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SOME JESSEN–BECKENBACH INEQUALITIES

JOSIP E. PEČARIĆ AND PAUL R. BEESACK

1. Introduction. In 1966 E.F. Beckenback [**1**] (see also [**4**, p.52] or [**5**, p.81] proved the following generalization of Hölder's inequality:

Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ *be two n–tuples of positive real numbers, and* p, q *be real numbers such that* $p^{-1} + q^{-1} = 1 (p > 1)$. *If* $0 < m < n$, *then*

$$(1) \qquad \Big(\sum_1^n a_i^p\Big)^{1/p}\Big(\sum_1^n a_i b_i\Big)^{-1} \geq \Big(\sum_1^n \tilde{a}_i^p\Big)^{1/p}\Big(\sum_1^n \tilde{a}_i b_i\Big)^{-1},$$

where

$$\tilde{a}_i = a_i (1 \leq i \leq m), \quad \tilde{a}_i = \Big\{ b_i \sum_{j=1}^m a_j^p \Big/ \sum_{j=1}^m a_j b_j \Big\}^{q/p} \quad (m+1 \leq i \leq n).$$

Equality holds in (1) *if and only if* $\tilde{a}_i \equiv a_i$. *The inequality in* (1) *is reversed if* $p < 1, p \neq 0$. *For* $m = 1$, (1) *reduces to Hölder's inequality.*

In this paper we shall give some generalizations of this result with $\sum$ replaced by an isotonic linear functional. See especially Corollary 3, and Remark 4, below.

2. Main results. Let E be a nonempty set, let $\mathcal{A}$ be an algebra of subsets of E, and let L be a linear class of real–valued functions $g : E \to \mathbf{R}$ having the properties

L1: $f, g \in L \Rightarrow (af + bg) \in L$ *for all* $a, b \in \mathbf{R}$;
L2: $1 \in L$, *that is if* $f(t) = 1$ *for* $t \in E$, *then* $f \in L$;
L3: $f \in L, E_1 \in \mathcal{A} \Rightarrow fC_{E_1} \in L$,

where C_{E_1} is the characteristic function of $E_1 (C_{E_1}(t) = 1$ for $t \in E_1$, or 0 if $t \in E \backslash E_1)$. It follows from L2, L3 that $C_{E_1} \in L$ for all $E_1 \in \mathcal{A}$. Also note that L contains all constant functions by L1, L2.

We also consider isotonic linear functionals $A : L \to \mathbf{R}$. That is, we

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suppose:

A1: $A(af + bg) = aA(f) + bA(g)$ *for* $f, g \in L, a, b \in \mathbf{R}$;

A2: $f \in L, f(t) \geq 0$ *on* $E \Rightarrow A(f) \geq 0$ *(A is isotonic).*

Our main tool will be the following well-known result (see [**2**] for example).

Jessen's Inequality. Let L *satisfy properties* L1, L2 *on a nonempty set* E*, and suppose* ϕ *is a convex function on an interval* $I \subset \mathbf{R}$*. If* A *is an isotonic linear functional with* $A(1) = 1$ *then, for all* $g \in L$ *such that* $\phi(g) \in L$ *we have* $A(g) \in I$ *and*

$$\phi(A(g)) \leq A(\phi(g)). \tag{2}$$

We shall also make use of the fact that if L also satisfies L3, then for each $E_1 \in \mathcal{A}$ such that $A(C_{E_1}) > 0$, the functional A_1 defined for all $g \in L$ by $A_1(g) = A(gC_{E_1})/A(C_{E_1})$ is an isotonic linear functional with $A_1(1) = 1$. (See also Lemma 4(1′) of [**2**].)

THEOREM 1. *Let* L *satisfy properties* L1, L2, L3 *on a nonempty set* E*, and suppose* ϕ *is convex on a closed interval* $I \subset \mathbf{R}$*. Let* A *be an isotonic linear functional with* $A(1) = 1$*, and let* J *be an interval such that* $\phi(I) \subset J$*, and* $F : J^2 \to \mathbf{R}$ *be a nondecreasing function of its first variable. Given* $E_1 \in \mathcal{A}$ *such that* $A(C_{E \setminus E_1}) > 0$*, then for any* $g \in L$ *such that* $\phi(g) \in L$ *we have*

$$F[A(\phi(g)), \phi(A(g))] \geq \inf_{x \in I} F[A(\phi(g_{E_1,x})), \phi(A(g_{E_1,x}))], \tag{3}$$

