ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

DIRECT SUMS AND PRODUCTS OF ISOMORPHIC ABELIAN GROUPS

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Introduction. Suppose G is a reduced abelian group and I and J are infinite sets. When can the direct product G^I equal the direct sum $A^{(J)}$ for some subgroup A? If G is a torsion group, then G must be torsion by Corollary 2.4 in [**3**] and the answer is easy to determine. In Theorem 1 we provide an answer for all cases where $|G|$ or $|I|$ is non-measurable. We then present, in Example 2, a group decomposition $G^I = A^{(J)}$ where G is reduced and unbounded. There is another unusual decomposition of G^I which occurs whenever $|I|$ is measurable and seems worth mentioning. We do this in Example 3.

In this paper all groups are abelian. By G^I and $G^{(I)}$ we mean the direct product and direct sum respectively of copies of G indexed by I. If I is a set, then $|I|$ is measurable if there is a $\{0,1\}$-valued countably additive function μ on $P(I)$, the power set of I such that $\mu(I) = 1$ and $\mu(\{i\}) = 0$ for each $i \in I$. The letter N denotes the set of natural numbers. Unexplained terminology may be found in [**2**].

THEOREM 1. *Let G be a reduced group and let I and J be infinite sets. If $|G|$ or $|I|$ is non-measurable, then $G^I = A^{(J)}$ for some subgroup A if and only if $G = B \oplus C$, where $B^I \cong T^{(J)}$ for some bounded subgroup T and $C^I \cong C^{(J)} \cong C^k$ for some positive integer k.*

PROOF. Sufficiency is clear so we assume $G^I = A^{(J)}$ and derive the stated conditions. Write $X = \prod_I G_i = \oplus_J A_j$ where $\phi_i : G_i \to G$ is an isomorphism for each i and $A_J \cong A$ for each j.

(A) Suppose $|G|$ is non-measurable. Let $f_j : X \to A_j$ be the obvious protection and let $(S, +, \cdot)$ be the Boolean ring on $S = P(I)$. Also let $K = \{s \in S$: there is an n_s in N such that $n_s f_j(\prod_s G_i) = 0$ for almost all $j\}$ and set $H = \langle \prod_s G_i : s \in K\rangle$. Clearly K is an ideal in S. Thus H consists of the elements in G with support in K. The crucial fact for our proof is that K is a γ-ideal in S (i.e., if $\{s_n : n \in N\}$ is an

Received by the editors on July 12, 1983, and in revised form on September 25, 1985

orthogonal family in S, then $\sum_{n>k} s_n \in K$ for some k in N). For a proof of this fact, based on the work of S. Chase, see Theorem 1.5 in [3]. From Theorem 1.5 and Lemma 1.2 in [3] then we deduce: (a) S/K is finite and there are orthogonal elements $u_1, \ldots, u_k$ in S which map onto the atoms of S/K; (b) if $\{s_m : m \in M\}$ is a set of orthogonal elements in S and $|M|$ is non-measurable, then $\sum_{M'} s_m \in K$ for some cofinite subset M' of M; (c) if $K = \cup_n K_{n'}$ where $K_1 \subseteq K_2, \subseteq \ldots,$ then $K = K_k$ for some k. From (c) we conclude that $mH \subseteq \oplus_{J_1} A_j$ for some m in N and some finite subset J_1 in J. For each u_n in (a) above let $L_n = \{x_n(g) : g \in G\}$ where $x_n(g) = \sum_{u_n} \phi_i^{-1}(g)$. Plainly L_n is a subgroup of X isomorphic to G. We claim $X = L_1 \oplus \cdots \oplus L_k \oplus H$. Let $x = \sum_I x_i$ be an element in X and write $x = \sum_G (\sum_{s_g} x_i)$ where $s_g = \{i \in I : \phi_i(x_i)\} = g$. Since the family $\{s_g : g \in G\}$ partitions I by (b) above $\sum_{G'}(\sum_{s_g} x_i) \in H$ for some cofinite subset G' in G. Moreover, for $g \in G \backslash G', s_g = \sum a_n u_n + v$ with $a_n = 0$ or 1 and $v \in k$; thus $\sum_{s_g} x_i = \sum_n a_n x_n(g) + \sum_v x_i - \sum_n a_n (\sum_{u_n v} x_i)$, which is in $\sum L_n + H$. Therefore $x \in \sum L_n + H$ and $X = \sum L_n + H$. Suppose that $y_1 + \cdots + y_k + z = 0$ where each y_n is in L_n and z is in H. Since u_n is not in K but the support of z is in K, there is an i_n in u_n at which z has 0 component. Since the u_n are orthogonal, the definition fo L_n implies each Y_n is 0. Therefore $z = 0$ also and $X = L_1 \oplus \cdots \oplus L_k \oplus H$, as desired. Let I_1 be a set of k elements, one from each u_n. Then $X = \oplus_1^k L_n \oplus \prod_{I \backslash I_1} G_i = \oplus_1^k L_n \oplus H$ so $H \cong \prod_{I \backslash I_1} G_i$. We may then assume $m \prod_{I \backslash I_1} G_i \subseteq \oplus_{J_1} A_j$. Let $r = |J_1|$ and let $G = B \oplus C$ and $A = T \oplus U$ where B and T are maximal m-bounded direct summands of G and A. We can now write

(1) $X = B^I \oplus D \oplus E = T^{(J)} \oplus V \oplus W$ where $D \cong C^k, E \cong C^I, V \cong U^r, W \cong U^{(J)}$ and $mE \subseteq W$.

