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FOURIER TRANSFORM FOR INTEGRABLE BOEHMIANS

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ABSTRACT. Basic properties of the Fourier transform for intgrable Boehmians are discussed. An inversion theorem is proved.

Introduction. The Fourier transform for Boehmians has been defined independently by J. Burzyk (oral communication) and D. Nemzer [**5**]. The definition given by J. Burzyk is very general and in this case the Fourier transform of a Boehmian is not necessarily a function (like the Fourier transform of a tempered distribution). D. Nemzer was particularly interested in the Fourier transform of Boehmians with compact support. This note will discuss basic properties of the so-called integrable Boehmians. In this case the Fourier transform is always a continuous function and has all basic properties of the Fourier transform in $\mathcal{L}_1$. In particular, we will prove an inversion theorem which has the form of a classical theorem in $\mathcal{L}_1$.

1. Integrable Boehmians. A general construction of Boehmians was given in [**2**]. In this note we are interested in a special case of that construction. Denote by $\mathcal{L}_1$ the space of complex valued Lebesgue integrable functions on the real line $\mathbf{R}$. By $||\cdot||$ we mean the norm in $\mathcal{L}_1(||f|| = \int_R |f(x)|dx)$. If $f, g \in \mathcal{L}_1$ then the convolution product $f * g$, i.e.,

$$(f * g)(x) = \int_R f(u)g(x-u)du,$$

is an element of $\mathcal{L}_1$ and $||f * g|| \leq ||f|| \cdot ||g||$.

A sequence of continuous real functions $\delta_n \in \mathcal{L}_1$ will be called a *delta sequence* if

$$\begin{cases} \int_R \delta_n(x)dx = 1 & \text{for every } n \in N, \\ ||\delta_n|| < M & \text{for some } M \in \mathbf{R} \text{ and all } n \in N, \\ \lim_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)|dx = 0 & \text{for each } \varepsilon > 0. \end{cases}$$

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If (ς_n) and (ψ_n) are delta sequences, so is $(\varsigma_n * \psi_n)$. If $f \in \mathcal{L}_1$ and (δ_n) is a delta sequence, then $\|f * \delta_n - f\| \to 0$ as $n \to \infty$. Delta sequences are also called approximate identities or summability kernels.

A pair of sequences (f_n, ς_n) is called a *quotient of sequences*, and denoted by f_n/ς_n, if $f_n \in \mathcal{L}_1 (n = 1, 2, \ldots)$, (ς_n) is a delta sequence, and $f_m * \varsigma_n = f_n * \varsigma_m$ for all $m, n \in N$. Two quotients of sequences f_n/ς_n and g_n/ψ_n are equivalent if $f_n * \psi_n = g_n * \varsigma_n$ for every $n \in N$. The equivalence class of a quotient of sequences will be called an *integrable* Boehmian. The space of all integrable *Boehmians* will be denoted by $\mathcal{B}_{\mathcal{L}_1}$.

The space $\mathcal{B}_{\mathcal{L}1}$ is a convolution algebra when the multiplication by scalar, addition, and convolution are defined as follows:

$$\lambda[f_n/\varsigma_n] = [\lambda f_n/\varsigma_n],$$

$$[f_n/\varsigma_n] + [g_n/\psi_n] = [(f_n * \psi_n + g_n * \varsigma_n)/\varsigma_n * \psi_n],$$

$$[f_n/\varsigma_n] * [g_n/\psi_n] = [f_n * g_n/\varsigma_n * \psi_n].$$

A function $f \in \mathcal{L}_1$ can be identified with the Boehmian $[f * \delta_n/\delta_n]$ where (δ_n) is any delta sequence. It is convenient to treat $\mathcal{L}_1$ as a subspace of $\mathcal{B}_{\mathcal{L}_1}$. Note that if $F = [f_n/\delta_n]$, then $F * \delta_n = f_n$ and hence $F * \delta_n \in \mathcal{L}_1$ for every $n \in N$.

We say that a sequence of Boehmians F_n is *Δ-convergent* to a Boehmian F ($\Delta\text{-}\lim F_n = F$) if there exists a delta sequence (δ_n) such that $(F_n - F) * \delta_n \in \mathcal{L}_1$ for every $n \in N$ and $\|(F_n - F) * \delta_n\| \to 0$ as $n \to \infty$. From a general theorem proved in **[3]** it follows that $\mathcal{B}_{\mathcal{L}_1}$ with Δ–convergence is a complete metric (quasi-normed) space.

