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SOURCES, SINKS AND SADDLES FOR EXPANSIVE HOMEOMORPHISMS WITH CANONICAL COORDINATES

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ABSTRACT. We study sources, sinks and saddles for expansive homeomorphisms of compact metric spaces which have canonical coordinates. If the phase space is connected and locally connected then each point is a saddle. We show by example that local connectedness is a necessary hypothesis.

1. Introduction. Canonical coordinates were introduced by R. Bowen [**1**]. (see also S. Smale [**9**].) He used expansive homeomorphisms having canonical coordinates to study Axiom A diffeomorphisms [**1,2,3,4**]. This notion was a fruitful one for ergodic theory [**3,5,8**], entropy calculations [**1,5**] and topological dynamics [**4,7**].

Since canonical coordinates "move around with a point" one may extend certain notions which are valid for fixed or periodic points to this setting. We generalize the notions of source, sink and saddle to any point in the phase space of an expansive homeomorphism which has canonical coordinates.

2. Canonical Cordinates. Let f be an expansive homeomorphism of the compact metric (d) space X. Fix an expansive constant $c > 0$ for f. We now define canonical coordinates and related concepts and collect some useful facts, mostly without (the easy) proofs.

DEFINITION 2.1. We define the local stable set of f at $x \in X$ for $\delta \geq 0$ as follows.

$$W^s(x,\delta) = W^s(x,\delta,f) = \{y : d[f^n(x)],[f^n(y)] \leq \delta \text{ for } n \geq 0\}.$$

We explicitly denote the homeomorphism f only when it is necessary. We define the local unstable set of f at $x \in X$ for $\delta \geq 0$ by

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$W^u(x,\delta,f) = W^u(x,\delta) = W^s(x,\delta,f^{-1})$.

LEMMA 2.2. *For $x \in X$ and $\delta \geq 0$, the following statements are valid.*
(1) $fW^s(x,\delta)] \subset W^s[f(x),\delta]$.
(2) $f^{-1}[W^u(x,\delta)] \subset W^u[f^{-1}(x),\delta]$
(3) $W^s(x,\delta)$ *is a compact set.*
(4) $W^u(x,\delta)$ *is a compact set.*

LEMMA 2.3. *If $\delta < c/2$, then $|W^s(x,\delta) \cap W^u(y,\delta)| \leq 1$ for $(x,y) \in X \times X$.*

DEFINITION 2.4. We say f has canonical coordinates provided that for each $\delta > 0$ there exists $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $W^s(x,\delta) \cap W^u(y,\delta) \neq \emptyset$.

From now on, let δ and $\varepsilon(\delta)$ denote such positive numbers.
We note that if f has canonical coordinates, it follows from definitions that f^{-1} does, too.

REMARK 2.5. It follows from definitions and a uniform continuity argument that f has canonical coordinates if and only if f^m does for each integer m.

LEMMA 2.6. *If f has canonical coordinates and $\delta < c/2$ then $d(x,y) < \varepsilon$ implies $|W^s(x,\delta) \cap W^u(y,\delta)| = 1$.*
DEFINITION 2.7. If f has canonical coordinates, $\delta < c/2$ and $d(x,y) < \varepsilon$ we define the point $[x,y] \in X$ by the equation $[x,y] = W^s(x,\delta) \cap W^u(y,\delta)$.

LEMMA 2.8. *Suppose f has canonical coordinates, $\delta < c/2$ and $d(x,y) < \varepsilon$. The following statements hold.*
(1) $[x,x] = x$.
(2) *If $x \in W^u(y,\delta)$, then $[x,y] = x$.*
(3) *If $y \in W^s(x,\delta)$, then $[x,y] = y$.*

We adopt the notation $N(B,\nu)$ for the open neighborhood of radius $\nu > 0$ of $B \subset X$.

The proposition below is one of the reasons that the term "canonical coordinates" is a good one.

PROPOSITION 2.9. *Suppose f has canonical coordinates and $\varepsilon < \delta < c/2$. There is a positive number $\nu(\varepsilon,\delta,f)$ such that for each $p \in X$, the function $\Gamma_p : W^u(p,\varepsilon/2) \times W^s(p,\varepsilon/2) \to X$ defined by $\Gamma_p(x,y) = [x,y]$ is a homeomorphism onto a (compact) neighborhood of p containing the neighborhood $N(p,\nu)$.*

PROOF. Since f has canonical coordinates, $x \in W^u(p,\varepsilon/2)$ and $y \in W^s(p,\varepsilon/2)$ implies by the triangle inequality that $d(x,y) < \varepsilon$ so that $[x,y]$ is defined. Hence Γ_p is a function.

