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INTEGRAL REPRESENTATIONS OF LINEAR FUNCTIONALS ON FUNCTION MODULES

GERHARD GIERZ

ABSTRACT. An integral representation for linear functionals on function modules is given under the condition that the function module is 'uniformly separable'. This result is a generalization of Riesz' Representation Theorem for linear functionals on $C(X)$. The results apply to spaces of (weighted) vector valued functions and to Grothendieck's G-spaces.

1. Introduction. Function modules were first introduced by R. Godement [**5**], I. Kaplansky [**7**], and M.A. Naimark [**13**] under the name of Continuous Sums. They considered spaces E of functions σ defined on a topological space X with values in given Banach spaces $E_x, x \in X$, satisfying the following axioms:

(1) E is a closed linear subspace of the Banach space $\{\sigma \in \prod_{x\in X} E_x : \sup_{x\in X} ||\sigma(x)|| < \infty\}$, equipped with the norm $||\sigma|| = \sup_{x\in X} ||\sigma(x)||$.

(2) The function $x \mapsto ||\sigma(x)|| : X \to \mathcal{R}$ is upper semicontinuous for every $\sigma \in E$.

(3) $E_x = \{\sigma(x) : \sigma \in E\}$ for every $x \in X$.

(4) E is a $C_b(X)$-module with respect to the multiplication $(f, \sigma) \mapsto f\sigma$ where $(f\sigma)(x) = f(x)\sigma(x)$ and where $C_b(X)$ denotes the algebra of all bounded continuous scalar valued functions on X.

Let us agree to call E a *function module over* X. For a given $x \in X$, the Banach space E_x is called the *stalk over* x.

Function modules are important in the representation theory of C^*-algebras (see Dauns and Hofmann [**3**]). For compact Hausdorff spaces X, the notion of function modules is also equivalent to the notion of spaces of section in a Banach bundle over X (see [**4**] for the details of this equivalence). Examples for function modules are the Banach spaces $C_b(X), C_b(X, F)$ (the space of all continuous functions with values in a given Banach space F) as well as spaces of continuous functions equipped with a weighted norm.

The object of the present note is to study the dual space of a function module. For all the examples mentioned above, integral representations

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of linear functionals are known (e.g. [**2**], [**16**]). In this paper, we will concern ourselves with the following.

PROBLEM. Let E be a function module with base space X and with stalks $E_x, x \in X$, and let $\phi : E \to \mathcal{R}$ be a bounded linear functional on E. Can we find a family of bounded linear functionals $\eta_x : E_x \to \mathcal{R}, x \in X$, and a finite Borel measure μ on X such that

(1) $x \mapsto \eta_x(\sigma(x)) : X \to \mathcal{R}$ is Borel measurable for every $\sigma \in E$,

(2) $\phi(\sigma) = \int_X \eta_x(\sigma(x))d\mu(x)$ for every $\sigma \in E$?

In the appendix of [**4**] it was shown that this is possible if X is compact and metric, or, more generally, if every Borel measure on X admits a strong lifting in the sense of [**6**]. Recently, A. Seda [**15**] has shown that the strong lifting property of the base space X is also necessary in order to represent every linear function on every function module over X by an integral.

The results just mentioned are not the best possible ones. Since V. Losert [**10**] has constructed a compact space without the strong lifting property, Seda's result implies that the result from [**4**] does not even cover Riesz' representation theorem on linear functionals on $C(X)$. In the center of these notes stands an integral representation for linear functionals which includes Riesz' theorem.

There are various different approaches to our results. For example, (3.2) is equivalent to a disintegration theorem for measures due to G. Mokobodzki [**11**]. A second proof of (3.1) would utilize the fact that the dual of a function module E is norm isomorphic to the space of all $C(X)$-module homomorphisms on E which take values in the space $\mathcal{M}(X)$ of all regular Borel measures on X. Our approach shows that it is really the M-structure of a function module which makes an integral representation of linear functionals possible.

Banach spaces are always denoted by the letters E, F, etc. The dual space of E (i.e., the space of all bounded linear functionals on E) will be denoted by E'. The polar or annihilator of a subspace $F \subset E$ will be denoted by E°. The word 'compact' always includes the Hausdorff separation axiom.