In practice it is often more convenient to use other types of convergence in $\mathcal{B}_{\mathcal{L}_1}$: we say that a sequence of Boehmians F_n is *δ-convergent* to F($\delta\text{-}\lim F_n = F$) if there exists a delta sequence (δ_n) such that $F_n * \delta_k \in \mathcal{L}_1$ and $F * \delta_k \in \mathcal{L}_1$ for every $n, k \in N$ and $\|(F_n - F) * \delta_k\| \to 0$ for each $k \in N$. The following equivalence explains how these two types of convergence are related (see **[3]**).

(*) $\Delta\text{-}\lim F_n = F$ if and only if each subsequence of (F_n) contains a subsequence which is δ-convergent to F.

The above fact can be used to prove the following: if $\Delta\text{-}\lim F_n = F$ and $\Delta\text{-}\lim G_n = G$, then $\Delta\text{-}\lim F_n * G_n = F * G$. If (δ_n) is a delta sequence, then δ_n/δ_n represents an integrable Boehmian. Since the Boehmian $[\delta_n/\delta_n]$ corresponds to the Dirac delta distribution, we denote it by δ. All derivatives of δ are also integrable Boehmians. Since, there are delta sequences (δ_n) such that all functions δ_n are infinitely differentiable and have bounded support, we can define the k−derivative of δ by $\delta^{(k)} = [\delta_n^{(k)}/\delta_n]$. It is easy to check that $\delta^{(k)} \in \mathcal{B}_{\mathcal{L}_1}$ for any $k \in N$. The k-th derivative of a Boehmian $F \in \mathcal{B}_{\mathcal{L}_1}$ can be defined as $F^{(k)} = F * \delta^{(k)}$. From the continuity of the convolution in $\mathcal{B}_{\mathcal{L}_1}$ it follows that if $\Delta-\lim F_n = F$, then $\Delta-\lim F_n^{(k)} = F^{(k)}$ for any $k \in N$.

Let $F = [f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}_1}$. Then for each $n \in N$ we have $f_1 * \delta_n = f_n * \delta_1$. Since $\int_R \delta_n(x)dx = 1$ for each $n \in N$, we have also

$$\int_R f_1(x)dx = \int_R (f_1 * \delta_n)(x)dx = \int_R (f_n * \delta_1)(x)dx = \int_R f_n(x)dx.$$

This property allows as to define the integral of a Boehmian: if $F = [f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}_1}$ then $\int_R F(x)dx = \int_R f_1(x)dx$. For a function from $\mathcal{L}_1$ this integral is the same as the Lebesgue integral. However, there are functions which are inegrable as Boehmians but not integrable as functions. To see this, consider a continuously differentiable function from $\mathcal{L}_1$ such that its derivative is not in $\mathcal{L}_1$.

2. Fourier transform. To define the Fourier transform of an integrable Boehmian we will use Burzyk's method.

LEMMA 1. *If* $[f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}_1}$, *then the sequence*

$$\hat{f}_n(x) = \int_R f_n(t)e^{-itx}dt$$

converges uniformly on each compact set in $\mathbf{R}$.

PROOF. If (δ_n) is a delta sequence, then $(\hat{\delta}_n)$ converges uniformly on each compact set to the constant function 1. Hence, for each compact K, $\hat{\delta}_k > 0$ on K for almost all $k \in K$ and

$$\hat{f}_n = \hat{f}_n \frac{\hat{\delta}_k}{\hat{\delta}_k} = \frac{(f_n * \delta_k)\hat{}}{\hat{\delta}_k} = \frac{(f_k * \delta_n)\hat{}}{\hat{\delta}_k} = \frac{\hat{f}_k}{\hat{\delta}_k}\hat{\delta}_n \text{ on } K.$$

In view of the above lemma, the *Fourier transform of an integrable Boehmian* $F = [f_n/\delta_n]$ can be defined as the limit of $(\hat{f}_n)$ in the space of continuous functions on R. Thus, the Fourier transform of an integrable Boehmian is a continuous function.