We now show that Γ_p is continuous. Suppose $x_m \to x$ in $W^u(p,\varepsilon/2)$ and $y_m \to y$ in $W^s(p,\varepsilon/2)$. Let v be a limit point of $\{\Gamma_p(x_m,y_m)\}$. Then there is a sequence $(x_{m(j)},y_{m(j)}) = v_j$ such that $v = \lim_{j\to\infty} v_j$. Since $\Gamma_p(v_j) = [x_{m(j)},y_{m(j)}] = W^s[x_{m(j)},\varepsilon/2] \cap W^u[y_{m(j)},\varepsilon/2]$, we have the following inequalities for $n \geq 0$, because $v_j \in W^s[x_{m(j)},\varepsilon/2]$ for $j \geq 0$.

$$d[f^n(v),f^n(x)] = \lim_{j\to\infty} d\{f^n(v_j),f^n[x_{m(j)}]\} \leq \varepsilon/2.$$

Hence $v \in W^s(x,\varepsilon/2)$ and similarly $v \in W^u(y,\varepsilon/2)$. Hence $v = [x,y] = \Gamma_p(x,y)$. Therefore Γ is continuous.

We show that Γ_p is $(1-1)$. Suppose $z = \Gamma(x,y) = \Gamma(x,w)$. Then $z \in W^s(x,\varepsilon/2)$ so $x \in W^s(z,\varepsilon/2)$ hence $x \in W^s(z,\varepsilon/2) \cap W^u(p,\varepsilon/2)$. Similarly, $v \in W^s(z,\varepsilon/2) \cap W^u(p,\varepsilon/2)$. Since $\varepsilon/2 < c/2$ it follows from Lemma 2.3 that $v = x$. Similarly, $y = w$. Therefore Γ_p is $(1-1)$.

Since the domain of Γ_p is compact and the range metric, Γ_p is a homeomorphism onto its image.

We show that $\Gamma_p[W^u(p,\varepsilon/2) \times W^s(p,\varepsilon/2)]$ is a uniformly large neighborhood of p. Choose $\nu > 0$ such that if $d(x,y) < \nu$ then $W^s(x,\varepsilon/2) \cap W^u(y,\varepsilon/2) \neq \emptyset$. If $d(z,p) < \nu$ we have (x,y) such that

$$x = W^s(z,\varepsilon/2) \cap W^u(p,\varepsilon/2) \text{ and}$$
$$y = W^s(p,\varepsilon/2) \cap W^u(z,\varepsilon/2).$$

Then $z = [x,y] = \Gamma_p(x,y) \in \Gamma[W^u(p,\varepsilon/2) \times W^s(p,\varepsilon/2)]$. That is $N(p,\nu) \subset \Gamma_p[W^u(p,\varepsilon/2) \times W^s(p,\varepsilon/2)]$.

3. Sources, sinks and saddles. In this section, f is an expansive homeomorphism of the compact metric (d) space X and f has canonical coordinates. Let $c > 0$ denote an expansive constant for f and let positive $2\varepsilon < \delta < c/2$ be the numbers guaranteed to exist for canonical coordinates.

DEFINITION 3.1. A point $x \in X$ is called a source for f if $W^s(x, a) = \{x\}$ for some positive α. It is called a sink for f if it is a source for f^{-1}. If it is neither a source nor a sink for f, it is called a saddle point for f.