2. Preliminary Results. From now on, we fix a function module E with compact base space X and with stalks $E_x, x \in X$. For every closed subset $A \subset X$ let

$$N_A = \{\sigma \in E : \sigma(x) = 0 \text{ for all } x \in A\}.$$

Then N_A is an M-ideal of E (see [**4**, 13.6]), i.e., there is a projection $p_A : E' \to N_A^\circ$ such that

$$(L) \qquad ||\phi|| = ||p_A(\phi)|| + ||\phi - p_A(\phi)||$$

holds. In general, a projection satisfying (L) is called an L-projection; the collection of all L-projections is a complete Boolean algebra (see [**1**]).

We continue this section with a short summary of the results of the appendix of [**4**]:

If $U \subset X$ is open, define $p_U : E' \to E'$ by

$$p_U = \mathrm{id}_{E'} - p_{X \setminus U}.$$

Then p_U is the complement of the L-projection $p_{X\setminus U}$ in the Boolean algebra of all L-projections on E'. For $M \subset X$ in general, let

$$p_*(M) = \sup\{p_A : A \subset M, A \text{ closed}\},$$
$$p^*(M) = \inf\{p_U : M \subset U, U \text{ open}\},$$

where suprema and infima are taken in the Boolean algebra of all L-projections. If $p_*(M) = p^*(M)$, we let

$$p_M = p_*(M) = p^*(M).$$

The set $\mathcal{M}(E) = \{M \subset X : p_*(M) = p^*(M)\}$ is a σ-complete Boolean algebra containing the Borel sets of X. If $\phi : E \to \mathcal{R}$ is a bounded linear functional, then

$$\mu_\phi : \mathcal{M}(E) \to \mathcal{R}$$
$$M \mapsto ||p_M(\phi)||$$

is a finite σ-additive measure on $\mathcal{M}(E)$ and

$$\nu_\phi : \mathcal{M}(E) \to E',$$
$$M \mapsto p_M(\phi)$$

is a σ-additive vector-valued measure. Following ideas of J.Kupka [**9**] one can show that there is a function $\eta_\phi : X \to E'$ such that

$$\nu_\phi(M) = p_M(\phi) = \int_M \eta_\phi(x) d\mu_\phi(x)$$

in the sense that

$$p_M(\phi)(\sigma) = \int_M \eta_\phi(x)(\sigma) d\mu_\phi(x)$$

for every $\sigma \in E$. Moreover, $||p_M(\phi)|| = \mu_\phi(M) = \int_M ||\eta_\phi(X)|| d\mu_\phi(x)$, which yields $||\eta_\phi(x)|| = 1 \mu_\phi$-almost everywhere.

Now note that the evaluation map $\varepsilon_x : E \to E_x, \sigma \mapsto \sigma(x)$ is a quotient map of Banach spaces with kernel $N_{\{x\}}$ (this result is due to M. Dupre, see also [**4**, 2.10]). Hence, by duality, we may identify E'_x with a subspace of E'. Once we have carried out this identification, equations like $\phi(\sigma) = \phi(\sigma(x))$ become meaningful, provided that $\phi \in E'_x$. Thus, if we can show that $\eta_\phi(x) \in E'_x$ for almost all $x \in X$, then we will have represented ϕ via integration in the desired fashion.

PROPOSITION 2.1. *Let $\phi \in E$ be given.*

(i) *For every $\sigma \in E$ we have $\eta_\phi(x)(\sigma) \leq ||\sigma(x)||$ μ_ϕ-almost everywhere. Especially, $|\phi(\sigma)| \leq \int_X ||\sigma(x)|| d\mu_\phi$.*

(ii) *For every $f \in C(X), \sigma \in E$ we have $\eta_\phi(f\sigma) = (f\eta_\phi)(\sigma)$ μ_ϕ-almost everywhere.*

PROOF. In order to verify (i), we have to show that, for every measurable $M \in \mathcal{M}(E)$, the inequality

$$\left| \int_M \eta_\phi(x)(\sigma) d\mu_\phi(x) \right| \leq \sup_{x \in M} ||\sigma(x)|| \int_M d\mu_\phi(x)$$

holds. This inequality may be rewritten as

$$(*) \qquad |p_M(\phi)(\sigma)| \leq ||p_M(\phi)|| \sup_{x \in M} ||\sigma(x)||.$$

We prove (*): Let $A \subset X$ be closed. Then $p_A : E' \to N_A^\circ$ is an L-projection onto $N_A^\circ = (E/N_A)'$. Since we have $||\sigma + N_A|| = \sup_{x \in A} ||\sigma(x)||$ (see [**4,4.5**]), we obtain $|p_A(\phi)(\sigma)| \leq ||p_A(\phi)||$ $||\sigma + N_A|| = ||p_A(\phi)|| \sup_{x \in A} ||\sigma(x)||$.