where

$$g_{E_1,x}(t) = g(t)C_{E_1}(t) + xC_{E \setminus E_1}(t).$$

PROOF. For brevity, set $E_2 = E \setminus E_1$; we are assuming $A(C_{E_2}) > 0$. We clearly have both

$$g = gC_{E_1} + gC_{E_2}, \phi(g) = \phi(g)C_{E_1} + \phi(g)C_{E_2}.$$

Also,

$$\phi(A(g)) = \phi(A(gC_{E_1}) + A(gC_{E_2})) = \phi(A(gC_{E_1}) + \sigma z),$$

where

$$\sigma = A(C_{E_2}), z = A(gC_{E_2})/A(C_{E_2}).$$

In addition,

$$A(\phi(g)) = A(\phi(g)C_{E_1} + \phi(g)C_{E_2}) = A(\phi(g)C_{E_1}) + A(\phi(g)C_{E_2}) \\ \geq A(\phi(g)C_{E_1}) + \sigma\phi(z),$$

on using the remark following (2), but with E_1 replaced by E_2.

Now, $z \in I$ because if $I = [\alpha, \beta]$ then $\alpha \leq g(t) \leq \beta$ for $t \in E$ since $\phi(g)$ is in L(hence is defined). Thus $\alpha C_{E_2}(t) \leq g(t)C_{E_2}(t) \leq \beta C_{E_2}(t)$ for all $t \in E$, whence $\alpha A(C_{E_2}) \leq A(gC_{E_2}) \leq \beta A(C_{E_2})$ so $\alpha \leq z \leq \beta$. A simple modification shows that $z \in I$ if either $\alpha = -\infty$ or $\beta = +\infty$. It now follows from the above inequality and the nondecreasing character of $F(., y)$ that

$$\begin{aligned} F[A(\phi(g)), \phi(A(g))] &\geq F[A(\phi(g)C_{E_1}) + \sigma\phi(z), \phi(A(gC_{E_1}) + \sigma z)] \\ &\geq \inf_{x\in I} F[A(\phi(g)C_{E_1}) + \sigma\phi(x), \phi(A(gC_{E_1}) + \sigma x)] \\ &= \inf_{x\in I} F[A(\phi(g_{E_1}, x)), \phi(A(g_{E_1}, x))] \end{aligned}$$

since

$$A(g_{E_1}, x) = A(gC_{E_1}) + xA(C_{E_2}) = A(gC_{E_1}) + \sigma x,$$
$$\phi(g_{E_1}, x)(t) = \phi(g(t)C_{E_1}(t) + xC_{E_2}(t)) = \phi(g(t))C_{E_1}(t) + \phi(x)C_{E_2}(t).$$

REMARK 1. There are clearly many variations and generalizations of Theorem 1 which have essentially the same proof. For example, if $F(., y)$ is nonincreasing for each $y \in J$, then in place of (3) we have

$$(3') \qquad F[A(\phi(g)), \phi(A(g))] \leq \sup_{x\in I} F[A(\phi(g_{E_1,x})), \phi(A(g_{E_1,x}))].$$

This also follows from (3) just by replacing F there by $F_1 = -F$.

For another, more extensive, generalization suppose $E_1 \in \mathcal{A}$ for $1 \leq i \leq n$ with $E_i \bigcap E_j = \emptyset (i \neq j)$ and $E = \bigcup_1^n E_i$. By setting $\sigma_i = A(C_{E_1}), z_i = A(gC_{E_1})/A(C_{E_i})$ (where we assume all $\sigma_i > 0$), we find under the hypotheses of Theorem 1 that if $x = (x_1, \ldots, x_n)$,

$$(4) \qquad F[A(\phi(g)), \phi(A(g))] \geq \inf_{x\in I^n} F[\sum_1^n \sigma_i\phi(x_i), \phi(\sum_1^n \sigma_i x_i)].$$

If we set

$$g_{E_1,\dots E_n,x}(t) = \sum_{i=1}^{n} x_i C_{E_i}(t),$$

the right–hand side of (4) can be written as

$$\inf_{x\in I^n} F[A(\phi(g_{E_1,\dots,E_n,x})), \phi(A(g_{E_1,\dots,E_n,x}))].$$

Note, however, that this value is independent of the function g and so provides a lower bound for the left–hand side of (4) which is valid for *all* admissible $g \in L$. Similarly, if $F(.,y)$ is nonincreasing, then instead of (4) we have

$$(4') \qquad F[A(\phi(g)), \phi(A(g))] \leq \sup_{x\in I^n} F[\sum_1^n \sigma_i\phi(x_i), \phi(\sum_1^n \sigma_i x_i)].$$