LEMMA 3.2. *The following statements are equivalent.*

(1) *The point x is a source.*

(2) *There exists arbitrarily small positive α such that $W^u(x, \alpha)$ contains a neighborhood of x.*

(3) *There exist $\alpha > 0$ and a neighborhood V of x such that $W^s(x, \alpha) \cap V = \{x\}$.*

PROOF. Suppose (1). Let $\beta > 0$ be given. By (1) there exists a positive δ_0 such that $W^s(x, \delta_0) = \{x\}$. Choose $\delta = \min\{\delta_0, c/2\}$. Choose $2\rho = \varepsilon(\delta) < \min\{\beta, \delta\}$. Then $W^s(x, \rho) \subset W^s(x, \delta_0) = \{x\}$, hence $W^s(x, \rho) = \{x\}$. By proposition 2.8, $\Gamma_x[W^u(x, p) X W^s(x, p)]$ contains a neighborhood of $\{x\}$. We use the definition of Γ_x and Lemma 2.8 (2) to compute as follows.

$$\begin{aligned}\Gamma_x[W^u(x, \rho) \times W^s(x, \rho)] &= \Gamma_x[W^u(x, \rho) \times \{x\}] \\ &= \{[x, y] : y \varepsilon W^u(x, \rho)\} \\ &= \{y \varepsilon W^u(x, \rho)\} = W^u(x, \rho).\end{aligned}$$

Hence $W^u(x, \rho)$ contains a neighborhood of x and $\rho = \varepsilon/2 < \beta$.

Now suppose (2). Choose positive $\delta < c/2$ and $2\rho = \varepsilon(\delta) < \delta$ such that $W^u(x, \rho)$ contains a neighborhood of x. By Lemma 2.8 (1) we have the following.

$$\{x\} = W^s(x, \rho) \cap W^u(x, \rho) \supset W^s(x, \rho) \cap V \supset \{x\}.$$

This establishes (3).

Now suppose (3). Let V be the neighborhood of x and $\alpha > 0$ the number guaranteed to exist by (3) and choose positive $\gamma < \alpha$ such that the open neighborhood $N(x,\gamma)$ satisfies $N(x,\gamma) \subset V$. We have $W^s(x,\gamma/2) \subset N(x,\gamma)$ and hence $\{x\} \subset W^s(x,\gamma/2) \subset W^s(x,\gamma) \cap N(x,\gamma) \subset W^s(x,\alpha) \cap V = \{x\}$. Hence $W^s(x,\gamma/2) = \{x\}$. Then x is a source.

The following Lemma is a Corollary of Lemma 3.2, obtained by replacing f by f^{-1}.

LEMMA 3.3. *The following statements are equivalent.*

(1) *The point x is a sink.*

(2) *There exists arbitrarily small positive α such $W^s(x,\alpha)$ contains a neighborhood of x.*

(3) *There exist positive α and an open neighborhood V of x such that $W^u(x,\alpha) \cap V = \{x\}$.*

LEMMA 3.4. *The set of sources is open. The set of sinks is open.*

PROOF. Let x be an a source. Choose positive $\delta < c/2$ such that $W^u(x,\delta)$ contains a neighborhood of x. Then $\operatorname{Int} W^u(x,\delta)$ is a neighborhood of x. Choose $y \varepsilon \operatorname{Int} W^u(x,\delta)$. Let $z \neq y$ be in $\operatorname{Int} W^u(x,\delta)$. For $n \leq 0$, one has $d[f^n(y), f^n(z)] \leq d[f^n(y), f^n(x)] + d[f^n(x), f^n(z)] \leq 2\delta < c$. Therefore $z \varepsilon W^u(y, 2\delta)$. Since $z \neq y$, by Corollary 2.4 we have $z \notin W^s(y,2\delta)$. Let $V = \operatorname{Int} W^u(x,\delta)$. We have $\{y\} = W^s(y,2\delta) \cap V$ and hence by Lemma 3.2 y is a source. Thus $\operatorname{Int} W^u(x,\delta)$ is an open set of sources containing x. Therefore, the set of sources is open.

To see that the set of sinks is open, we rewrite the proof replacing f by f^{-1}.

LEMMA 3.5. *Let X be locally connected. Then the set of saddle points is open.*

PROOF. Let x be a mixed point. Choose $0 < \rho = \varepsilon/2 < \varepsilon < \delta < c/2$ as in Proposition 2.9. Since Γ_x is continuous, we may choose a positive $\eta < \varepsilon$ such that $d(p,q) < \eta$ implies $d([p,y],[q,y]) < \varepsilon$ for all $y \in W^s(x,p)$ and $d([f,p],[y,q]) < \varepsilon$ for all $y \in W^u(x,\rho)$. Further, by Proposition 2.9 we know that there exists a positive ν