If $M \in \mathcal{M}(E)$ is arbitrary, then we have

$$p_M(\phi) = \lim\{p_A(\phi) : A \subset M, A \text{ closed}\}$$

(see [**4,21.8**]), hence the result follows in this case from continuity.

For a proof of (ii), we have to verify that, for every $M \in \mathcal{M}(E)$, the equation

$$\int_M \eta_\phi(f\sigma)d\mu_\phi = \int_M (f\eta_\phi)(\sigma)d\mu_\phi$$

holds. Thus, let $\varepsilon > 0$. For every integer n we define $A_n = \{x \in M : n\varepsilon \leq f(x) < (n+1)\varepsilon\}$. Using (*) again, we obtain

$$\begin{aligned}
&\left|\int_{A_n} (\eta_\phi(f\sigma) - (f\eta_\phi(\sigma))d\mu_\phi\right| \\
&\leq \left|\int_{A_n} (\eta_\phi(f\sigma - (n+1/2)\varepsilon\sigma))d\mu_\phi\right| + \left|\int_{A_n} ((n+1/2)\varepsilon - f)\eta_\phi(\sigma)d\mu_\phi\right| \\
&\leq |p_{A_n}(\phi)((f-(n+1/2)\varepsilon)(\sigma)| + \int_{A_n} \frac{\varepsilon|\eta_\phi(\sigma)|}{2}d\mu_\phi \\
&\leq ||p_{A_n}(\phi)|| \sup_{x\in A_n} ||(f(x) - (n+1/2)\varepsilon)\sigma(x)|| + \varepsilon||p_{A_n}(\phi)||/2 \\
&\leq \varepsilon||p_{A_n}(\phi)||/2 + \varepsilon||p_{A_n}(\phi)||/2 \\
&= \varepsilon||p_{A_n}(\phi)||.
\end{aligned}$$

Since M is the union of the A_n and since $A \mapsto ||p_A(\phi)||$ is countably additive, we obtain

$$\left|\int_M (\eta_\phi(f\sigma) - (f\eta_\phi))(\sigma))d\mu_\phi\right| \leq \varepsilon||p_M(\phi)||.$$

Since $\varepsilon > 0$ was arbitrary, this is as desired.

3. The main result and applications to spaces of vector valued functions. Let us consider again a bounded linear function $\phi : E \to \mathcal{R}$ on a function module E. We construct the function $\eta_\phi : X \to \mathcal{R}$ and the measure μ_ϕ as in §2. We then know from (2.1.(i)) that $|\eta_\phi(\sigma)| \leq ||\sigma(x)||$ for all $x \in X\backslash N$, where N is a set of measure 0 depending on σ. Let us suppose for a moment that the set N would not depend on σ. Then we could set $\eta_\phi(x) = 0$ for $x \in N$. We would obtain $|\eta_\phi(x)(\sigma)| \leq ||\sigma(x)||$. Especially, $\sigma(x) = 0$ would imply $\eta_\phi(\sigma) = 0$, i.e., $\eta_\phi(x) \in N^\circ_{\{x\}} = E'_x$. We could write $\phi(\sigma) = \int_X \eta_\phi(x)(\sigma(x))d\mu_\phi(x)$ and we would have found an integral representation of ϕ.

PROPOSITION 3.1. *Let E be again a function module over a compact base space X and let $\phi : E \to \mathcal{R}$ be bounded linear functional. Furthermore, assume that E admits a subspace F such that*

(a) $|\eta_\phi(x)(\sigma)| \leq ||\sigma(x)||$ *for all* $x \in X, \sigma \in F$;

(b) $\{\sigma(x) : \sigma \in F\}$ *is dense in the stalk* E_x *for every* $x \in X$. *Then there is a maping* $\xi : X \to E'$ *such that*

(i) $||\xi(x)|| \leq 1$ *for all* $x \in X$,

(ii) $\xi(x) \in E'_x$ *for all* $x \in X$,

(iii) $p_M(\phi)(\sigma) = \int_M \xi(x)(\sigma(x))d\mu_\phi(x)$ *for all* $\sigma \in E, M \in \mathcal{M}(E)$.