There are other variations which are intermediate between (3) and (3′). For example, if $1 \leq m < n$, and we set $\tilde{x}_m = (x_{m+1}, \dots, x_n)$ and

$$g_{E_1,\dots,E_n,\tilde{x}_m}(t) = \sum_{i=1}^{m} g(t)C_{E_i}(t) + \sum_{j=m+1}^{n} x_j C_{E_j}(t),$$

then we can prove under the hypotheses of Theorem 1 that

$$(5) \qquad \begin{aligned} F[A(\phi(g)), \phi(A(g))] \leq \sup_{\tilde{x}_m\in I^{n-m}} & F[A(\phi(g_{E_1,\dots,E_n,\tilde{x}_m})), \\ & \phi(A(g_{E_1,\dots,E_n,\tilde{x}_m}))]. \end{aligned}$$

Finally, we observe that in (3), (4) or (5) the lower bounds on the right–hand sides depend on the subsets $E_i \subset E$ (with $E_i \in \mathcal{A}$), and a possible larger bound (hence a better result) might be obtained by allowing the sets E_i to vary. For example, by (3) we have

$$(6) \quad F[A(\phi(g)), \phi(A(g))] \geq \sup_{E_1\in\mathcal{A}_1} \left\{ \inf_{x\in I} F[A(\phi(g_{E_1,x})), \phi(A(g_{E_1,x}))] \right\},$$

where $\mathcal{A}_1 = \{E_1 \in \mathcal{A} : A(C_{E\setminus E_1} > 0\}$.

We now give an upper bound for $F[A(\phi(g)), \phi(A(g))]$ which, unlike that in (3′), holds under the same hypotheses on F as in Theorem 1.

THEOREM 2. *Let all the conditions of Theorem 1 be satisfied, but with $I = [m, M]$ a compact interval, so $m \le g(t) \le m$ for all $t \in E$. Then if $\sigma = A(C_{E\setminus E_1}) > 0$,*

$$\begin{aligned}(7)\qquad & F[A\phi(g)), \phi(A(g))] \le \sup_{0\le\theta\le\sigma} F[A(\phi(g)C_{E_1}) \\ & + \theta\phi(m) + (\sigma-\theta)\phi(M), \phi(A(gC_{E_1}) + \theta m + (\sigma-\theta)M)].\end{aligned}$$

PROOF. As in the proof of Theorem 1 we set $E_2 = E\setminus E_1$. Now let $d(t) = (M - g(t))/(M - m)$, so $g(t) = md(t) + M(1 - d(t))$, and set $\beta = A(dC_{E_2})$. Then

$$\begin{aligned}\phi(A(g)) &= \phi(A(gC_{E_1}) + A[(md + M(1-d))C_{E_2}]) \\ &= \phi(A(gC_{E_1}) + m\beta + M(\sigma - \beta)).\end{aligned}$$

Also, using the convexity of ϕ on I,

$$\begin{aligned}A(\phi(g)) &= A[\phi(g)C_{E_1} + \phi(g)C_{E_2}] \\ &= A[\phi(g)C_{E_1} + \phi(md + M(1-d))C_{E_2}] \\ &\le A[\phi(g)C_{E_1} + \{d\phi(m) + (1-d)\phi(M)\}C_{E_2}] \\ &= A(\phi(g)C_{E_1} + \phi(m)A(dC_{E_2}) + \phi(M)A((1-d)C_{E_2}) \\ &= A(\phi(g)C_{E_1}) + \beta\phi(m) + (\sigma-\beta)\phi(M).\end{aligned}$$

Since $0 \le d(t) \le 1$, we have $0 \le \beta = A(dC_{E_2}) \le A(C_{E_2}) = \sigma$. The result (7) now follows from this, the nondecreasing character of $F(., y)$, and the last two displayed results.

COROLLARY 1. *If $F(., y)$ is nonincreasing on J for each $y \in J$, but all other conditions of Theorem 2 are satisfied, then*

$$\begin{aligned}(7')\qquad & F[A(\phi(g)), \phi(A(g))] \ge \inf_{0\le\theta\le\sigma} F[A(\phi(g)C_{E_1}) \\ & + \theta\phi(m) + (\sigma-\theta)\phi(M), \phi(A(gC_{E_1}) + \theta m + (\sigma-\theta)M],\end{aligned}$$

where $\sigma = A(C_{E\setminus E_1})$

This follows by applying (7) to the function $F_1 = -F$.

