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PROJECTIVE MAPPINGS ON DIFFERENTIABLE MANIFOLDS

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ABSTRACT. A mapping $f : M \to M'$ between two C^∞-manifolds is quasi-projective if it carries geodesics to geodesics, and if, in addition, it preserves the projective parameter, it is called projective. Such a mapping is known to relate the symmetric affine connections of M and M', and is characterised by a relation between the Schwartzian differential (for the parameters) and the Ricci curvatures of $M$and M'. We use these facts to establish the non-existence and existence of projective maps. For instance we show that f is not projective if there does not exist a solution to the non-linear non-homogeneous equation given by the Schwartzian differential and the Ricci tensor; it is projective if f is a diffeomorphism and the Schwartzian differential formed by the projective parameters is zero. We also use the collection of projective maps on M to define an action integral on it and show that the extremal of this action is a Levi-Civita connection. Finally, we prove that if energy-momentum tensors and sectional curvatures are suitably restricted then a quasi-projective (projective) mapping can be volume (distance) decreasing.

Given a C^∞-manifold M, consider the group Diff(M) of diffeomorphisms of M. Let $\mathcal{I}(M)$ and $\mathcal{A}(M)$ denote the group of isometrics and affine mappings on M. Then the following inclusion relation is a classical fact:

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathrm{Diff}(M).$$

A diffeomorphism which (1) carries geodesics to geodesics and (2) preserves the projective parameter p (up to linear fractional transformations) is called projective. The set of all these diffeomorphisms is a group denoted $\mathcal{P}(M)$. It is easy to see that

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathcal{P}(M) \subset \mathrm{Diff}(M).$$

The class of diffeomorphisms which does not necessarily satisfy (2) also forms a group denoted $\tilde{\mathcal{P}}(M)$, evidently $\tilde{\mathcal{P}}(M) \supset \mathcal{P}(M)$. Thus we have

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathcal{P}(M) \subset \tilde{\mathcal{P}}(M) \subset \mathrm{Diff}(M).$$

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Let (M, ∇) and (M', ∇') be two different C^∞-manifolds (dim M not necessarily equal to dim M') with symmetric affine connections. It is known that a smooth mapping $f : (M, \nabla) \to (M', \nabla')$ which carries geodesics of M to geodesics of M' preserves the projective parameter p only if f is a diffeomorphism. The connection ∇' in this case is the image of ∇ under the diffeomorphism, and ∇, ∇' are called projectively equivalent. We must mention, however, that f being a diffeomorphism is not a sufficient condition for the projective invariance of p. We shall see in §1 that a 1-form ω on M (an essential ingredient of projective equivalence) has to be a gradient in this case.

For all manifolds M which are diffeomorphic to spherical or pseudo-spherical space-forms, the question of $\mathcal{P}(M)$ and $\tilde{\mathcal{P}}(M)$ is pretty much settled throughout their covering manifolds. But for manifolds other than that, not much is known, even though the concept of projective connection (normal projective connection) and projective parameter can almost be referred to as ancient in the present context [**20,2,5,17,21,17**].

Recently, Kobayashi [**12,13**] has used the concept of projective equivalence to define the pseudo-distance d_M (associated with symmetric affine connections) by mapping Poincare-type intervals (i.e., the interval $-1 < u < 1$ with metric $ds^2 = 4du^2/(1-u^2)^2$) into the manifold M. He has shown that if Ric_M (Ricci tensor of M) is negative and bounded away from zero, then d_M is a complete distance.

Replacing the interval by a complete Riemannian-manifold these mappings have been shown to be distance decreasing when Ricci curvatures of M and M' are suitably restricted [**12,9**]. In each of these papers the equality (to be explained later)

$$\{p, s\} = R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \tag{i}$$

has played an important role. Even though this equality was derived some five decades earlier, in our understanding, it has not been exploited exhaustively. Since projective diffeomorphism between two manifolds leads to a relation between their Ricci tensors, and Ricci tensors in turn are related to matter, we expect that this simple study with (i) as its starting point will be of physical interest.