PROOF. For a given $x \in X$ let $F_x = \{\sigma(x) : \sigma \in F\}$. We define a linear functional on F_x by

$$\xi_x : F_x \to \mathcal{R}$$
$$\sigma(x) \mapsto \eta_\phi(x)(\sigma).$$

Then property (a) implies that ξ_x is well defined and has norm no larger than 1. Let $\xi(x) : E_x \to \mathcal{R}$ be the unique continuous extension of ξ_x to F_x. Clearly, $||\xi(x)|| \leq 1$ for all $x \in X$. It remains to show that the function $x \mapsto \xi(x)(\sigma(x))$ is μ_ϕ-integrable for every $\sigma \in E$ and that (iii) holds. First notice that it is enough to verify that for every $\sigma \in E$ we have $\xi(x)(\sigma(x)) = \eta_\phi(x)(\sigma), \mu_\phi$-almost everywhere. We will consider three cases.

Case 1. ($\sigma \in F$). In this case we even have $\xi(x)(\sigma(x)) = \eta_\phi(x)(\sigma)$ for all $x \in X$.

Case 2. ($\sigma = \sum_{i=1}^n f_i\sigma_i$, where $f_i \in C(X), \sigma_i \in F, 1 \leq i \leq n$). We obtain

$$\begin{aligned}
\xi(x)(\sigma(x)) &= \xi(x)(\sum_{i=1}^n (f_i\sigma_i)(x)) = \sum_{i=1}^n f_i(x)\xi(x)(\sigma_i(x)) \\
&= \sum_{i=1}^n f_i(x)\eta_\phi(x)(\sigma_i) = \sum_{i=1}^n (f_i\eta_\phi)(x)(\sigma_i) \\
&= \sum_{i=1}^n \eta_\phi(x)(f_i\sigma_i) \quad \mu_\phi\text{-almost everywhere by (2.1)} \\
&= \eta(x)(\sum_{i=1}^n f_i\sigma_i) = \eta(x)(\sigma).
\end{aligned}$$

Case 3. ($\sigma \in E$ arbitrary). Since by the Stone-Weierstrass theorem for bundles (see [**4,4.3**]) the elements of the form $\sum_{i=1}^n f_i\sigma_i, f_i \in C(X), \sigma_i \in F$ are norm dense in F, this case follows from Case 2 and the fact that a countable union of sets of measure 0 is again of measure 0.

In order to formulate the next theorem, we need another notation. Let us suppose that we are given a function module E over X and let us assume that E admits a countable subspace $F \subset E$ such that $F_x = \{\sigma(x) : \sigma \in F\}$ is dense in the stalk over x for every $x \in X$. In this case we will call E a *uniformly separable function module.* Examples for uniformly separable function modules are spaces of the form $C(X), C(X,G)$, where G is separable in the usual sense, and spaces of section in locally trivial n-dimensional vector bundles.

THEOREM 3.2. *Let E be a uniformly separable function module and let $\phi : E \to \mathcal{R}$ be a bounded linear functional. Then there exists a regular Borel measure μ_ϕ on X and a family $\xi_\phi(x) \in E'_x$ of linear functionals on the stalks of norm at most 1 such that*

(i) $x \mapsto \xi_\phi(x)(\sigma(x))$ *is Borel-measureable for every* $\sigma \in E$,
(ii) $\phi(\sigma) = \int_X \xi_\phi(x)(\sigma(x))d\mu_\phi(x)$.

PROOF. Since E is uniformly separable, we can find a countable subspace $F_0 \subset E$ over the field of rationals such that $\{\sigma(x) : \sigma \in F_0\}$ is dense in E_x for every $x \in X$. Let η_ϕ be constructed as in §2. Using (2.1), we can find a set $N \subset X$ of μ_ϕ-measure 0 such that $|\eta_\phi(x)(\sigma)| \le ||\sigma(x)||$ holds for all $x \in X \backslash N$ and all $\sigma \in F_0$. We alter the function $\eta_\phi : X \to E'$ on N by letting it be constant 0 there. Hence we may assume that

$$|\eta_\phi(x)(\sigma)| \le ||\sigma(x)|| \text{ for all } x \in X \text{ and all } \sigma \in F_0.$$

Clearly, this last inequality carries over to the uniform closure F of F_0 which is a real subspace of E satisfying the conditions (a) and (b) of (3.1). Hence (3.1) yields all the assertions of (3.2) with the exception of the Borel-measurability of the functions $x \mapsto \xi_\phi(x)(\sigma(x)), \sigma \in E$. The fact that we indeed can choose ξ_ϕ to be weak-*-Borel measurable follows as in [**4**, 21.21].