We note that the special case $E_1 = \emptyset$ of Theorem 2 was proved as Theorem 1 in [**6**]. If we note that $\sigma = \sigma_{E_1}$ and denote the right-hand side of (7) by $H(E_1)$, we obtain the best (least) upper bound for $F[A(\phi(g)),\phi(A(g))]$ under the hypotheses of Theorem 2 as

$$F[A(\phi(g)),\phi(A(g))] \le \inf_{E_1\in \mathcal{A}_1} H(E_1),$$

where $\mathcal{A}_1 = \{E_1 \in \mathcal{A} : A(C_{E\setminus E_1}) = \sigma_{E_1} > 0\}$.

A generalization of Jessen's inequlaity for convex functions of several variables was given in 1937 by E.J. McShane [**3**].

McShane's Inequality. Let ϕ *be a convex function on a closed, convex set* $U \subset \mathbf{R}^n$. *Let* L *satisfy properties* L1, L2 *on a nonempty set* E, *and let* $A : L \to \mathbf{R}$ *be an isotonic linear functional with* $A(1) = 1$. *Set* $\tilde{L} = \{G = (g_1,\dots,g_n) : g_i \in L \text{ for } 1 \le i \le n\}$, *and define* $\underline{A} : \tilde{L} \to \mathbf{R}^n$ *by* $\underline{A}(G) = (A(g_1),\dots,A(g_n))$. *Then* $\underline{A}$ *is a linear operator on the linear class* $\tilde{L}$. *For any* $G \in \tilde{L}$ *for which* $\phi(G) \in L$ *we have* $\underline{A}(G) \in U$, *and*

$$\phi(A(G)) \le A(\phi(G)). \tag{8}$$

THEOREM 3. *Let* L *satisfy properties* L1, L2, L3 *on a nonempty set* E, *and let* ϕ, $A, \underline{A}, G$ *be as in McShane's Inequality, and* J *be an interval such that* $\phi(U) \subset J$ *and* $F : J^2 \to \mathbf{R}$ *be a nondecreasing function of its first variable. Given* $E_1 \in \mathcal{A}$ *such that* $A(C_{E\setminus E_1}) > 0$ *then, for any* $G \in \tilde{L}$ *such that* $\phi(G) \in L$, *we have*

$$F[A(\phi(G)),\phi(\underline{A}(G))] \ge \inf_{\underline{x}\in U} F[A(\phi(G_{E_1,\underline{x}})),\phi(\underline{A}(G_{E_1,\underline{x}}))],$$

where $G_{E_1,\underline{x}}(t) = G(t)C_{E_1}(t) + \underline{x}C_{E\setminus E_1}(t)$.

PROOF. The proof is similar to the proof of Theorem 1, and we merely outline the differences. Set $E_2 = E\setminus E_1$ and $\sigma = A(C_{E_2})$. Then

$$\phi(\underline{A}(G)) = \phi(\underline{A}(GC_{E_1}) + \sigma\underline{z}),\ \underline{z} = \underline{A}(GC_{E_2})/A(C_{E_2}),$$
$$A(\phi(G)) = A(\phi(G)C_{E_1}) + A(\phi(G)C_{E_2}) \ge A(\phi(G)C_{E_1}) + \sigma\phi(\underline{z}),$$

since $A_1(g) = A(gC_{E_2})/A(C_{E_2})$ is an isotonic linear functional on L with $A_1(1) = 1$; hence McShane's inequality (8) applies to the operator $\underline{A}_1 : \tilde{L} \to \mathbf{R}^n$ defined by $\underline{A}_1(G) = (A_1(g_1)\dots,A_1(g_n)) = \underline{A}(GC_{E_2})/A(C_{E_2})$. Moreover, by McShane's result, we have $\underline{z} =$

$\underline{A}_1(G) \in U$. The rest of the proof remains unchanged except for notation.