We raise and answer the following questions:

1) What are the conditions for a C^∞-map between two manifolds to be projective; also when does $f \in \tilde{\mathcal{P}}(M)$ ($F \in \tilde{\mathcal{P}}(M)$ will be referred to as 'quasi-projective'.) (§1)?

2) Are there manifolds which cannot be projectively related (§2)?

3) Is there any relation between projective maps and other better known maps such as affine, harmonic (§1) or conformal (§4)?

4) How does a projective map influence the metric (§3)?

5) Does the presence of matter affect projective maps (§5)?

6) What is the requirement on the geometry of a manifold so that an incomplete connection of one manifold may be projectively equivalent to a complete one on this manifold (§4)? The answer to this question suggests that projective maps can be used as a mechanism to obtain manifolds with complete connection from those with incomplete ones.

As the study of projective maps is indeed the study of a vector space defined by $(f^*\nabla' - \nabla)$ that reduces to the subspace given by $\omega \otimes df$ for a 1-form ω on M, we treat $\omega \otimes df$ as basic and obtain its local description in §1. The techniques that we use in this study are variational or Bochner-type. We begin the paper with preliminaries designated as §0.

0. Throughout the paper all objects are C^∞, and all manifolds which may or may not be equi-dimensional carry symmetric affine connections. The vector fields/tangent vectors are denoted as U, V or X, Y and 1-forms as ω, η. The very same alphabets with appropriate indices are used to denote their local counterparts. The set $\{e_i\}$ denotes an orthonormal basis. The notations for the Riemannian (curvature), the Ricci and the scalar curvature are the standard R, Ric and S.

Recall (Theorem of Weyl) that the torsion free linear connections ∇' and ∇ on M are projectively equivalent if and only if there exists a 1-form ω on M such that

$$\nabla'_U V - \nabla_U V = \omega(U)V + \omega(V)U. \tag{0.1}$$

This result translated for distant manifolds reads: A mapping f : $(M, \nabla) \to (M', \nabla')$ is quasi-projective if and only if there exists a 1-form ω on M such that [**9,16**]

$$\nabla'_{f_* U} f_* V - f_* \nabla_U V = \omega(U) f_* V + \omega(V) f_* U. \tag{0.2}$$

In terms of the tangent vector γ to a curve $\gamma \equiv \gamma(t)$ this equation can be put as

$$\nabla'_{f_*\dot\gamma} f_*\dot\gamma - f_*\nabla_{\dot\gamma}\dot\gamma = 2\omega(\dot\gamma) f_*(\dot\gamma),$$

and therefore, if $f_*\dot\gamma \neq 0$ and γ is a geodesic, this gives,

$$\omega(\dot\gamma) = (\nabla'_{f_*\dot\gamma} f_*\dot\gamma)\ \frac{1}{2f_*\dot\gamma}. \tag{0.3}$$

For suitable choice of frames of references, (0.2) reduces to the familiar classical equation (see §2)

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_k\omega_j + \delta^i_j\omega_k \tag{0.4}$$

for quasi-projectively related connections. Also, simple computations based on (0.2) lead to the following relation between the Riemannian tensors of M and M':

$$\begin{aligned} R'(f_*U, f_*V)f_*W =& f_*\{R(U,V)W + d\omega(U,V)W \\ &+ (\omega)(U,W)V - (\omega)(V,W)U\}, \end{aligned} \tag{0.5}$$

where

$$(\omega)(U,V) = (\nabla\omega - \omega\otimes\omega)(U,V) \tag{0.6}$$

and

$$(d\omega)(U,V) = (\nabla\omega(U,V) - (\nabla\omega)(V,U). \tag{0.7}$$

The trace: $\operatorname{tr}(U \to R(U,V)W) \equiv$ Ric (V,W) gives the relation