such that $N(x,\nu) \subseteq \Gamma_x[W^u(x,\eta/2) \times W^s(x,\eta/2)]$. Since $N(x,\nu)$ is homeomorphic (via Γ_x) to an open subset of a cartesian product, there exists a product space rectangle $R_1 \times R_2 \subset R \subset N(x,\nu)$ such that R is open in $N(x,\nu), R_1$ is an open neighborhood of x in $W^u(x,\eta/2), R_2$ is an open neighborhood of x in $W^s(x,\eta/2)$, and $R = \Gamma_x(R_1 \times R_2) \subset N(x,\nu)$. Each coordinate projection $\pi_j : R \to R_j$ is an open mapping, and hence each R_j is the continuous image of the locally connected space R via an opening mapping π_j. Consequently, each R_j is a locally connected space. It follows that there exists connected relatively open subsets $A \subset R_1 \subset W^u(x,\eta/2)$ and $B \subset R_2 \subset W^s(x,\eta/2)$ such that $V = \Gamma_x(A \times B) \subset R_1 \times R_2 = R \subset N(x,\nu)$ with V open in X. Now consider points $v \in V$. If $v \in W^u(x,\rho) \cap V$, we have by lemma 2.8 (2), $v = [v,x] = \Gamma_x(v,w) \in V = \Gamma_x(A \times B)$, so that $v \varepsilon A \subset W^u(x,\eta/2)$. It follows that $W^u(x,\rho) \cap V = W^u(x,\eta/2) \cap V = A$. Since x is not a sink, it follows from lemma 3.3 that A contains points not equal to x. Similarly, the connected set B contains points not equal to x. Choose a point $(a,b) \in A \times B$. Since A and B are connected and not singletons, we can find sequences a_m in $A - \{a\}$ and b_m in $B - \{b\}$ such that $a_m \to a$ and $b_m \to b$. Then for $m \geq 0$ and $n \leq 0$, we have

$$\begin{aligned} d[f^n(a_m), f^n(a)] &\leq d[f^n(a_m), f^n(x)] + d[f^n(x), f^n(a)] \\ &\leq \eta/2 + \eta/2 = \eta. \end{aligned}$$

Hence, by the choice of $\eta, [a_n, b] \in W^u[a,b],\varepsilon)$ for every n. Suppose that $W^s([a,b],\varepsilon)$ contains a neighborhood of $[a,b]$. Then for sufficiently large m we have $[a_m,b] \in W^u([a,b],\varepsilon) \cap W^s([a,b],\varepsilon)$. Since $\varepsilon < c/2$, this implies $[a_m,b] = [a,b]$ and since Γ_x is $(1-1)$, we infer $a_m = a$ contrary to the choice of a_m. Hence $W^s([a,b],\varepsilon)$ contains no neighborhood of $[a,b]$ and by Lemma 3.3, $[a,b]$ is not a sink.

Similarly $[a,b]$ is not a source.

Hence $\Gamma(A \times B) = V$ consists of saddle points and V is a neighborhood of x. Hence the set of mixed points is open.

REMARK 3.6. The hypothesis of local connectedness in Lemma 3.5 was used to ensure that A and B did not contain isolated points. Other, less natural, hypotheses will also work.

THEOREM 3.7. *Given X such that X is compact, connected and locally connected, let $f : X \to X$ be an expansive homeomorphism such* *that f has canonical coordinates. Then either X consists of just one point, or else every point of X is a saddle point.*

PROOF. Suppose there are at least two points in X. Given $z \in X$, suppose z is both a source and a sink. Then lemmas 3.2 and 3.3 show that there exists $\delta(z) > 0$ such that $\delta(z) < c/2$ and $W^s(z, \delta(z)) \cap W^u(z, \delta(z))$ is a neighborhood of z. Lemma 2.3 then shows that z is an isolated point in X. Since X is connected and has more than one point, there is no such z. Consequently, we now deduce from 3.4 and 3.5 that the decomposition of X into sources, sinks and saddles points is a partitioning of X into 3 disjoint open subsets. We are reduced to proving that X cannot consist entirely of sources or sinks. Suppose X consists entirely of sources. By Lemma 3.2 (2) each point x in X has a neighborhood $\operatorname{Int} W^u[x, \delta(x)]$, and we may assume each $\delta(x) < c/2$. Cover X with a finite collection of these, $\Omega = \{\operatorname{Int} W^u(x_1) : i = 1, \cdots, m\}$. By [**6**, Theorem 10.36], there is a positively asymptotic pair $\{y, z\}$. Replacing y and z with $f^m(y)$ and $f^m(z)$ if necessary, we may assume without loss of generality that $d[f^n(y), f^n(z)] < \min\{\nu, c\}$ for $n \geq 0$, where ν is a Lebesgue number for Ω. Hence, omitting subscripts, we find x such that $\{y, z\} \subset W^u(x, \delta)$ where $\delta < c/2$. Thus for $n \leq 0 d[f^n(y), f^n(z) \leq d[f^n(y), f^n(x)] + d[f^n(z)] < \delta + \delta < c$. Hence, $d[f^n(y), f^n(z)] < c$ for all integers n, contrary to the choice of c as an expansive constant for f. So there are no sources. Similarly, there are no sinks. Thus X consists of saddle points.