3. Some applications. First we shall give four applications of Theorem 1.

COROLLARY 2. *Let L satisfy properties* L1, L2, L3 *on a nonempty set E and let A be an isotonic functional on L. Suppose $E_1 \in \mathcal{A}$ has $A(C_{E_2}) > 0$, where $E_2 = E \backslash E_1$. Then for each nonnegative $g \in L$ such that $g^p \in L (p > 1)$ and $A(gC_{E_1}) > 0$ we have*

$$A(g^p)^{1/p}/A(g) \geq A(g_{E_1}^p)^{1/p}/A(g_{E_1}), \tag{9}$$

where

$$g_{E_1}(t) = g(t)C_{E_1}(t) + \{A(g^pC_{E_1})/A(gC_{E_1})\}^{1/(p-1)} \cdot C_{E_2}(t).$$

PROOF. First observe that $A(g) \geq A(gC_{E_1}) > 0$ and $A(g_{E_1}) > A(gC_{E_1}) > 0$, so both sides of (9) are well-defined. Apply Theorem 1 with A replaced by $A_1(g) = A(g)/A(1), F(x, y) = x^{1/p}/y^{1/p}, \phi(x) = x^p$, with $I = J = [0, \infty)$. Then (3) reduces to

$$A(g^p)^{1/p}/A(g) \geq \inf_{x \in I} A(g_{E_1,x}^p)^{1/p}/A(g_{E_1,x}), \tag{10}$$

where

$$g_{E_1,x}(t) = g(t)C_{E_1}(t) + xC_{E_2}(t).$$

Hence

$$g_{E_1,x}^p(t) = g^p(t)C_{E_1}(t) + x^pC_{E_1}(t).$$

By elementary calculus one finds that the minimum value of

$$k(x) = \{A(g^pC_{E_1}) + x^pA(C_{E_2})\}^{1/p}/\{A(gC_{E_1}) + xA(C_{E_2})\}$$

for $x \geq 0$ occurs for $x = \{A(g^pC_{E_1})/A(gC_{E_1})\}^{1/(p-1)}$, whence (9) follows from (10).

REMARK 2. As an example of (4) of Remark 1 for the case of Corollary 2, we take $n = 2$. Under the additional assumption that $\sigma_1 = A_1(C_{E_1}) > 0$ (with $A_1 = A/A(1)$), (4) reduces to

$$A(g^p)^{1/p}/A(g) \geq \inf_{x_1,x_2 \geq 0} \frac{\{A(C_{E_1})x_1^p + A(C_{E_2})x_2^p\}^{1/p}}{A(C_{E_1})x_1 + A(C_{E_2})x_2}.$$

For $x_1 = 0$, the term on the right–hand side has the value $A(C_{E_2})^{-1/q}$, where $q^{-1} + p^{-1} = 1$. For $x_1 > 0$, by setting $x = x_2/x_1$, we are concerned with

$$\inf_{x\geq 0}\{A(C_{E_1}) + A(C_{E_2})x^p\}^{1/p}\{A(C_{E_1}) + A(C_{E_2})x\}.$$

By a comparison with $k(x)$ above, this infimum is attained for $x = 1$, and has the value $A(1)^{1/p}/A(1) = A(1)^{-1/q}$. Since $A(1) \geq A(C_{E_2})$ we can conclude that

$$A(g^p)^{1/p}/A(g) \geq A(1)^{-1/q},$$

or

$$A(g) \leq A(g^p)^{1/p} \cdot A(1)^{1/q}.$$

This is, of course, just a special case of the generalized Hölder inequality given in [**2**;Th. 7].

COROLLARY 3. *Let L satisfy properties* L1, L2, L3 *on a nonempty set E, and let A be an isotonic linear functional on L. Suppose the nonnegative functions $f, g : E \to \mathbf{R}$ are such that $f^p, g^q, fg \in L$, where $p > 1, p^{-1} + q^{-1} = 1$. Suppose also that $E_1 \in \mathcal{A}$ has $A(fgC_{E_1}) > 0$ and $A(g^qC_{E_2}) > 0$ where $E_2 = E\backslash E_1$. Then*

$$A(f^p)^{1/p}/A(fg) \geq A(\tilde{f}^p_{E_1})^{1/p}/A(\tilde{f}g). \tag{11}$$

where

$$\tilde{f}_{E_1}(t) = f(t)C_{E_1}(t) + \{g(t)A(f^pC_{E_1})/A(fgC_{E_1})\}^{q/p} \cdot C_{E_2}(t).$$

