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OSCILLATORY AND ASYMPTOTIC BEHAVIOR IN CERTAIN THIRD ORDER DIFFERENCE EQUATIONS

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1. Introduction. In this paper, the difference equation

$$(E^-) \qquad \Delta^3 U_n - P_n U_{n+2} = 0,$$

where Δ denotes the differencing operation $\Delta U_n = U_{n+1} - U_n$, will be studied subject to the condition $P_n > 0$ for each integer $n \geq 1$. An example is given which shows that it is possible for (E^-) to have only nonoscillatory solutions. Our main result is a discrete analogue of Taylor [**15**, Theorem 6], and is concerned with a characterization of the existence of oscillatory solutions of (E^-) in terms of the behavior of nonoscillatory solutions. We also refer to the works of Hanan [**3**], Jones [**5,6**] and Lazer [**7**].

We will use primarily the terminology of Fort's Book [**1**] in our discussion. A real sequence $U = \{U_n\}$ which satisfies (E^-) for each $n \geq 1$ we term a solution of (E^-). Hereafter the term "solution" shall mean a "nontrivial solution." By the graph of a solution U we will mean the polygonal path connecting the points $(n, U_n), n \geq 1$. Any point where the graph of U intersects the real axis is called a node. A solution of (E^-) will be called oscillatory if it has arbitrarily large nodes; otherwise it is said to be nonoscillatory. Owing to the linearity of (E^-), we assume without loss of generality that all nonoscillatory solutions are eventually positive. Whenever (E^-) has an oscillatory solution we say that (E^-) is oscillatory. It is understood below that the variables n, m, N, M, i, j, k represent positive integers.

2. Preliminary Results. Our first result shows that initial values can be used to construct nonoscillatory solutions of (E^-). Since the proof is an easy argument, using the technique of mathematical induction, it will be omitted.

LEMMA 2.1. *If U is a solution of (E^-) satisfying*

$$U_m \geq 0, \Delta U_m \geq 0, \Delta^2 U_m > 0,$$

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for some choice of $m \geq 1$, *then*

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n > 0$$

for each $n \geq m + 2$.

The above result shows that (E^-) always has nonoscillatory solutions. Furthermore, the positivity of the coefficient function $\{P_n\}$ places rather strong restrictions on the behavior of the nonoscillatory solutions of (E^-).

THEOREM 2.2. *Let* U *be a nonoscillatory solution of* (E^-). *Then for all sufficiently large* n,

$$U_n \Delta U_n \Delta^2 U_n \neq 0$$

and either

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n > 0, \tag{1}$$

or

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n < 0. \tag{2}$$

PROOF. Assume that U is a nonoscillatory solution of (E^-), where $U_n > 0$ for each $n \geq N$. Note that $\Delta^3 U_n = \Delta(\Delta^2 U_n) = P_n U_{n+2} > 0$ for all $n \geq N$ hence $\Delta^2 U_n$ is increasing and eventually of one sign. So, it follows that M exists, $M \geq N$ for which ΔU_n and $\Delta^2 U_n$ are sign definite, for all $n \geq M$. Hence $U_n \Delta U_n \Delta^2 U_n \neq 0$, for every $n \geq M$. The cases

$$U_n > 0, \Delta U_n < 0, \Delta^2 U_n > 0, \;\; n \geq M$$

and

$$U_n > 0, \Delta U_n < 0, \Delta^2 U_n < 0, \;\; n \geq M$$

are clearly impossible, since $\Delta^i U_n \Delta^{i+1} U_n > 0$ for all n sufficiently large implies sgn $\Delta^{i-1} U_n =$ sgn $\Delta^i U_n$ eventually, and the proof is complete.

Denote by S^- the 3-dimensional vector space of solutions of (E^-). For each $U \in S^-$ define

$$G_n = G[U_n] = (\Delta U_n)^2 - 2U_{n+1}\Delta^2 U_n. \tag{3}$$

Computing the difference of G_n and making the substitution from (E^-), we find

$$\Delta G_n = -(\Delta^2 U_n)^2 - 2P_n U_{n+2}^2. \tag{4}$$

As a consequence of (4) we have the following result.

