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ARITHMETIC PROGRESSIONS IN LACUNARY SETS

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ABSTRACT. We make some observations concerning the conjecture of Erdös that if the sum of the reciprocals of a set A of positive integers diverges, then A contains arbitrarily long arithmetic progressions. We show, for example, that one can assume without loss of generality that A is lacunary. We also show that several special cases of the conjecture are true.

1. Introduction. The now famous theorem of Szemerédi [7] is often stated:

(a) *If the density of a set A of natural numbers is positive, then A contains arbitrarily long arithmetic progressions.*

Let us call a set A of natural numbers k-good if A contains a k-term arithmetic progression. Call A ω-good if A is k-good for all $k \geq 1$. We define four density functions as follows: For a set A and natural numbers m, n, let $A[m, n]$ be the cardinality of the set $A \bigcap \{m, m+1, m+2, \ldots, n\}$. Then define

$$\underline{\delta}(A) = \liminf_{n} \frac{A[1, n]}{n},$$
$$\overline{\delta}(A) = \limsup_{n} \frac{A[1, n]}{n},$$
$$\underline{u}(A) = \lim_{n} \min_{m \geq 0} \frac{A[m+1, m+n]}{n} \text{ and}$$
$$\overline{u}(A) = \lim_{n} \max_{m \geq 0} \frac{A[m+1, m+n]}{n}.$$

It can be seen that the limits in the definitions of $\underline{u}$ and $\overline{u}$ always exist. These four "asymptotic" set functions are called the lower and upper "ordinary" and the lower and upper "uniform" density of the set A respectively. They are related by

$$\underline{u}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{u}(A)$$

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for any set A.

Szemerédi actually proved:

(b) *If* $\overline{u}(A) > 0$, *then* A *is* ω*-good.* Hence we also have

(c) *If* $\overline{\delta}(A) > 0$, *then* A *is* ω*-good.*

In fact, Szemerédi proved the following "finite" result (which we state in a general form to be used later):

(d) *Let* $\varepsilon > 0$ *and* $k \in N = \{1, 2, 3, \ldots\}$. *Then there exists an* $n_0 \in N$ *such that if* P *is any arithmetic progression of length* $|P| \geq n_0$ *and* $A \subseteq P$ *with* $|A| \geq \varepsilon|P|$, *then* A *is* k*-good.*

It is not hard to prove (without assuming the truth of any of the statements) that (b), (c) and (d) are equivalent.

Erdös [**1**] has conjectured that the following stronger statement holds:

(e) *If* $A \subseteq N$ *and* $\sum_A \frac{1}{a} = \infty$, *then* A *is* ω*-good.*

By $\sum_A (1/a)$ we mean of course $\sum_{a \in A}(1/a)$. The proof (or disproof) of (e) is, at present, out of sight. In fact, it has not even been proved that $\sum_A (1/a) = \infty$ implies that A is 3-good (compare Roth [**6**]). That (e)=>(c) can be seen as follows: If $\overline{\delta}(A) = \varepsilon > 0$, then there exists a sequence of natural numbers $0 = n_0 < n_1 < n_2 < \ldots$, such that, for each i,

$$\frac{A[1, n_i]}{n_i} > \frac{\varepsilon}{2} \text{ and } \frac{n_{i-1}}{n_i} < \frac{\varepsilon}{4}.$$

Then

$$\sum_A \frac{1}{a} \geq \sum_{\substack{a \in A \\ a \leq n_k}} \frac{1}{a} \geq \sum_{i=1}^{k} \frac{A[n_{i-1}+1, n_i]}{n_i} \geq \sum_{i=1}^{k} \frac{A[1, n_i] - n_{i-1}}{n_i}$$

$$\geq k\left(\frac{\varepsilon}{2} - \frac{\varepsilon}{4}\right) = \frac{k\varepsilon}{4} \to \infty (k \to \infty)$$

and so $\sum_A (1/a) = \infty$. Assuming (e), it follows that A is ω-good.

Hence Erdös' conjecture is indeed stronger than Szemerédi's theorem. Note also that Erdös' conjecture, if true, would immediately answer in the affirmative the long-standing question of whether or not the primes are ω-good.

In the next section we make some observations regarding this conjecture, and we show that several special cases of the conjecture are true.

Other observations can be found in Gerver [**3,4**] and Wagstaff [**8**].