THEOREM 2. *Let* $F, G \in \mathcal{B}_{\mathcal{L}_1}$. *Then*
(a) $(\lambda F)\hat{} = \lambda \hat{F}$ *(for any complex* λ*) and* $(F+G)\hat{} = \hat{F} + \hat{G}$,
(b) $(F * G)\hat{} = \hat{F}\hat{G}$,
(c) $(F(x-a))\hat{} = e^{-iax}\hat{F}$,
(d) $(F^{(n)})\hat{} = (-ix)^n \hat{F}$,
(e) *If* $\hat{F} = 0$, *then* $F = 0$,
(f) *If* $\Delta\text{-}\lim F_n = F$, *then* $\hat{F}_n \to \hat{F}$ *uniformly on each compact set.*

PROOF. Properties (a) through (d) follow directly from the corresponding properties for the Fourier transform in $\mathcal{L}_1$. (Note that $F \in \mathcal{B}_{\mathcal{L}_1}$ implies $F^{(n)} \in \mathcal{B}_{\mathcal{L}_1}$). To prove (e) we can use uniqueness of the Fourier transform in $\mathcal{L}_1$ or Theorem 4. From (*) it follows, that to prove (f) ir suffices to show that $\delta\text{-}\lim F_n = F$ implies $\hat{F}_n \to \hat{F}$ uniformly on each compact set. Let (δ_n) be a delta sequence such that $F_n * \delta_k, F * \delta_k \in \mathcal{L}_1$ for all $n, k \in N$ and $\|(F_n - F) * \delta_k\| \to 0$ as $n \to \infty$ for each $k \in N$ and let K be a compact set in $\mathbf{R}$. Then $\hat{\delta}_k > 0$ on K for some $k \in N$. Since $\hat{\delta}_k$ is a continuous function, it is enough to show that $\hat{F}_n \cdot \hat{\delta}_k \to \hat{F} \cdot \hat{\delta}_k$ uniformly on K. But $\hat{F}_n \cdot \hat{\delta}_k - \hat{F} \cdot \hat{\delta}_k = ((F_n - F) * \delta_k)\hat{}$ and $\|(F_n - F) * \delta_k\| \to 0$ as $n \to \infty$. The proof is complete.

To prove the inversion theorem we are going to use the following property of the Fourier transform in $\mathcal{L}_1$ (see, e.g., [**1**]).

LEMMA 3. *Let* $f \in \mathcal{L}_1$ *and*

$$f_n(x) = \frac{1}{2\pi} \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) \hat{f}(t) e^{itx} dt.$$

Then (f_n) *converges to* f *in the* $\mathcal{L}_1$ *norm.*

THEOREM 4. *Let* $F \in \mathcal{B}_{\mathcal{L}_1}$ *and*

$$f_n(x) = \frac{1}{2\pi} \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) \hat{F}(t) e^{itx} dt.$$

Then $\delta\text{-}\lim f_n = F$ *(hence also* $\Delta\text{-}\lim f_n = F$*).*

PROOF. Let $F = [g_n/\delta_n]$ and $k \in N$. Then

$$\begin{aligned}(f_n * \delta_k)(x) &= \int_R f_n(x-u)\delta_k(u)du \\ &= \frac{1}{2\pi}\int_R\int_{-n}^{n}\left(1-\frac{|t|}{n}\right)e^{it(x-u)}\hat{F}(t)\delta_k(u)dtdu \\ &= \frac{1}{2\pi}\int_{-n}^{n}\left(1-\frac{|t|}{n}\right)e^{itx}\hat{F}(t)\int_R e^{-itu}\delta_k(u)dudt \\ &= \frac{1}{2\pi}\int_{-n}^{n}\left(1-\frac{|t|}{n}\right)e^{itx}\hat{F}(t)\hat{\delta}_k(t)dt \\ &= \frac{1}{2\pi}\int_{-n}^{n}\left(1-\frac{|t|}{n}\right)e^{itx}F * \delta_k(t)dt.\end{aligned}$$

Therefore, by Lemma 3, $||f_n * \delta_k - F * \delta_k|| \to 0$ as $n \to \infty$. Since k is an arbitrary positive integer, we have proved that $\delta\text{-}\lim f_n = F$.

By (e) and (f) in Theorem 2, the family of linear continuous functionals on $\mathcal{B}_{\mathcal{L}_1}$ separates points. As a consequence we have the following

THEOREM 5. *If a function $\mathcal{F}(t)$ defined on the interval $[0,1]$ with values in $\mathcal{B}_{\mathcal{L}_1}$ is such that the derivative $\mathcal{F}'(t)$ exists and is equal to* 0 *at each point, then $\mathcal{F}$ is a constant function.*

PROOF. See [**6**, p. 155].

A similar problem for the field of Mikusinski operators instead of $\mathcal{B}_{\mathcal{L}_1}$ is still open.

REMARK. The space $\mathcal{B}_{\mathcal{L}_1}$ contains some elements which are not Schwartz distributions. Some connections between Boehmians and other types of generalized functions are discussed in [**3**] and [**4**].

Another approach to the Fourier transform of convolution quotients is presented in [**7**] and [**8**].

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