LEMMA 2.3. *If U is a solution of (E^-), then the functional defined by* (3) *is decreasing. Moreover there can exist at most one value of n such that $U_n = U_{n+1} = 0$.*

A solution U of (E^-) satisfying $U_k = U_{k+1} = 0$ is said to have a double zero at k. Hence the latter part of Lemma 2.3 states that a nontrivial solution of (E^-) can have at most one double zero. Note that at a double zero, say $n = k$, of a solution U, the functional G_n vanishes and hence must be negative for $n > k$. This clearly shows that U cannot have two double zeros.

The next theorem is of fundamental importance and will be used extensively in the next section.

THEOREM 2.4. *There exists $U \in S^-$ satisfying $G_n > 0$ for each $n \geq 1$.*

PROOF. Let X, Y, Z be a basis for S^-. For every positive integer m define $U_n^m = A_1^m X_n + A_2^m Y_n + A_3^m Z_n$ where the A_i^m are chosen in a way that $U_{m+1}^m = U_{m+2}^m = 0$ and $(A_1^m)^2 + (A_2^m)^2 + (A_3^m)^2 = 1$. Let $U_m^m > 0$. It follows from Lemma 2.3 that $G[U_n^m] > 0$ for all $1 \leq n \leq m$. Put $A_m = (A_1^m, A_2^m, A_3^m)$ where the A_i^m are as above. Then $||A_m|| = 1$ for each m. Due to the compactness of the unit ball in R^3 it follows that the sequence $\{A_m\}$, has a convergent subsequence $\{A_{m_i}\}$ such that $A_{m_i} \to A = (A_1, A_2, A_3)$ as $i \to \infty$, where $(A_1)^2 + (A_2)^2 + (A_3)^2 = 1$. Let U be defined by $U_n = A_1 X_n + A_2 Y_n + A_3 Z_n$. Then clearly U is a nontrivial solution of (E^-). Now $G[U_n] > 0$ for all $n \geq 1$, for if not there is an integer j such that $G[U_j] < 0$. Since $U_j^{m_i} \to U_j$, we can infer that $G[U_{j_i}^{m_i}] \to G[U_j] < 0$. Choose a positive integer M such that for all $i > M, G[U_j^{m_i}] < 0$, and $m_i > j$. Since $G[U_n]$ is decreasing and $G[U_{m_i}^{m_i}] > 0$, we have for $i > M, 0 < G[U_{m_i}^{m_i}] < G[U_j^{m_i}] < 0$. From this contradiction we see that $G[U_n] > 0$ for each n. This completes the proof of the theorem.

We now introduce a quasi-adjoint difference equation

$$\Delta^3 V_n + P_{n-1} V_{n+1} = 0. \tag{E^+}$$

The adjoint of (E$^-$) as defined in [**1**] and [**11**], is the equation

$$\Delta^3 \alpha_n + P_{n+1}\alpha_{n+1} = 0.$$

However, the remainder of this paper contains results which show that the solutions of the difference equations (E$^-$) and (E$^+$) satisfy relations similar to those that exist between two adjoint differential equations. For this reason we say that equations (E$^-$) and (E$^+$) are quasi-adjoint.

Turning to the equation (E$^+$), we show that (E$^+$) always has nonoscillatory solutions. Our result can be derived from a theorem of Hartmen and Wintner [**2**], but we shall give a proof based on the following lemma.

LEMMA 2.5. *If V is a solution of (E^+) satisfying*

$$V_m \geq 0, \Delta V_m \leq 0, \Delta^2 V_m > 0$$

for some integer $m > k \geq 1$, then

$$V_k > 0, \Delta V_k < 0, \Delta^2 V_k > 0$$

for each $1 \leq k < m$.

PROOF. We show the lemma true for $k = m-1$. Note that $\Delta(\Delta^2 V_{m-1}) = -P_{m-2}V_m \leq 0$. Thus $\Delta^2 V_m \leq \Delta^2 V_{m-1}$, and we have that $\Delta^2 V_{m-1} > 0$. Similarly, $\Delta^2 V_{m-1} > 0$ implies $\Delta V_{m-1} < 0$, which in turn implies $V_{m-1} > 0$. Hence the result holds for $k = m-1$. Repeating this process for each $1 \leq k < m-1$ proves the lemma.

THEOREM 2.6. *Let $m \geq 1$. There exists a solution V of (E^+) satisfying*

$$V_n > 0, \Delta V_n < 0, \ \Delta^2 V_n > 0 \tag{5}$$

for each $n \geq N \geq m$.

