ is $\mu$–Pettis integrable.

In the following proposition we present a characterization of the norm equality mentioned in the paragraph preceding Proposition 2.2.

PROPOSITION 2.8. *Let $T : X^{**} \to Y$ be an operator. Then*

$$||\tilde{T}^*(y^*)|| = ||T^*(y^*)||$$

*for all* $y^* \in Y^*$ *iff* $\tilde{T}(B_X)$ *is norm dense in* $T(B_{X^{**}})$.

PROOF. Suppose that $\tilde{T}(B_X)$ is norm dense in $T(B_{X^{**}})$, and let $y^* \in Y^*$. Then

$$\begin{aligned} ||\tilde{T}^*(y^*)|| &= \sup\{\langle \tilde{T}(x), y^* \rangle : x \in B_X\} \\ &= \sup\{\langle T(x^{**}), y^* \rangle : x^{**} \in B_{X^{**}}\} = ||T^*(y^*)||. \end{aligned}$$

Conversely, suppose $||\tilde{T}^*(y^*)|| = ||T^*(y^*)||$ for all $y^* \in Y^*$ and that $\tilde{T}(B_X)$ is not dense in $T(B_{X^{**}})$. Since $\tilde{T}(B_X)$ is convex, it follows that $A = t(b_{x^{**}}) \setminus$ (weak closure $(\tilde{T}(B_X)) \neq \emptyset$. Let $y \in A$, and let $y^* \in B_{Y^*}$ so that $y^*(y) > \sup\{y^*(\tilde{(}B_X))\}$. But then $||T^*(y^*)|| > ||\tilde{T}^*(y^*)||$, and we have a contradiction.

The reader should compare Propositions 2.4 and 2.8. Since 2.8 requires only that $\tilde{T}(K(\{0\}, \varepsilon) \cap X)$ be dense in $T(K(\{0\}, \varepsilon))$ for each $\varepsilon > 0$ and 2.4 requires that $\tilde{T}(K(F, \varepsilon) \cap X)$ be dense in $T(K(F, \varepsilon))$ for each finite set $F$ and each $\varepsilon > 0$, it is not surprising that the norm equality of 2.8 does not characterize $(w^*, w)$–continuity even for very simple operators.

Let $X$ be a Banach space so that there exist $x^* \in X^*$ and $x^{***} \in X^{***}$ which satisfy

(i) $||x^*|| = ||x^{***}||$,
(ii) $x^{***}(x) = x^*(x)$ for all $x \in X$,
(iii) $x^{***} \neq x^*$.

Define $T : X^{**} \to \mathbf{R}$ by $T(x^{**}) = x^{***}(x^{**})$. Then $\tilde{T}(x) = x^*(x), T(X^{**}) = \mathbf{R} = \tilde{T}(X)$, and $\tilde{T}(B_X) = \{\alpha \in \mathbf{R} : -||x^*|| \leq \alpha \leq ||x^*||\} = \{\alpha \in \mathbf{R} : -||x^{***}|| \leq \alpha \leq ||x^{***}||\} = T(B_X^{***})$, i.e.,

$$||\tilde{T}^*(y^*)|| = ||T^*(y^*)||$$

for each $y^* \in Y^*$. But certainly $T$ is not $(w^*, w)$–continuous. For, if it were, $x^{***}$ would be a $w^*$–continuous linear functional on $X^{**}$, and we would be forced to conclude that $x^{***} = x^*$.

**3. Completely Continuous Operators and $\ell^1$.** In the addendum to [**18**], Rosenthal gave E. Odell's characterization of those spaces $X$ not containing $\ell^1$: The Banach space $X$ fails to contain an isomorphic copy of $\ell^1$ iff every completely continuous operator $T : X \to Y$ is compact. In this section we continue the theme of §2 and develop a proof based on ideas associated with the Pettis integral. We also feel that our argument may be of independent interest since in highlights some of same structure revealed in Riddle and Uhl [15], and it clearly demonstrates the power of martingale convergence theorems. We also present other connections between completely continuous operators and $\ell^1$.