2. Main results.
(2.1). First we consider the "finite form" of Erdös' conjecture.

THEOREM 1. *Fix k, and assume that for all sets $A \subseteq N$, if $\sum_A (1/a) = \infty$ then A is k-good. Under this assumption, there exists T such that if $\sum_A (1/a) > T$, then A is k-good.*

(Gerver **[3]** *has this result under the stronger hypothesis that if $\sum_A (1/a) = \infty$ then A is $(k+1)$-good.)*

PROOF. We may assume $k \geq 3$. Suppose the theorem is false. We will construct a set A such that $\sum_A (1/a) = \infty$ and A is not k-good. Choose a finite set A_0 such that A_0 is not k-good and $\sum_A (1/a) > 1$. Let p_1 be prime, $p_1 > 2\max A_0$, and choose a finite subset A_1 of $\{tp_1 | t \geq 1\}$ such that A_1 is not k-good and $\sum_{A_1} (1/a) > 1$. Let p_2 be prime, $p_2 > 2\max A_1$, and choose a finite subset A_2 of $\{tp_2 | t \geq 1\}$ such that A_2 is not k-good and $\sum_{A_2} (1/a) > 1$. Continuing in this way, we obtain finite sets $A_0, A_1, \ldots$ such that for each $i \geq 0, A_1$ is not k-good, $\min A_{i+1} \geq p_{i+1} > 2\max A_i$, each element of A_{i+1} is a multiple of p_{i+1}, and $\sum_{A_i} (1/a) > 1$.

Let $A = \bigcup A_i$. It is clear that $\sum_A (1/a) = \infty$. To show that A is not k-good, it suffices to show that every 3-term arithmetic progression contained in A must be contained in a single set A_i.

To this end, suppose that $x < y < z$, with $x, y, z \in A$ and $z - y = y - x$. Let $y \in A_i$. Then $z \in A_i$ also, since otherwise $z - y \geq \min A_{i+1} - \max A_i > \max A_i > y - x$. Thus $y, z \in A_i \subset \{tp_i | t \geq 1\}$. Hence x is divisible by p_i, so $x \geq p_i > \max A_{i-1}$, and $x \in A_i$. This finishes the proof of Theorem 1.

COROLLARY 1. *The following statement is equivalent to statement* (e):

(f) *For each $k \in N$, there exists $T \in N$ such that if $\sum_A (1/a) > T$, then A is k-good.*

We state next a lemma which will be useful later.

LEMMA 1. *Let $F_1, F_2, \ldots$ be a sequence of finite subsets of N such that for each i, F_i is not k-good and* $\min F_{i+1} \geq 2\max F_i$. *Then $F = \bigcup F_i$ is not $(k+1)$-good.*

(The proof of Lemma 1 is contained in the proof of Theorem 1 above).

(2.2). Now we define an increasing sequence, $a_1 < a_2 < a_3 < \dots$, of natural numbers to be lacunary if $d_n = a_{n+1} - a_n \to \infty$ as $n \to \infty$ and to be M-lacunary if, furthermore, $d_n \le d_{n+1}$ for all n. We shall think of such a sequence simultaneously as a sequence and as a subset of N. Any lacunary sequence A has $\overline{u}(A) = 0$ (see [**2**]), so that Szemerédi's theorem does not apply.

A subsequence of a lacunary sequence is lacunary, but the corresponding statement, unfortunately, does not hold for M-lacunary sequences. It is known that if the real series $\sum t_i$ is not absolutely convergent, then there exists a lacunary sequence B such that $\sum_{i \in B} t_i$ diverges (see Freedman and Sember [**2**]). It follows that if $A \subseteq N$ and $\sum_A (1/a) = \infty$, then there exists a lacunary sequence $B \subseteq A$ such that $\sum_B (1/b) = \infty$. Thus we have the following

THEOREM 2. *The following statement is equivalent to statement* (e).
(g) *If A is a lacunary sequence and $\sum_A (1/a) = \infty$, then A is ω-good.*

Hence we need only investigate lacunary sequences when contemplating the Erdös conjecture.

It can also be shown that if $\sum t_i = \infty$ and $t_i \ge 0$ for all i, then there exists an M-lacunary sequence B such that $\sum_{i \in B} t_i = \infty$. (We omit the rather cumbersome proof of this statement.) But notice that this does not imply that statement (h) below is equivalent to statement (e)! This is too bad - because we now prove (h).