PROOF. Let r, s, t be a basis for S^+, the solution space of (E$^+$). For each positive integer k, define $V_n^k = B_1^k r_n + B_2^k s_n + B_3^k t_n$, where the B_i^k are chosen in such a fashion that $V_k^k = V_{k+1}^k = 0$ and $(B_1^k)^2 + (B_2^k)^2 + (B_3^k)^2 = 1$. Assuming $V_{k+2}^k > 0$, and proceeding as in the proof of Theorem 2.4, we can find a sequence $\{k_i\}$ of positive integers such that $\lim_{k_i \to \infty} V_n^{k_i} = V_n$ defines a nontrivial solution V of

(E$^+$). We see by Lemma 2.5 that $V_n \geq 0, \Delta V_n \leq 0, \Delta^2 V_n \geq 0, \Delta^3 V_n = -P_{n-1}V_{n+1} \leq 0$ for all n. If $V_{n_o} = 0$ for $n_o \geq m$, then since V_n is nonincreasing, $V_n = 0$ for all $n \geq n_o$, contradicting the fact that V is nontrivial. Hence $M_o \geq m$ exists, such that $V_n > 0$ for every $n \geq M_o$, in which case $\Delta^3 V_n = -P_{n-1}V_{n+1} < 0$ for all $n \geq M_o$. It then follows from another application of Lemma 2.5 that $V_n \Delta V_n \Delta^2 V_n \neq 0$ for all $n \geq 1$ and furthermore $V_n > 0, \Delta V_n < 0, \Delta^2 V_n > 0$, for each n. This completes the proof of the theorem.

Following [**15**], we term solutions of (E^-) which satisfy (1) as strongly increasing, and those which satisfy (2) as minimally increasing. Those solutions of (E^+) which satisfy (5) we term as strongly decreasing.

3. Oscillation properties of (E$^-$). In this section, we will examine the asymptotic behavior of certain solutions of (E$^-$). We will also consider some general relationships that exist between the solutions of (E$^-$), and those of (E$^+$). In the event that (E$^-$) has oscillatory solutions, our main result will show that even stronger restrictions are placed on the nonoscillatory solutions of (E$^-$), than required by Theorem 2.2. In fact, we will show that minimally increasing solutions cannot be "introduced" into the solution space S^- without "forcing" out all of the oscillatory solutions.

THEOREM 3.1. *Let U be a solution of (E^-) satisfying $G_n > 0$ for each $n \geq 1$. Then*
(i) $\sum^\infty (\Delta^2 U_n)^2 < \infty$ *and*
(ii) $\sum^\infty P_n U_{n+2}^2 < \infty$.

PROOF. Since $G_n > 0$ for each $n \geq 1$, differencing G_n and summing from 1 to $m-1$ yields

$$0 < G_m = G_1 - \sum_1^{m-1} (\Delta^2 U_j)^2 - 2\sum_1^{m-1} P_j U_{j+2}^2.$$

Thus,

$$\sum_1^{m-1} (\Delta^2 U_j)^2 + 2\sum_1^{m-1} P_j U_{j+2}^2 < G_1.$$

Letting m tend to infinity establishes each of (i) and (ii) since G_1 is independent of m.

COROLLARY 3.2. *Suppose* $\liminf_{n\to\infty} P_n > 0$. *If* U *is a solution of* (E^-) *satisfying* $G_n > 0$ *for each* n, *then* $\sum^\infty U_n^2 < \infty$.

We now exhibit the discrete LaGrange bilinear concomitant for solutions of (E^-) and (E^+). For $(U,V) \in S^- \times S^+$ define

$$(6) \qquad F_n = F[U_n, V_n] = U_{n+1}\Delta^2 V_{n+1} - \Delta U_n \Delta V_{n+1} + V_{n+1}\Delta^2 U_n.$$

It is easy to verify by differencing F_n and making the appropriate substitutions from (E^-) and (E^+), that $\Delta F_n = 0$, for each $n \geq 1$. Hence we have the following theorem.

THEOREM 3.3. *If* $U \in S^-$ *and* $V \in S^+$, *then the function defined by (6) is a constant that is determined by the initial values of* U *and* V.