PROOF. We shall apply Corollary 2 to the functional $A_1(g_1)$ defined, for certain $g_1 : E \to \mathbf{R}$ by $A_1(g_1) = A(kg_1)/A(k)$, with $k = g^q \in L$. We have $A(k) \geq A(g^qC_{E_2}) > 0$. By Lemma 4 (1′) of [**2**], with $\phi(u) = u^p$ convex on $I = [0, \infty]$, we have

$$\{A(g^qg_1)/A(g^q)\}^p \leq A(g^qg_1^p)/A(g^q)$$

for all functions $g_1 : E \to \mathbf{R}$ for which $g^qg_1 \in L$ and $g^qg_1^p \in L$. We note that this is precisely the inequality corresponding to (2) for the functional $A_1(g_1)$ and $\phi(u) = u^p$, and this in turn implies the validity of Theorem 1, hence also of Corollary 2 for A_1. We may thus apply

Corollary 2 with the function g replaced by $g_1 = fg^{-q/p}$ since we do have $A_1(g_1 C_{E_1}) > 0$ and $A_1(C_{E_2}) > 0$ as required

Now $g^q g_1 = fg$ and $g^q g_1^p = f^p$, so

$$A_1(g_1^p) = A(f^p)/A(g^q), \ A_1(g_1) = A(fg)/A(g^q).$$

It is easy to verify that (9), with A, g replaced by A_1, g, reduces to

$$A(f^p)^{1/p}/A(fg) \geq A(g^q \tilde{g}_{E_1}^p)^{1/p}/A(g^q \tilde{g}_{E_1}), \tag{12}$$

where

$$\tilde{g}_{E_1}(t) = g_1(t)C_{E_1}(t) + \{A(f^p C_{E_1})/A(fgC_{E_1})\}^{1/(p-1)} \cdot C_{E_2}(t).$$

Hence, using the fact that $1/(p-1) = q-1 = q/p$, we find that

$$g^q \tilde{g}_{E_1} = g\{fC_{E_1} + [gA(f^p C_{E_1}/A(fgC_{E_1})]^{q/p} C_{E_2}\} = g\tilde{f}_{E_1},$$
$$g^q \tilde{g}_{E_1}^p = f^p C_{E_1} + [gA(f^p C_{E_1})/A(fgC_{E_1})]^q C_{E_2} = \tilde{f}_{E_1}^p,$$

so (11) follows from (12).

REMARK 3. As in Remark 2, the inequality (4) for the case $n = 2$, reduces in this case to

$$A(f^p)^{1/p}/A(fg) \geq \inf_{x_1, x_2 \geq 0} A(g^q \tilde{g}_{E_1,E_2,x})^{1/p}/A(g^q \tilde{g}_{E_1,E_2,x}),$$

with

$$\tilde{g}_{E_1,E_2,x} = x_1 C_{E_1} + x_2 C_{E_2}.$$

Again, the infimum is attained for $x_1 = x_2$, and now has the value $A(g^q)^{1/p}/A(g^q) = A(g^q)^{-1/q}$. The inequality thus reduces to the generalized Hölder inequality

$$A(fg) \leq A(f^p)^{1/p} \cdot A(g^q)^{1/q}.$$

REMARK 4. Beckenback's inequality (1) is the special case of Corollary 3 corresponding to the choice $E = \{1, 2, \ldots, n\}, E_1 = \{1, 2, \ldots, m\}$ (where $1 \leq m < n$), $L = \mathbf{R}^n$, the vector space of all real n–vectors $a = (a_1, \ldots, a_n)$, and $A(a) = \sum_1^n a_i$.

COROLLARY 4. *Let the conditions of Corollary 2 be satisfied, except that now* $A(gC_{E_1}) = 0$ *may hold. Given* $p > 1, q = p/(p-1)$, *and* $\beta \geq 0$ *such that* $A(C_{E_2})\beta^q < 1$ *we have*

$$(13)\qquad A(g^p)^{1/p} - \beta A(g) \geq A(\tilde{g}^p_{E_1,\beta})^{1/p} - \beta A(\tilde{g}_{E_1,\beta}),$$

where

$$\tilde{g}_{E_1,\beta} = GC_{E_1} + x_\beta C_{E_2} \text{ with } x_\beta = \{\beta^q A(g^p C_{E_1})/[1 - \beta^q A(C_{E_2})]\}^{1/p}.$$

The right–hand side of (13) *equals* $A(g^p C_{E_1})^{1/p}[1 - \beta^q A(C_{E_2})]^{1/q} - \beta A(gC_{E_1})$.