We recall that an operator $T : X \to Y$ is said to be completely continuous if $T$ maps weak Cauchy sequences in $X$ into norm convergent sequences in $Y$. Further, if $f : \Omega \to X^*$ is a function which belongs to $w^* - L^1(\mu, X^*)$ (i.e., $xf \in L^1(\mu)$ for each $x \in X$) and $T_f : X \to L^1(\mu)$ is defined as usual, then we define the Pettis norm $||f||_{\mathcal{P}}$ of $f$ to be $||T_f|| = \sup\{\int |xf| d\mu : ||x|| \leq 1\}$. We refer the reader to Fremlin [7] for a discussion of perfect measure spaces; however, we shall only need that Lebesgue measure on $[0, 1]$ generates a perfect measure space.

The following proposition plays an important role in our proof of Odell's theorem.

PROPOSITION 3.1. [2] *Suppose that* $(\Omega, \Sigma, \lambda)$ *is a perfect probability space and that* $f : \Omega \to B_{X^*}$ *is* $w^*$*–measurable. The following are equivalent.*

(i) $T_f$ *is a compact operator.*

(ii) *If* $(f_\alpha, \Sigma_\alpha)$ *is any* $B_{X^*}$*–valued* $w^*$*–martingale (i.e.,* $(xf_\alpha, \Sigma_\alpha)$ *is a martingale for all* $x \in X$*) so that* $f_\alpha \to f$ *in* $w^* - L^1(\lambda, X^*)$*, then* $f_\alpha \to f$ *in Pettis norm.*

(iii) *There exists a net* $(f_\alpha)$ *of simple functions with values in* $B_{X^*}$ *so that* $f_\alpha \to f$ *in Pettis norm.*

LEMMA 3.2. *Suppose that* $(\Omega, \Sigma, \mu)$ *is a probability space,* $f_n : \Omega \to B_{X^*}$ *is* $w^*$*–measurable for each* $n \in N$*, and for each* $x \in X$ *there is a real valued function* $g_x$ *on* $\Omega$ *so that* $xf_n \to g_x$ *a.e.* $[\mu]$*. Then there is an* $f : \Omega \to B_{X^*}$ *so that* $xf = g_x$ *a.e.* $[\mu]$ *for each* $x \in X$*.*

PROOF. Suppose the hypotheses are satisfied, and note that $(f_n)$ is a net in the compact space $(B_{X^*}, w^*)^\Omega$. Hence there is a subnet $(f_{n'}$ of $(f_n)$ and a function $f : \Omega \to B_{X^*}$ so that $(f_{n'})$ converges to $f$ pointwise in the $w^*$–topology. But then $xf = g_x$ a.e. $[\mu]$ for each $x \in X$.

We note that the sets of $\mu$–measure zero may well change with choices of $x \in X$. By appealing to the mean martingale convergence theorem and Theorem 8, p. 129 of [4], we immediately obtain the following corollary.

COROLLARY 3.3. *Let $(f_n, \Sigma_n)_{n=1}^{\infty}$ be a $B_{X^*}$–valued $w^*$–sequential martingale. Then there is a function $f : \Omega \to B_{X^*}$ so that $xf_n \to xf$ a.e. $[\mu]$ whenever $x \in X$.*

Before proceeding, it is convenient to have another definition and an example.

DEFINITION. A sequence $(x_{n,i}), n = 0, 1, 2, \ldots, i = 1, \ldots, 2^n$ in $X$ is called a tree if $x_{n,i} = (x_{n+1,2i-1} + x_{n+1,2i})/2$ for each $n$ and $i$. A tree $(x_{n,i})$ is said to be a bounded $\delta$–Rademacher tree if $||\sum_{i=1}^{2^n}(-1)^i x_{n,i}|| \geq 2^n\delta$ for each $n$, and there is a positive constant $M$ so that $||x_{n,i}|| \leq M$ for each $n$ and $i$.

Now let $(I_{n,i}), n = 0, 1, \ldots, i = 1, \ldots, 2^n$ denote the dyadic subintervals of $[0, 1]$, and put $x_{n,i} = 2^n \chi_{I_{n,i}}$ in $L^1([0, 1])$, i.e., $x_{0,1} = \chi_{[0,1]}, x_{1,1} = 2\chi_{[0,1/2]}, x_{1,2} = 2\chi_{[1/2,1]}, \ldots$. It is not difficult to see that $(x_{n,i})$ forms a bounded 1–Rademacher tree in $L^1([0, 1])$.