Let X, Y be independent solutions of (E^-). The Wronskian

$$W_n = W(X_{n-1}, Y_{n-1}) = \begin{vmatrix} X_{n-1} & Y_{n-1} \\ \Delta X_{n-1} & \Delta Y_{n-1} \end{vmatrix},$$

is easily checked to define a nontrivial solution W of (E^+). Moreover, if X and Y do not enjoy the same oscillatory character, then $W = \{W_n\}$ is a nontrivial oscillatory solution of (E^+). Similarly, if R and S are solutions of (E^+), that are of a distinct oscillatory nature, then $\{W(R_{n-1}, S_{n-1})\}$ is a nontrivial oscillatory solution of (E^-). We therefore have the following result which is a discrete analogue of Hanan [**3**, Theorem 4.7].

THEOREM 3.4. *Equation* (E^+) *is oscillatory if and only if equation* (E^-) *is oscillatory.*

Let r, s, t be solutions of (E^-). Expanding the Wronskian

$$R_n = R(r_n, s_n, t_n) = \begin{vmatrix} r_n & s_n & t_n \\ \Delta r_n & \Delta s_n & \Delta t_n \\ \Delta^2 r_n & \Delta^2 s_n & \Delta^2 t_n \end{vmatrix}$$

along its third column we obtain the following relationship between F_n, W_n and R_n:

$$R_n = F[W(r_{n-1}, s_{n-1}), t_{n-1}].$$

THEOREM 3.5. *If V is a nonoscillatory solution of (E^+), then two independent solutions of (E^-) satisfy the selfadjoint second order difference equation*

$$\Delta(\frac{\Delta U_n}{V_{n+1}}) + (\frac{\Delta^2 V_{n+1}}{V_{n+1}V_{n+2}})U_{n+1} = 0. \tag{7}$$

PROOF. Since V is fixed in S^+, we have $K_F = \{U \in S^- | F[U_n, V_n] = 0\}$ is the kernel of the linear functional $F_n : S^- \to \hat{R}$, where $\hat{R}$ denotes the set of real numbers. If $V_n > 0, n \geq N$, then $X \in K_F$ implies

$$\Delta(\frac{\Delta X_n}{V_{n+1}}) + (\frac{\Delta^2 V_{n+1}}{V_{n+1}V_{n+2}})X_{n+1} = 0.$$

The result follows since

$$\dim K_F + \dim \hat{R} = \dim S^-.$$

We now derive an oscillation condition for (E^-) in terms of equation (7).

THEOREM 3.6. *The following two statements are equivalent:*
(i) *Equation (E^-) is oscillatory.*
(ii) *Equation (7) is oscillatory.*

PROOF. Suppose that condition (i) holds, then by Theorem 3.4, (E^+) is oscillatory. Let r be an oscillatory solution of (E^+). Consider $R(r_n, V_n, V_n)$, where V is the nonoscillatory solution of (E^+) whose existence was shown in Theorem 2.6. Thus, $F[W(r_{n-1}, V_{n-1}), V_{n-1}] = 0$, and we find that $\{W(r_{n-1}, V_{n-1})\}$ is an oscillatory solution of

$$\Delta(\frac{\Delta W_{n-1}}{V_n}) + (\frac{\Delta^2 V_n}{V_n V_{n+1}})W_n = 0.$$

This proves the first part of the theorem.

Suppose that condition (ii) holds, where V is a nonoscillatory solution of (E^+), with $V_n > 0, n \geq N$. If U is an oscillatory solution of (7) then $U \in K_F$, and in particular $U \in S^-$. This completes the proof of the theorem.

REMARK. Since the nodes of linearly independent solutions of (7) separate each other, and those of linearly dependent solutions coincide, it follows that either all solutions of (7) are oscillatory, or all solutions of (7) are nonoscillatory [8]. We therefore have the following corollary of Theorem 3.6.

COROLLARY 3.7. *If* (E^-) *is oscillatory, then* S^- *has a basis consisting of one nonoscillatory solution and two oscillatory solutions.*

We now turn to our main result.