THEOREM 3. *The following statement is true.*
(h) *If A is M-lacunary and $\sum_A (1/a) = \infty$, then A is ω-good.*

PROOF. Let $A = \{a_1 < a_2 < a_3 < \dots\}$ be an M-lacunary sequence with infinite reciprocal sum. Assume there is a k such that $d_i < d_{i+k}$ for each i, where $d_n = a_{n+1} - a_n, n \ge 1$. We show that $a_{i+jk} \ge j^2/2$ for all $i \ge 1, j \ge 0$. Indeed,

$$\begin{aligned} a_{i+jk} &= a_i + d_i + d_{i+1} + \dots + d_{i+jk-1} \\ &\ge d_i + d_{i+k} + d_{i+2k} + \dots + d_{i+(j-1)k} \\ &> 1 + 2 + 3 + \dots + j > j^2/2. \end{aligned}$$

(Note that to obtain the first inequality we have merely omitted some terms from the sum).

But then

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \sum_{j=0}^{\infty} \frac{1}{a_{1+jk}} + \sum_{j=0}^{\infty} \frac{1}{a_{2+jk}} + \cdots + \sum_{j=0}^{\infty} \frac{1}{a_{k+jk}}$$
$$\leq k(1 + \sum_{j=1}^{\infty} \frac{2}{j^2}) < \infty, \text{ a contradiction.}$$

Hence, for each k, there is an i such that $d_i = d_{i+k}$, whence $a_i, a_{i+1}, \ldots, a_{i+k+1}$ are in arithmetic progression and A is ω-good.

The following is an immediate corollary.

COROLLARY 2. *If A is a finite union of M-lacunary sets and $\sum_A (1/a) = \infty$, then A is ω-good.*

(2.3). We now use some slightly expanded arguments to show that statement (g) holds for some special sequences which are not M-lacunary (but are nearly so).

THEOREM 4. *Let $A = \{a_1 < a_2 < a_3 < \ldots\}$ be any set. Suppose there are intervals $I_n = [s_n, t_n]$ with $t_n < s_{n+1}$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_{s_n}}} < \infty, \quad \sum_{k \in \bigcup I_n} \frac{1}{a_k} = \infty.$$

Suppose further that for each $n, d_k \leq d_{k+1}$ if $s_n \leq k < t_n$. Then A is ω-good.

PROOF. We will arrive at a contradiction if we assume that there is a $K \in N$, such that $d_i < d_{i+K}$ whenever $i, i+K$ belong to the same interval I_j. Then, for any K, we have that there exists an i such that $d_i = d_{i+1} = \cdots = d_{i+K}$ so that $a_i, a_{i+1}, \ldots, a_{i+K+1}$ are in arithmetic progression.

To we get the required contradiction we proceed as follows: If n,

$n+K, n+2K, \ldots, n+cK \in I_i$, then

$$\begin{aligned}
&\frac{1}{a_n} + \frac{1}{a_{n+K}} + \frac{1}{a_{n+2K}} \cdots \frac{1}{a_{n+cK}} \\
&\quad \le \frac{1}{a_n} + \frac{1}{a_n + d_n} + \frac{1}{a_n + d_n + d_{n+K}} + \cdots \\
&\quad + \frac{1}{a_n + d_n + d_{n+K} + \cdots + d_{n+(c-1)K}} \\
&\quad < \sum_{j=0}^{\infty} \frac{1}{a_n + (j^2/2)} < \frac{b}{\sqrt{a_n}} \le \frac{b}{\sqrt{a_{s_i}}} \quad (b \text{ constant}).
\end{aligned}$$

Hence,

$$\sum_{K \in I_i} \frac{1}{a_k} < \frac{Kb}{\sqrt{a_{s_i}}} \text{ and } \sum_{k \in \bigcup I_i} \frac{1}{a_k} < Kb \sum_{i=1}^{\infty} \frac{1}{\sqrt{a_{s_i}}} < \infty,$$

contrary to assumption.

Using a similar technique we can prove the following theorem.