The following proposition shows that the presence of a bounded $\delta$–Rademacher tree in $X^*$ ensures the existence of a $w^*$–martingale for which the conclusion of Proposition 3.1 does not hold.

PROPOSITION 3.4. [1], [15]. *Let $X$ be a Banach space so that $X^*$ contains a bounded $\delta$–Rademacher tree for some $\delta > 0$. Then there is a $B_{X^*}$–valued simple $w^*$–sequential martingale which does not converge in Pettis norm.*

PROOF. Suppose the hypotheses are satisfied, and let $(x^*_{n,i})$ be a $\delta$–Rademacher tree in $B_{X^*}$. Let $(I_{n,i})_{i=1}^{2^n}, n = 0, 1, 2, \ldots,$ be the dyadic subintervals of $[0, 1]$ and let $\Sigma_n$ be the finite $\sigma$–algebra generated by $(I_{n,i})_{i=1}^{2^n}, n = 0, 1, 2, \cdots$. Now define $f_n : [0, 1] \to B_{X^*}$ by

$f_n = \sum_{i=1}^{2^n} x_{n,i}^* \chi_{I_{n,i}}, n = 0, 1, 2, \ldots$, and let $\lambda$ be Lebesgue measure on $[0,1]$. The tree property of $(x_{n,i}^*)$ guarantees that $(f_n, \Sigma_n)$ is a $B_{X^*}$ simple sequential martingale.

We assert that $(f_n)$ is not Cauchy in Pettis norm. To see this, begin by putting $A_n = \cup_{i=1}^{2^n} I_{n+1,2i}$. Then

$$\left|\left|\int_{A_n} (f_{n+1} - f_n) d\lambda\right|\right| = \left|\left|\sum_{i=1}^{2^n} \int_{I_{n+1,2i}} (f_{n+1} - f_n) d\lambda\right|\right|$$

$$\frac{1}{2^{n+1}}\left|\left|\sum_{i=1}^{2^n} (x_{n+1,2i}^* - x_{n,i}^*)\right|\right| = \frac{1}{2^{n+1}}\left|\left|\sum_{i=1}^{2^n} \frac{1}{2}(x_{n+1,2i}^* - x_{n+1,2i-1}^*)\right|\right|$$

$$= \frac{1}{2}\left(\frac{1}{2^{n+1}}\left|\left|\sum_{i=1}^{2^{n+1}} (-1)^i x_{n+1,i}^*\right|\right|\right) \geq \frac{\delta}{2}.$$

But certainly $||\int_{A_n}(f_{n+1} - f_n)d\lambda|| = \frac{1}{2^{n+1}}||\sum_{i=1}^{2^n}(x_{n+1,2i}^* - x_{n,i}^*)|| = \sup\{\langle \frac{1}{2^{n+1}}(\sum_{i=1}^{2^n} x_{n+1,2i}^* - x_{n,i}^*), x\rangle : x \in B_X\} \leq ||f_{n+1} - f_n||_{\mathcal{P}}$.

Combining 3.1, 3.4, Rosenthal's theorem, and a result of Pelczynski [**14**], [**15**], we obtain Odell's characterization.

THEOREM 3.5. *The Banach space $X$ fails to contain an isomorphic copy of $\ell^1$ iff every completely continuous operator $T : X \to Y$ is compact for each space $Y$.*

PROOF. Suppose that $X$ does not contain a copy of $\ell^1, T : X \to Y$ is completely continuous, and $(x_n)$ is a sequence in $B_X$. Then $(x_n)$ has a weak Cauchy subsequence, say $(x_{n_i})_{i=1}^{\infty}$. Consequently $(T(x_{n_i}))$ is norm convergent, and we see that $T$ is compact.

Conversely, suppose that $X$ does contain an isomorphic copy of $\ell^1$. Then by Pelczynski's theorem [**14**], [**15**], $X^*$ contains an isomorphic copy of $L^1([0,1])$. Since $L^1([0,1])$ contains a bounded 1–Rademacher tree, it follows that $X^*$ contains a bounded $\delta$–Rademacher tree for some $\delta > 0$.