$$f_* \text{ Ric}' = \text{ Ric} + d\omega + (n-1)\omega. \tag{0.8}$$

When f is projective, (0.8) simplifies to

$$(f^* \text{ Ric}')(\dot\gamma,\dot\gamma) = \text{ Ric }(\dot\gamma,\dot\gamma) - \frac{n-1}{2}S_\phi, \tag{0.9}$$

where $S_\phi \equiv \{\phi, t\}$ denotes the Schwartzian differential of the affine parameter ϕ used for parametrisation of $f\circ\gamma \equiv \gamma'$.

Equations (0.8) and (0.9) are derived in the next section. The following results about projective mappings between equidimensional manifolds are known [**9**].

THEOREM A. *Let M be complete and let the Ricci curvatures of M and M' be bounded below and above by constants $-A$ and $-B < 0$. If*

M and M' are Riemannian and $A > 0$, then f is distance decreasing up to a constant $(A/B)^{\frac{1}{2}}$.

THEOREM B. *If f is quasi-projective, then under the same hypothesis f is volume decreasing up to a constant $(A/B)^{n/2}$.*

1. Let $f : (M_n, \nabla) \to (M'_{p'}, \nabla')$ be a mapping which carries the paths (geodesics) in M to paths (geodesics) in M'. We use the word path to emphasize that M_n and M'_p may not necessarily be Riemannian. They are c^∞-manifolds with symmetric affine connections. A local description of f is obtained as follows: Take coordinate neighborhoods $\{U, x^i\}(i = 1, 2 \ldots n)$ of M and $\{V, y^\alpha\}(\alpha = 1, 2 \ldots p)$ of M' in such a manner that $f(U) \subset V$. Suppose that in the choice of charts f is represented by the equations

$$(1.1) \qquad y^\alpha = y^\alpha(x^1, \ldots, x^n), \quad \alpha = 1, \ldots, p.$$

Then, writing

$$(1.2) \qquad B_i^\alpha = \frac{\partial y^\alpha(x^1 \ldots x^n)}{\partial x^i},$$

we note that $df \equiv f_*$ is represented by the matrix (B_i^α).

Accordingly, any function ψ on M' can be identified with a function $\psi \circ f$ on M, and a vector field X on M with local representation $X = X^i \frac{\partial}{\partial X^i}$ is mapped locally to $(B_i^\alpha X^i)\frac{\partial}{\partial y^\alpha}$ on $f(M)$. We denote the components of ∇ and ∇' by Γ^i_{jk} and $\Gamma^\alpha_{\beta\gamma}$ and write the covariant derivative of B_i^α with respect to ∇, thus

$$(1.3) \qquad \nabla_j B_i^\alpha = \frac{\partial B_i^\alpha}{\partial x^j} + B_i^\beta B_j^\gamma \Gamma^\alpha_{\beta\gamma} - B_k^\alpha \Gamma^k_{ij}.$$

We denote it as B^α_{ji} noting that $B^\alpha_{ji} = B^\alpha_{ij}$ and that $(B^\alpha_{ji} X^j Y^i)\frac{\partial}{\partial y^\alpha}$ is the local expression of some vector field Z defined along $f(M)$.

Now a geodesic $\gamma : I \to M$ given by $x^i = x^i(t)$ satisfies the equation

$$(1.4) \qquad \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = a(t)\frac{dx^i}{dt}$$

for some function $a(t)$, which by a suitable choice of parameter t (called affine) can be reduced to

$$(1.5) \qquad \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

In view of our definition of f, the image $f \circ \gamma : I \to f(M)$, written locally $y^\alpha = y^\alpha(x^i(t))$, satisfies a similar equation,

$$(1.6) \qquad \frac{d^2 y^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma} \frac{dy^\beta}{dt} \frac{dy^\gamma}{dt} = A(t) \frac{dy^\alpha}{dt}$$

for some function $A(t)$ on the geodesic $\gamma' = f \circ \gamma$. But using (1.3) we have

$$(1.7) \quad \frac{d^2 y^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma} \frac{dy^\beta}{dt} \frac{dy^\gamma}{dt} = B^\alpha_k \left(\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) + B^\alpha_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