THEOREM 5. *Let $A = \{a_1 < a_2 < a_3 < \ldots\}$ be a set. Suppose $I_n = [s_n, t_n]$ are intervals with $t_n < s_{n+1}$ such that $d_i \le d_{i+1}$ if $s_n \le i < t_n$ and $d_{t_n - 1} < d_{s_{n+1}}$. Then, if $\sum_{k \in \bigcup I_n} (1/a_k) = \infty$, A is ω-good.*

(2.4). We now define new density functions λ and $\overline{\lambda}$ in terms of lacunary sequences: For all sets A, let $\overline{\lambda}(A) = 0$ if A is finite or a finite union of lacunary sequences and otherwise let $\overline{\lambda}(A) = 1$. Define $\lambda(A) = 1 - \overline{\lambda}(N - A)$. These densities, taking only $0, 1$ values, may seem a little odd. The definition could be improved so that λ becomes "continuous" and has the correct value on an (infinite) arithmetic progression etc. However, this would not suit our purposes any better. One can prove that for any $A \subseteq N$

$$\underline{\lambda}(A) \le \underline{u}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{u}(A) \le \overline{\lambda}(A)$$

and so, in analogy to Szemerédi's Theorem, it is natural to ask about the arithmetic progressions in A if $\overline{\lambda}(A) > 0$.

THEOREM 6. *There exists a set A such that $\overline{\lambda}(A) > 0$ and A is not ω-good.*

PROOF. Let $B_i = \{1!, 2!, \ldots, i!\}$. B_i is not 3-good. Let (H_i) be the sequence of sets

$$(B_1, B_1, B_2, B_1, B_2, B_3, B_1, B_2, B_3, B_4, B_1, \ldots)$$

Let f_i be an increasing sequence of integers such that $f_1 = 0$ and

$$\min(f_{i+1} + H_{i+1}) \geq 2\max(f_i + H_i)$$

and define $A = \bigcup_i (f_i + H_i)$. By Lemma 1, A is not 4-good. (By choosing f_i sufficiently quickly increasing one can even make A not 3-good.) Finally, $\overline{\lambda}(A) = 1$ since otherwise $A = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_k$ where each L_j is a lacunary sequence. Whenever $H_i = B_{k+1}$ we have $|f_i + H_i| > k$ and so some L_j has at least two members in $f_i + H_i$. Hence we may find a fixed j such that

$$|L_j \bigcap (f_i + B_{k+1})| \geq 2$$

for infinitely many i. Then L_j has infinitely many differences $d_t < (k+1)!$, and so L_j is not lacunary.

(2.5). Let us consider "relative density", that is, "the density of A relative to B" where $A \subseteq B$. The definitions are:

$$\underline{\delta}(A|B) = \liminf_{i\to\infty} \frac{A[1, b_i]}{i} \text{ and}$$
$$\underline{u}(A|B) = \lim_{n\to\infty} \min_{m\geq 0} \frac{A[b_{m+1}, b_{m+n}]}{n}.$$

$\overline{\delta}(A|B)$ and $\overline{u}(A|B)$ are obtained by replacing "inf" with "sup" and "min" with "max" respectively. One can show, as before, for any $A, B, A \subseteq B$, that

$$\underline{u}(A|B) \leq \underline{\delta}(A|B) \leq \overline{\delta}(A|B) \leq \overline{u}(A|B).$$

Let B be M-lacunary and $\sum_B 1/b = \infty$. Then, by Theorem 3, B is ω-good. We ask whether $A \subseteq B$ and the relative density of A

positive imply that A is also ω-good. The answer is "yes" if $\underline{u}(A|B) > 0$ (Theorem 7), "no" if $\overline{\delta}(A|B) > 0$ (Theorem 8) and the question is open for $\underline{\delta}(A|B) > 0$.

THEOREM 7. *If B is M-lacunary, $\sum_B 1/b = \infty$, $A \subseteq B$ and $\underline{u}(A|B) > 0$ then A is ω-good.*

PROOF. By (the proof of) Theorem 3 there are arbitrarily large n, m such that

$$P = \{b_{m+1}, b_{m+2}, \dots, b_{m+n}\}$$

is an arithmetic progression. By the definition of $\underline{u}(A|B)$ we have $|A \bigcap P| \geq \varepsilon |P|$ where $\varepsilon = (1/2)\underline{u}(A|B)$ and $|P|$ is arbitrarily large. Thus, by Szemerédi's Theorem (d) we have, for any k, that $|A \bigcap P|$ is k-good if $|P|$ is sufficiently large. Hence A is ω-good.