Next we apply Proposition 3.4 and obtain a sequential martingale $(f_n, \Sigma_n)$ of $B_{X^*}$–valued simple functions which does not converge in Pettis norm. (Recall that $\Sigma_n$ is the finite $\sigma$–algebra generated by the dyadic intervals $(I_{n,i})_{i=1}^{2^n}$.) But, by Corollary 3.3, there is a function $f : [0,1] \to B_{X^*}$ so that $xf_n \to xf$ a.e. $[\lambda]$ and hence in $L^1([0,1])$. However, certainly $(f_n)$ does not converge (in particular) to $f$ in Pettis norm, and consequently the operator $T_f$ is not compact by Proposition 3.1. To finish the argument, we note that any such operator must be completely continuous. For if $(x_n)$ converges weakly to $x$ in $X$, then the bounded convergence theorem guarantees that $T_f(x_n) = x_n f \to xf = T_f(x)$ in $L^1([0,1])$.

We conclude the paper with two additional propositions which relate the behavior of completely continuous operators on $X$ to the presence of isomorphic copies of $\ell^1$ in $X$. We say that an operator $T : X \to Y$ is (a) *weakly precompact* if $T$ maps each bounded sequence into a sequence some subsequence of which is weakly Cauchy and (b) *weakly completely continuous* if $T$ maps weakly Cauchy sequences in $X$ into weakly convergent sequences in $Y$. Further, $T$ is said to preserve a copy of $\ell^1$ if there is an isomorphic copy of $\ell^1$ in $X$ on which $T$ acts an isomorphism. If $A \subseteq X$, we denote the closed linear span of $A$ by $[A]$.

PROPOSITION 3.6. *Suppose that $T : X \to Y$ is an operator.*

(i) *If $T$ is not weakly precompact, then $T$ preserves a copy of $\ell^1$.*

(ii) *If $T$ is completely continuous and $T^{**}$ is $1-1$, then $T$ is compact or $T$ preserves a copy of $\ell^1$.*

(iii) *If $T$ is weakly completely continuous and $T^{**}$ is $1-1$, then $T$ is weakly compact or $T$ preserves a copy of $\ell^1$.*

PROOF. Part (i) is the content of Theorem 3 of Lohman [**10**]. And since the arguments for (ii) and (iii) are so similar, we shall present a proof for (iii) and leave the proof of (ii) to the reader.

Suppose then that $X, Y$, and $T$ are as in the hypotheses and the $T$ is not weakly compact. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $B_X$ so that $(T(x_n))$ has no weakly convergent subsequence. Since $T$ is weakly completely continuous, it follows that $(x_n)$ has no weakly Cauchy subsequence. Using Rosenthal's theorem [**17**] and [**15**], p. 201, let $(x_{n_i})$ be a subsequence so that $(x_{n_i})$ is equivalent to the canonical basis $(e_i)$ of $\ell^1$. Without loss of generality, we suppose that the full sequence $(x_n)$ is equivalent to $(e_n)$. Now since $T^{**}$ is $1-1$, range $(T^*)$ is dense in $X^*$; thus $(T(x_m))$ has no weakly Cauchy subsequence.

Using Rosenthal's theorem again, it follows that there is a subsequence $(T(x_{n_i}))$ of $(T(x_n))$ so that

$$[x_{n_i}] \approx \ell^1 \approx [T(x_{n_i})],$$

and $T$ is an isomorphism on $[x_{n_i}]$.

PROPOSITION 3.7. *The follow are equivalent:*
(i) $\ell^1$ *is complemented in* $X$.
(ii) *If* $Y$ *is any infinite dimensional Banach space, then there is a completely continuous operator* $T : X \to Y$ *so that* $T$ *is not compact.*

PROOF. Suppose that $\ell^1$ is complemented in $X$ and that $Y$ is an infinite dimensional Banach space. Let $Z$ be an infinite dimensional separable subspace of $Y$, and let $L : \ell^1 \to Z$ be a bounded linear surjection. Then $L$ is completely continuous (since every weakly Cauchy sequence in $\ell^1$ is already norm convergence) and not compact. Let $P : X \to \ell^1$ be a bounded linear projection, and set $T = L \circ P : X \to Y$. The $T$ is completely continuous and not compact.

Now suppose that (ii) holds. Then there is a non–compact operator $T : X \to \ell^1$. Then, by Lemma 1 of Pelczynski [**13**], there is a bounded linear surjection $S : X \to \ell^1$. Hence, by another result of Pelczynski [**12**] and [**5**], p. 72, $X$ must contain a complemented subspace isomorphic to $\ell^1$.

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