This shows that if t is an affine parameter for γ', then

$$(1.8) \qquad A(t) \frac{dy^\alpha}{dt} = B^\alpha_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

Since f is defined on an arbitrary geodesic γ we can write (1.8) as

$$(1.9) \qquad A(t) B^\alpha_i \eta^i = B^\alpha_{ij} \eta^i \eta^j$$

for any direction $\eta = \eta_i(\frac{\partial}{\partial x^i})$ at any point (covered by $\{U, x^i\}$) of M. This implies that there exist local functions ω_i in U which are components of a 1-form ω on M and which satisfy

$$(1.10) \qquad B^\alpha_{ij} = \omega_j B^\alpha_i + \omega_i B^\alpha_j.$$

We have thus shown

PROPOSITION 1.1. *A mapping* $f : (M_n, \nabla) \to (M'_p, \nabla')$ *carries geodesics to geodesics if and only if* (1.10) *holds* [**17**].

From the above equation we can compute the relation between the curvature tensors on U and $f(U)$ and the components of a differential 1-form.

If M and M' are equidimensional and f is non-degenerate we can also write the Ricci-curvature relation from

$$(1.11) \qquad {}'R^\alpha_{\delta\gamma\beta} B^\delta_k B^\gamma_j B^\beta_i - R^h_{kji} B^\alpha_h = \nabla_k B^\alpha_{ji} - \nabla_j B^\alpha_{ki}$$

by multiplying it with $\overline{B}^i_\alpha$ (the inverse of B^α_i) and summing up for repeated indices. We thus have

$$(1.12)\qquad 'R_{\delta\gamma}B^\delta_k B^\gamma_j - R_{kj} = (\omega_{j,k} - \omega_{k,j}) + (n-1)(\omega_{j,k} - \omega_j\omega_k).$$

Returning to equation (1.8), we observe

PROPOSITION 1.2. *The parameter t which is affine parameter for ∇-geodesic γ will be affine for ∇'-geodesic $f \circ \gamma = \gamma'$ as well, if and only if*

$$(1.13)\qquad B^\alpha_{ij}\frac{dx^i}{dt}\frac{dx^i}{dt} = 0.$$

The mapping f in this case is called affine. We are however interested in the projective parameter and the mappings which preserve the projective parameter. We recall that simple examples of projective parameter are provided by the Poincare and Klein models of hyperbolic space H^n. In the first case the projective parameter $p = \tanh s(-1 < p < 1)$, where s is the hyperbolic arc-length of the geodesic circle orthogaonal to the rim. In the second case, geodesics are segments of straight lines, and p is the Euclidean arc-length.

To illustrate the point that we made in the introduction that all diffeomorphisms which satisfy condition (1) do not necessarily satisfy (2) we first consider quasi-projective mappings. We shall therefore prove

PROPOSITION 1.3. *A quasi-projective mapping $f : (M,\nabla) \to (M',\nabla')$ is projective only if f is a diffeomorphism between M and M'. In preparation for proving this result we recall the classical concept of projective invariance of the projective parameter. For projectively related connections*

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j\omega_k + \delta^i_k\omega_j$$

on a manifold, the Ricci tensors ($\sum_i R^i_{jki}$) are given by

$$(1.14)\qquad 'R_{ij} = R_{ij} + (\omega_{i,j} - \omega_{i,j}) + (n-1)(\omega_{i,j} - \omega_i\omega_j).$$