THEOREM 8. *There exists an M-lacunary sequence B with $\sum_B 1/b = \infty$ and an $A \subseteq B$ with $\overline{\delta}(A|B) > 0$ (= 1 in fact) such that A is not 3-good.*

PROOF. (leaving most of the details to the reader). Let $F = \{1!, 2!, 3!, \dots\}, b_1 = 1$ and define $b_{n+1} = b_n + d_n$ where the d_n's have the following properties: For all $i, d_i \in F$ and $d_i \leq d_{i+1}$. Furthermore, the set of natural numbers N can be partitioned into consecutive pairwise disjoint intervals $J_1, J_2, J_3, \dots$ such that if r is odd, then, for $i \in J_r, d_i = d_{i+1}$ and $\sum_{i \in J_r} 1/b_i \geq 1$, and, if r is even, then, for $i \in J_r, d_i < d_{i+1}, b_i > 2b_{i-1}$ and $|J_r| > (\max J_{r-1})^2$. Clearly $B = \{b_1, b_2, \dots\}$ is M-lacunary and $\sum_B 1/b = \infty$. Let $A = \{b_k | k \in \bigcup_r J_{2r}\}$. Then

$$\overline{\delta}(A|B) \geq \lim_r \frac{|J_{2r}|}{|J_{2r}| + \max J_{2r-1}} = 1.$$

One can also see that A is not 3-good since $a_i > 2a_{i-1}$ holds.

(2.6). Theorems 4,5, and 7 notwithstanding, it seems to be difficult to generalize the notion of M-lacunary even slightly and still prove the corresponding case of the Erdös conjecture. In this connection let us define a lacunary sequence A to be M_k-lacunary (where $k \geq 0$) if, for all $i, j, i \leq j$, we have $d_i \leq d_j + k$. Clearly the M_0-lacunary sequences are just the M-lacunary sequences. For no $k \neq 0$ are we able to prove that M_k-lacunary and $\sum_A(1/a) = \infty$ imply ω-good. We can show if A is M_1-or M_2-lacunary with $\sum_A(1/a) = \infty$ then A is 3-good. We prove first a lemma which may have independent interest:

LEMMA 2. *If $A = \{a_1 < a_2 < a_3 < \dots\}$ is any subset of N and $\sum_A 1/a = \infty$, then, for any $t > 0$, there exists an i such that $d_{i+j} \leq d_i$ for $j = 0, 1, \dots, t$. (Of course, $d_n = a_{n+1} - a_n$.)*

PROOF. The method is familiar by now: Suppose there is a t such that, for each i, there exists $j \in [1, t]$ with $d_i < d_{i+j}$. Then we can find a sequence (j_n) such that

$$d_1 < d_{1+j_1} < d_{1+j_1+j_2} < \dots (j_n \in [1, t]).$$

It follows that

$$\sum_A \frac{1}{a} \leq t \sum_{s=0}^{\infty} \frac{1}{a_{1+(j_1+\dots+j_s)}} \leq t \sum_{s=0}^{\infty} \frac{1}{a_1 + (1 + 2 + \dots + s)} < \infty.$$

THEOREM 9. *Let A be M_1-or M_2-lacunary and $\sum_A(1/a) = \infty$. Then A is 3-good.*

PROOF. By the definition of M_k-lacunary and Lemma 2 we have: for any $t > 0$ there is an i such that

$$d_i - e \leq d_{i+j} \leq d_i \quad j = 0, 1, \dots, t,$$

where $e = 1$ or 2. Hence, in the sequence (d_i), we have arbitrarily long blocks where the d_i take on only two (in case $e = 1$) or three (in case $e = 2$) values. Such long blocks must contain two consecutive subblocks with identical composition (see Pleasants [**5**]). These two subblocks will determine three terms of the sequence A in arithmetic progression.

This last result suggests a conjecture which is related to van der Waerden's theorem on arithmetic progressions and which would immediately imply that M_k-lacunary with $\sum_A(1/a) = \infty$ implies that A is 3-good.

Conjecture. Let x_i be a sequence of positive integers with $1 \leq x_i \leq K$. Then there are two consecutive intervals I, J, of the same length, with $\sum_{i \in I} x_i = \sum_{j \in J} x_j$. Equivalently, if $a_1 < a_2 < \dots$ satisfy

$a_{n+1} - a_n \leq K$, all n, then there exist $x < y < z$ such that $x + z = 2y$ and $a_x + a_z = 2a_y$.

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