PROOF. We apply Theorem 1 to the isotonic linear functional $A_1(g) = A(g)/A(1)$, with $F(x,y) = x^{1/p} - \beta A(1)^{1/q} y^{1/p}, \phi(x) = x^p, I = J = [0,\infty)$. The inequality (3) reduces to

$$(14)\qquad A(g^p)^{1/p} - \beta A(g) \geq \inf_{x \geq 0}\{A(g^p_{E_1,x})^{1/p} - \beta A(g_{E_1,x})\},$$

where $g_{E_1,x} = gC_{E_1} + xC_{E_2}$, so $g^p_{E_1,x} = g^p C_{E_1} + x^p C_{E_2}$. The expression in curly brackets is

$$K(x) = \{A(g^p C_{E_1}) + x^p A(C_{E_2})\}^{1/p} - \beta\{A(gC_{E_1}) + xA(C_{E_2})\}.$$

By elementary calculus, in case $0 \leq \beta < A(C_{E_2})^{-1/q}$, we find that the minimum value of $K(x)$ for $x \geq 0$ occurs for $x = x_\beta$, proving (13). This minimum value reduces, after some computation, to that stated in the final sentence of the Corollary.

By proceeding as in Remark 2 (using (4) with $n = 2$), we also find that

$$A(g^p)^{1/p} - \beta A(g) \geq 0 \text{ if } \beta^q A(1) \leq 1,$$
$$A(g^p)^{1/p} - \beta A(g) \geq A(C_{E_1})^{1/p}\{[1 - \beta^q A(C_{E_2})]^{1/q} - \beta A(C_{E_1})^{1/q}\}(< 0)$$

if $\beta^q A(C_{E_2}) < 1 \leq \beta^q A(1)$. A noted following (4) the above lower bounds are valid for all $g \in L$ satisfying the corresponding hypotheses.

COROLLARY 5. *Let the conditions of Corollary 3 be satisfied except that now* $A(fgC_{E_1}) = 0$ *may hold. If* $0 < \beta^q A(g^q C_{E_2}) < A(g^q)$, *then*

$$(15)\qquad A(f^p)^{1/p} A(g^q)^{1/q} - \beta A(fg) \geq A(\tilde{f}^p_{E_1,\beta})^{1/p} A(g^q)^{1/q} - \beta A(g\tilde{f}_{E_1,\beta}),$$

where

$$\tilde{f}_{E_1,\beta} = fC_{E_1} + x_\beta g^{q/p} C_{E_2}, \text{ with}$$
$$x_\beta = \{\beta^q A(f^p C_{E_1})/[A(g^q) - \beta^q A(g^q C_{E_2})]\}^{1/p}.$$

The right–hand side of (15) *equals*

$$A(f^p C_{E_1})^{1/p}[A(g^q) - \beta^q A(g^q C_{E_2})]^{1/q} - \beta A(fgC_{E_1}).$$

PROOF. Corollary 5 follows from Corollary 4, in precisely the same way as did Corollary 3 from Corollary 2 (and Lemma 4(1′) of [**2**]) by using $A_1(g_1) = A(g^q g_1)/A(g^q)$ with $g_1 = fg^{-q/p}$. We omit the details.

REMARK 5. In case $A(C_{E_2}) < 1$ (which holds if $A(1) = 1$ and $A(C_{E_1}) > 0$) we may take $\beta = 1$ in Corollaries 4 and 5. Then (13) and (15) reduce to

$$A(g^p)^{1/p} - A(g) \geq A(g^p C_{E_1})^{1/p} \cdot A(C_{E_1})^{1/q} - A(gC_{E_1}),$$
and
$$A(f^p)^{1/p} A(g^q)^{1/q} - A(fg) \geq A(f^p C_{E_1})^{1/p} A(g^q C_{E_1})^{1/q} - A(fgC_{E_1}),$$

respectively. The second of these inequalities is a genuine refinement of the generalized Hölder inequality [**2**; Th. 7] for isotonic functionals since

$$A(fgC_{E_1}) = A(fC_{E_1}, gC_{E_1}) \leq A(f^p C_{E_1})^{1/p} A(g^q C_{E_1})^{1/q}$$

holds, by [**2**; Th. 7]. Similarly, the right–hand side of the first inequality is also nonnegative. For the case $A(f) = \int_E f d\mu$, the above inequalities are weak versions of inequalities of W.N. Everitt (see, for example, [**4**; pp. 54, 86]).