It is apparent that both of these are symmetric simultaneously if and only if the covariant vector ω_i is a gradient. If s and $\tilde{s}$ denote the affine parameters for the geodesics represented by Γ^i_{jk} and Γ'^i_{jk}, an easy computation leads to the functional relation

$$\frac{d\tilde{s}}{ds} = e^{2\int \omega_j dx^j} \tag{1.15}$$

which shows its dependence on ω_i, the convector of the projective relation. On the other hand, the projective parameters p and $\tilde{p}$ which are solutions of the non-homeogeneous third-order non-linear differential equations

$$\{p, s\} = \frac{2}{n-1} R_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{1.16}$$

$$\{\tilde{p}, \tilde{s}\} = \frac{2}{n-1} {}'R_{ij} \frac{dx^i}{d\tilde{s}} \frac{dx^j}{d\tilde{s}} \tag{1.17}$$

(R_{ij} and ${}'R_{ij}$ both being symmetric) are related to each other by a linear fractional transformation, e.g.,

$$\tilde{p} = \frac{(ap+b)}{(cp+d)},$$

where a, b, c, d are real numbers ≥ 0 constrained by $ad - bc \neq 0$. (Transformations of this nature (referred to as conformal Mobius transformations in the case of the 2-plane) have a long history. For a more recent study see [**0**].) The elimination of a, b, c, d leads to the third order differential equation

$$\left(\frac{d^3\tilde{p}}{dp^3}\right) / \frac{d\tilde{p}}{dp} - \frac{3}{2}\left(\left(\frac{d^2\tilde{p}}{dp^2}\right) / \frac{d\tilde{p}}{dp}\right)^2 \equiv \{\tilde{p}, p\} = 0. \tag{1.18}$$

This equation, called the Schwartzian differential equation, has been derived and used in different context ever since 1834, (see [**7**]), though its use for the projective parameter was made for the first time in 1931 by Whitehead [**21**].

In the lemma given below we show that (1.16) and (1.17) lead to the projective invariance of the parameter p.

LEMMA 1.4. *Let p and $\tilde{p}$ be projective parameters for the system of geodesics given by projectively related connections. Then $\{\tilde{p}, p\} = 0$.*

PROOF. It is easy to check that any three variables u, v, w related by a Schwartzian differential satisfy the identity

$$\{w, v\} = \{w, u\}\left(\frac{du}{dv}\right)^2 + \{u, v\}. \tag{1.19}$$

We use it appropriately for variables $s, \tilde{s}$ and $p, \tilde{p}$. Thus,

$$\{\tilde{p}, \tilde{s}\} = \{\tilde{p}, p\}\left(\frac{dp}{d\tilde{s}}\right) + \{p, \tilde{s}\} \tag{1.19a}$$

and

$$\{p, \tilde{s}\} = \{p, s\}\left(\frac{ds}{d\tilde{s}}\right)^2 + \{s, \tilde{s}\}. \tag{1.19b}$$

Or

$$\{\tilde{p}, \tilde{s}\} = \{\tilde{p}, p\}\left(\frac{dp}{d\tilde{s}}\right)^2 + \{p, s\}\left(\frac{ds}{d\tilde{s}}\right)^2 + \{s, \tilde{s}\}. \tag{1.20}$$

We then substitute the values of $\{p, s\}$ and $\{\tilde{p}, \tilde{s}\}$ from (1.16) and (1.17) to obtain

$$\{\tilde{p}, p\}\left(\frac{dp}{ds}\right)^2 = \frac{2}{n-1}(-{}^{R}_{ij} + R_{ij})\frac{dx^i}{ds}\frac{dx^j}{ds} + \{s, \tilde{s}\}\left(\frac{d\tilde{s}}{ds}\right)^2. \tag{1.21a}$$