THEOREM 3.8. *The following two statements are equivalent:*

(i) *Equation* (E^-) *is oscillatory.*

(ii) *For every nonoscillatory solution* U *of* (E^-), *there exists an integer* N *for which*

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n > 0, \;\; n \geq N.$$

PROOF. Suppose that condition (i) holds and that (E^-) has a solution Y satisfying

$$Y_n > 0, \Delta Y_n > 0, \Delta^2 Y_n < 0, \;\; n \geq N.$$

By Corollary 3.7 and the above remark, there exist two independent oscillatory solutions of (E^-), every linear combination of which is oscillatory. Let f, g be such a pair of solutions with $f_N = 0, g_N \neq 0$. Let $\Phi_n = Y_n - dg_n$, where d is a constant chosen in such a way that $\Phi_N = 0$. Consider $W(\Phi_n, f_n)$. Now $W(\Phi_N, f_N) = 0$, hence there exist constants C_1, C_2 with $C_1^2 + C_2^2 \neq 0$ such that $C_1 \Phi_N + C_2 f_N = 0$ and $C_1 \Delta \Phi_N + C_2 \Delta f_N = 0$. Put $U_n = C_1 \Phi_n + C_2 f_n$. Then U has a double zero at N, and $U_n = C_1 Y_n + \Psi_n$, where Ψ defined by $\Psi_n = C_2 f_n - C_1 d g_n$, is an oscillatory solution of (E^-). Since U is nontrivial, we may suppose without loss of generality $\Delta^2 U_N > 0$. As a consequence of Lemma 2.1, $\lim_{n\to\infty} \Delta U_n = \infty$. Moreover, the relations $\Delta Y_n > 0, \Delta^2 Y_n < 0, \Delta^3 Y_n > 0, n \geq N$ imply that $\{\Delta Y_n\}$ is asymptotic to a finite constant. Now $T(x, \Delta U_n) = (\Delta^2 U_n)(x - n) + \Delta U_n, n \leq x \leq n + 1, n \geq 1$ defines the graph of $\{\Delta U_n\}$. Let $\{x_i\}$ be an increasing sequence of nodes of $\{\Delta \Psi_n\}$. Then at each x_i we have

$$T(x_i, \Delta U_n) = C_1 T(x_i, \Delta Y_n). \tag{8}$$

We have arrived at a contradiction since the left member of (8) becomes unbounded as $i \to \infty$, whereas the right member of (8) is bounded as

$i \to \infty$. This contradiction proves the first part of the theorem.

Suppose that condition (ii) holds, and that every nonoscillatory solution of (E^-) is strongly increasing. If U is a nonoscillatory solution of (E^-) such that conditions (1) hold for each $n \geq N$, then differencing G_n we obtain as a result of (4) the inequality

$$\Delta G_n \leq -(\Delta^2 U_n)^2. \tag{9}$$

Summing both sides of (9) from N to $m-1$, we obtain

$$G_m \leq G_N - \sum_{N}^{m-1} (\Delta^2 U_n)^2 \leq G_N - (\Delta^2 U_N)^2 \sum_{N}^{m-1} 1 \to -\infty$$

as $m \to \infty$. Hence $\lim_{n\to\infty} G_n = -\infty$ holds for every nonoscillatory solution of (E^-). By Theorem 2.4, there exists a solution of (E^-) satisfying $G_n > 0$ for each $n \geq 1$. Such a solution clearly satisfies $\lim_{n\to\infty} G_n \geq 0$, and hence must be oscillatory.

The following example shows that it is possible for every solution of (E^-) to be nonoscillatory.

EXAMPLE. Consider the equation

$$\Delta^3 U_n - \frac{1}{2(2^{n+2}-1)} U_{n+2} = 0. \tag{E}$$

An easy calculation will show that U, defined by $U_n = 1 - 2^{-n}$, is a minimally increasing solution of (E). As a consequence of Theorem 3.8 every solution of (E) is nonoscillatory.

We record as our final result a sufficient condition for (E^-) to be oscillatory in terms of the coefficient function $\{P_n\}$. This result is a discrete analogue of Jones [**5**, Theorem 2].

THEOREM 3.9. *If* $\sum^{\infty} P_n = \infty$, *then* ($E^-$) *is oscillatory.*

PROOF. In light of Theorem 2.4, it is enough to show that $\lim_{n\to\infty} G_n = -\infty$ for every nonoscillatory solution of (E^-). However, this clearly is the case if $\sum^{\infty} P_n = \infty$, for conditions (1) and (2) imply

$$G_m \leq G_N - 2(U_{N+2})^2 \sum_{N}^{m-1} P_n \to -\infty$$

as $m \to \infty$, where U is any nonoscillatory solution of (E^-). The theorem follows.

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