We conclude by giving an application of Theorem 2, namely

COROLLARY 6. *Let* L *satisfy properties* L1, L2, L3 *on a nonempty set* E, *and suppose* ϕ *is a differentiable function on* $I = [m, M](-\infty < m < M < \infty)$ *such that* ϕ' *is strictly increasing on* I. *Let* A *be an isotonic linear functional on* L *with* $A(1) = 1$, *and let* $E_1 \in \mathcal{A}$ *satisfy* $A(C_{E_2}) > 0$ *where* $E_2 = E \backslash E_1$. *If* $m \leq g(t) \leq M$ *for* $t \in E$, *where* $g \in L, \phi(g) \in L$, *and we set* $\sigma = A(C_{E_2}), \mu = (\phi(M) - \phi(m))/(M - m)$, *then we have either*

(a) $A(gC_{E_1}) + m\sigma \le \phi'^{-1}(\mu) \le A(gC_{E_1}) + M\sigma$, *or*
(b) $m < \phi'^{-1}(\mu) < A(gC_{E_1}) + m\sigma$, *or*
(c) $A(gC_{E_1}) + M\sigma < \phi'^{-1}(\mu) < M$.
Moreover, either
(16)
$$A(\phi(g)) - \phi(A(g)) \le A(\phi(g)C_{E_1}) + \sigma\phi(M) - \mu[A(gC_{E_1}) + \sigma M] + \mu\phi'^{-1}(\mu) - \phi(\phi'^{-1}(\mu))$$

in case (a); *or*

(17) $$A(\phi(g)) - \phi(A(g)) \le A(\phi(g)C_{E_1}) + \sigma\phi(m) - \phi(A(gC_{E_1}) + \sigma m)$$

in case (b); *or*

(18) $$A(\phi(g)) - \phi(A(g)) \le A(\phi(g)C_{E_1}) + \sigma\phi(M) - \phi(A(gC_{E_1}) + \sigma M)$$

in case (c).

PROOF. We apply Theorem 2 to $F(x, y) = x - y$ for $(x, y) \in \mathbf{R}^2$. By (7) we obtain
$$A(\phi(g)) - \phi(A(g)) \le \sup_{0 \le \theta < \sigma} H(\theta),$$
where
$$H(\theta) = A(\phi(g)C_{E_1}) + \theta\sigma(m) + (\sigma - \theta)\phi(M) - \phi(A(gC_{E_1}) + \theta m + (\sigma - \theta)M).$$

First we observe that if $h(\theta) = A(gC_{E_1}) + \theta m + (\sigma - \theta)M$, then $A(gC_{E_1}) + m\sigma \le h(\theta) \le A(gC_{E_1}) + M\sigma$ for $0 \le \theta \le \sigma$. Moreover $A(gC_{E_1}) + M\sigma \le A(MC_{E_1}) + MA(C_{E_2}) = MA(1) = M$, and similarly $A(gC_{E_1}) + m\sigma \ge m$. In addition by the strictly increasing character of ϕ' we have $\phi'(m) < \mu < \phi'(M)$, so $m < \phi'^{-1}(\mu) < M$. It follows that $\phi'^{-1}(\mu)$ must lie in precisely one of the three intervals listed as alternatives (a), (b), (c) in the statement of the Corollary.

In case $A(gC_{E_1}) + m\sigma \le \phi'^{-1}(\mu) \le A(gC_{E_1}) + M\sigma$, we find $H'(\theta) = 0$ precisely when $\phi'(h(\theta)) = \mu$, that is when $\theta = \theta_o$ where
$$(M - m)\theta_o = A(gC_{E_1}) + \sigma M - \phi'^{-1}(\mu).$$

Moreover $H(\theta) \le H(\theta_o)$ then holds. This leads to the bound in (16).

In case $m < \phi'^{-1}(\mu) < A(gC_{E_1}) + m\sigma (\le h(\theta)$ for $0 \le \theta \le \sigma)$, we have $H'(\theta) > 0$ for $0 \le \theta \le \sigma$, so $H(\theta) \le H(\sigma)$ and this leads to the bound in (17). Similarly if $A(gC_{E_1}) + M\sigma < \phi'^{-1}(\mu) < M$, we have $H'(\theta) < 0$ for $0 \leq \theta \leq \sigma$, so $H(\theta) \leq H(0)$ and we obtain the bound in (18).

REMARK 6. In the same way we could give generalizations of Theorem 35 and Corollaries 36, 37 from [**5**, pp. 136-138].

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FACULTY OF CIVIL ENGINEERING UNIVERSITY OF BEOGRAD, BULE-VAR REVOLUCIJE 73, 1100 BEOGRAD, JUGOSLAVIA

DEPARTMENT OF MATHEMATICS & STATISTICS, CARLETON UNIVER-SITY, OTTAWA, ONTARIO, CANADA K1S 5B6