Now B^I and $T^{(J)}$ are maximal m-bounded summands of X so

(2) $B^I \cong T^{(J)}$ and $C^I \cong U^{(J)}$. By the Exchange Property (Theorem 72.1 in [2]) for maximal m-bounded summands $B^I \oplus D \oplus E = B^I \oplus V \oplus W$. We may assume (replace $D \oplus E$ by its projection to $V \oplus W$) that $D \oplus E$ equals $X \oplus W$. Since mE is still in W, by Lemma 1.7 in [3] we have $D \oplus E = V \oplus W'$ where $W \cong W' \subseteq D$. By the modular law,

(3) $D = D \cap V \oplus W'$ and $V = K \oplus D \cap V$ for some K.

By (1) and (3) we obtain

(4) $C^k \cong D \cap V \oplus U^{(J)} \cong D \cap V \oplus V^{(J)} = D \cap V \oplus (K \oplus D \cap V)^{(J)} \cong V^{(J)} \cong U^{(J)} \cong (U^{(J)})^{(J)} \cong (C^k)^{(J)} \cong C^{(J)}$.

Now (2) and (4) yield $C^I \cong C^{(J)} \cong C^k$.

(B) Suppose $|I|$ is non-measurable. By Corollary 1.9 in [**3**] there are positive integers k and r and decompositions $G = B \oplus C, A = T \oplus U$ with B bounded such that: $B^I \cong T^{(J)}, C^I \cong U^{(J)}$, and $U^r = K \oplus L$ where $C^k \cong L \oplus U^{(J)}$. We can show, as in (3) of part (A), that $C^k \cong U^{(J)} \cong C^{(J)}$ and the proof is complete.

We now show that G need not be bounded to satisfy the conditions of Theorem 1.

EXAMPLE 2. If I and J are infinite sets, there exists a reduced unbounded group G such that $G \cong G^I \cong G^{(J)}$.

PROOF. Consider the cartesian product $(I \times J)^N$ with typical element $(i_1, j_1, i_2, j_2, \dots)$. Let H be any unbounded reduced group. Let G be the set of all functions $f : (I \times J)^N \to H$ such that, for each k and each fixed $i_1, j_1, \dots, i_k, f(i_1, j_1, \dots, i_k, j_k, i_{k+1}, j_{k+1}, \dots) = 0$ for almost all j_k (one can think of G as $\prod_I \oplus_J \prod_I \oplus_J \dots H$). Now G is a group under component-wise addition and it is easy to see that $G \cong (G^{(J)})^I$. But this implies $G \cong G^I \cong G^{(J)}$.

If $|I|$ is measurable, then G^I, for any group G, has an unusual decomposition we would like to mention. This decomposition generalizes examples found on page 184 in [**1**] and page 161, vol. II, of [**2**].

EXAMPLE 3. Let I be a set of measurable cardinality and let G be a group. There is a decomposition $G^I = L \oplus M$ where $L \cong G$ and $G^{(I)} \subsetneq M \cong G^I$.

PROOF. Write $G^I = \prod_I G_i$ where $\phi_i : G_i \to G$ is an isomorphism for each i. Let $\mu : P(I) \to \{0, 1\}$ be a countably additive function such that $\mu(I) = 1$ and $\mu(\{i\}) = 0$ for each $i \in I$. If $x = \sum_I x_i$ is an element in G^I, write $x = \sum_G (\sum_{s_g} x_i)$ where $s_g = \{i \in I : \phi_i(x_i) = g\}$. The s_g's partition I and $\mu(s_g) = 1$ for at most one g. Define $f : G^I \to G$ by $f(x) = \sum_G \mu(s_g)g$ for each x in G^I. If two subsets of $P(I)$ have measure 1, so does their intersection. It follows that, for each x, y in $G^I, f(x+y) = f(x) + f(y)$ so f is a homomorphism. Let M be the kernel of f and let $L = \{\sum_I \phi_i^{-1}(g) : g \in G\}$, the diagonal subgroup of G^I. It is easy to see that $G^I = L \oplus M, L \cong G$ and $G^{(I)} \not\subseteq M$. If $j \in I$, then $G^I = L \oplus \prod_{i \neq j} G_i$ and $M \cong \prod_{i \neq j} G_i \cong G^I$.

I would like to thank Professor R.S. Pierce for his many helpful comments on an earlier version of this paper.

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