Let us close these notes by pointing out a few applications of (3.1) and (3.2). Firstly, in the case where $E = C(X,G)$ we do not have to insist on E being uniformly separable, or, equivalently, on G being separable. In this case, the constant functions $c_u, u \in G$, form a subspace $F \subset C(E,G)$ satisfying the assumptions of (3.1).

COROLLARY 3.3. *Let X be a compact space and let G be a Banach space. Then for every bounded linear functional ϕ on $C(X,G)$ there* *exists a positive regular Borel measure* μ_ϕ *and a weak-*-*μ_ϕ*-integrable function* $\xi_\phi : X \to G'$ *such that* $\phi(\sigma) = \int_X \xi_\phi(x)(\sigma(x))d\mu_\phi(x)$ *for every* $\sigma \in C(X,G)$. *If* G *is separable, then* ξ_ϕ *can be chosen to be weak-*-Borel measurable.*

Our next corollary deals with weighted function spaces. Again, let X be a compact space and let $\omega : X \to \mathcal{R}$ be a strictly positive upper semicontinuous function. Let $C_\omega(X)$ be the completion of $C(X)$ in the norm

$$||f||_\omega = \sup_{x\in X} \omega(x)|f(x)|.$$

Then $C_\omega(X)$ is a uniformly separable function module over X. All the stalks of this function module are isomorphic to $\mathcal{R}$.

COROLLARY 3.4. $C_\omega(X)' \simeq \mathcal{M}(X)$. *Under this identification, a regular Borel measure* μ *operaters on* $C_\omega(X)$ *by* $\sigma \mapsto \int_X \omega(x)\sigma(x)d\mu(x)$.

It should be pointed out that these corollaries are of course not new. These results may be found in the papers of R. Buck [**2**], J. Wells [**17**], W. Summers [**16**], and G. Kleinstuck [**8**].

Our last aplication deals with Grothendieck's G-spaces. Recall that a closed linear subspace $G \subset C(K), K$ compact, is called a *G-space*, provided that there are triples $(x_i, y_i, r_i) \in K \times K \times \mathcal{R}, i \in I$, such that $G = \{f \in C(K) : f(x_i) = r_i f(y_i) \text{ for all } i \in I\}$. In order to avoid technical difficulties, we will assume that 0 is not in the weak-*-closure of the extreme points of the dual unit ball of G; as a matter of fact, every G-space can be 'approximated' by G-spaces with this property. In this case G can be represented as a function module over a compact space X with one-dimensional stalks in such a way that all extreme points of the dual unit ball are given by point evaluations at points $x \in X$ (see [**12**, 3.6(vi)] and [**12**,4.1]). Let us call this representation the *canonical representation* of G.

An integral representation for linear functionals on G-spaces is known in the separable case [**12**]. We are now able to extend these results in the following way:

COROLLARY 3.5. *Let* G *be a* G*-space such that* 0 *is not a weak-*-limit of the extreme points of the dual unit ball. Assume that* G *is represented in the canonical way as a function module over a compact space* X. *Then for every bounded linear function* ϕ *on* G *there exists a* *positive Borel measure μ on X and a family $(\xi_x \in E'_x \simeq \mathcal{R})_{x\in X}$ such that*

(i) $x \mapsto \xi_x(\sigma(x))$ *is Borel measurable for every* $\sigma \in G$.

(ii) $\phi(\sigma) = \int_X \xi_x(\sigma(x))d\mu(x)$.

PROOF. We only have to show that the canonical function module representing G is uniformly separable. To this end, let $U \subset G'$ be a weak-*-open neighborhood of 0 missing all extreme points of the dual unit ball. We then can pick $\sigma_1, \ldots, \sigma_n \in G$ such that $\{\psi \in G' : |\psi(\sigma_i)| < 1, 1 \leq i \leq n\} \subset U$. Hence for every extreme point π of the dual unit ball there exists an i such that $|\pi(\sigma_i)| \geq 1$. For the canonical function module, the extreme points are exactly the mappings $\pm\varepsilon_x, x \in X$, where ε_x denotes point evaluation. We now may conclude that for every $x \in X$ there is an $1 \leq i \leq n$ such that $\sigma_i(x) \neq 0$. It follows that the rational linear span of $\{\sigma_1, \ldots, \sigma_n\}$ is a countable set $F \subset G$ such that $\{\sigma(x) : \sigma \in F\}$ is dense in the stalk $E_x \simeq \mathcal{R}$ for every $x \in X$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521.