4. An example. In this section we construct an example of an expensive homeomorphism f on a connected but not locally connected compact metric space for which both sources and saddles exist. The details are ugly, available from the authors and omitted.

Let $g : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear homeomorphism of the plane whose matrix is $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let V be an eigenvector for the larger eigenvalue $\lambda(> 1)$ of M. In $\mathbf{R} \times \mathbf{R} \times \mathbf{I}$, let $H = \{(\alpha v, 1/(1 + \alpha) : \alpha \geq 0\}$, and let $g(\alpha v, 1/1(1+\alpha)) = (\lambda \alpha v, 1/(1 + \lambda\alpha))$. Thus g defines a homeomorphism of $R^2 \cup H$ to itself where H is a curve in $\{\lambda v : \lambda \geq 0\} \times \mathbf{I}$ which is asymptotic to the $\{\lambda v : \lambda \geq 0\}$ halfline and on which g commutes with the restriction of the projection onto R^2. We factor out the integer lattice $\mathbf{Z} \times \mathbf{Z}$ in R^2 to get a homeomorphism $f : T^2 \cup K \to T^2 \cup K$ where T^2 denotes the 2–torus. Now K is a curve in $T^2 \times \mathbf{I}$ which is asymptotic to a certain dense halfline in the torus.

The space $T^2 \cup K$ is obviously a connected compact metric space. It is not locally connected since each open set which intersects T^2 contains a countable set of disjoint intervals in K.

The homeomorphism f is expansive as we see by considering cases. First, $f|T$ is known to be expansive. If $x \in T$ and $y \in K$, then $f^n(x) \in T^2, n \leq 0$, and $f^n(y) \to p$ (the projection of $(0,0,1)$) as $n \to -\infty$, and $d(p,T) > 0$ in any metric, d. Finally, if $x \neq y \subset K$ and $f^n(x)$ is close to $f^n(y)$ for all $n \geq 0$, we may find $N > 0$ such that $f^n(z)$ is close to $f^n(w)$ for $n \geq N$, where z and w are the projections of x and y onto T^2. Since g is positively expansive on $\{\lambda v : \lambda \geq 0\}$, f is positively expansive on its projection, contrary to the behavior of $f^N(z)$ and $f^N(w)$. So f is expansive.

Clearly, each point in T^2 is a saddle and since $f^n(y) \to p$ for each $y \in K$, one can easily show that K consists of sources.

It is easy to verify that $f|K$ and $f|T$ have canonical coordinates. Thus, to see that f has canonical coordinates, let $x \in T^2$ and $y \in K$. If x and y are sufficiently close , x and w (the projection of y on T^2) are close enough so that $W^s(w,\varepsilon/2) \cap W^u(x,\varepsilon/2) \neq \emptyset$ and $y \in W^s(w,\varepsilon/2)$. Hence $W^s(y,\varepsilon) \cap W^u(x,\varepsilon) \neq \emptyset$. Also, if x and y are sufficiently close then x and w (the projection of y on T^2) are close enough so that $W^s(x,\varepsilon/2) \cap W^u(w,\varepsilon/2) = \emptyset$. Let $p = [x,w]$ and let $z \in K$ be the point that projects to p. If x and y are sufficiently close then $z \in W^s(p,\varepsilon/2) \cap W^u(y,\varepsilon)$; hence $z \in W^s(x,\varepsilon) \cap W^u(y,\varepsilon) \neq \emptyset$. So f has canonical coordinates.

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