This can be written as

$$\{\tilde{p}, p\}\left(\frac{dp}{ds}\right)^2 = \frac{2}{n-1}(-{}'R_{ij} + R_{ij})\frac{dx^i}{ds}\frac{dx^i}{ds} - \{\tilde{s}, s\}. \tag{1.21b}$$

in view of the identity $\{s, \tilde{s}\}(\frac{d\tilde{s}}{ds})^2 = -\{\tilde{s}, s\}$. But

$$\{\tilde{s}, s\} = \frac{d^2}{ds^2}\left(\log\left(\frac{d\tilde{s}}{ds}\right)\right) - \frac{1}{2}\left(\frac{d}{ds}\left(\log\frac{d\tilde{s}}{ds}\right)\right)^2. \tag{1.22}$$

Hence using the value of $\frac{d\tilde{s}}{ds}$ given by (1.15) and the fact that the 1-form ω is a gradient there, we have

$$\begin{aligned}
\{\tilde{s}, s\} &= \frac{d}{ds}\left(2\omega_i\frac{dx^i}{ds}\right) - \frac{1}{2}\left(4\omega_i\frac{dx^i}{ds}\omega_j\frac{dx^j}{ds}\right) \\
&= 2\left(\frac{\partial\omega_i}{\partial x^i}\frac{dx^j}{ds}\frac{dx^j}{ds} + \omega_i\frac{d^2x^i}{ds^2}\right) - 2\left(\omega_i\omega_j\frac{dx^i}{ds}\frac{dx^j}{ds}\right) \\
&= 2\left(\omega_{i,j} + \Gamma^m_{ij}\omega_m\right)\frac{dx^i}{ds}\frac{dx^j}{ds} + \omega_i(-\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}) \\
&\quad - 2\omega_i\omega_j\frac{dx^i}{ds}\frac{dx^j}{ds} = 2(\omega_{i,j} - \omega_i\omega_j)\frac{dx^i}{ds}\frac{dx^j}{ds}.
\end{aligned} \tag{1.23}$$

Finally, in view of (1.14) and (1.23), we observe that the right side of (1.21) is zero, and since $\frac{dp}{ds} \neq 0$ we have the required result

$$\{\tilde{p}, p\} = 0.$$

A transformation of M with this property is called a projective transformation.

The following simple corollary is self-evident from (1.14), (1.15) and (1.21b).

COROLLARY 1.5. *Every affine transformation is a projective transformation.*

Before proceeding further, we would like to point out that to write the equations of geodesics using the projective parameter p, we shall have to replace the connection coefficients Γ^i_{jk} by connection coefficients $\prod^\alpha_{\beta\gamma}$ of an $(n+1)$-dimensional manifold. This is done by introducing a variable x^0 given by

$$x^0 = \frac{1}{2\lambda} \log \frac{ds}{dp}$$

and $(n^2+n)/2$ point functions Γ^0_{jk} satisfying the equation

$$\{p, s\} = 2\lambda \Gamma^0_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}.$$

The functions Γ^0_{jk} are symmetric tensors and λ is an arbitrary constant. The coordinate transformation rule in the $(n+1)$-dimensional manifold is given by

$$\begin{aligned} d\overline{x}^0 &= dx^0 + \frac{1}{m}\omega_i dx^i \\ \overline{x}^i &= \overline{x}^i(x^1 \dots x^n) \ \left(\frac{\partial(\overline{x}' \dots \overline{x}^n)}{\partial(x^1 \dots x^n)} \neq 0\right). \end{aligned}$$

We must add, however, that this embedding is not required for the study that we are pursuing in this paper.

PROOF OF PROPOSITION 1.3. We note that, in proving the above lemma, (1.21b) played a crucial role. This means that to prove this proposition we must establish a similar relation between the Ricci tensors of M and M'.

Since f is quasi-projective we have (1.10) and the subsequent equation (1.11) satisfied by the Riemannian tensors of M and M'. Evidently, equation (1.12) relating the Ricci tensors follows only when f is a diffeomorphism.

We obtain in the next proposition the sufficiency condition for the diffeomorphism f to be a projective map.

PROPOSITION 1.6. *Let $f : (M, \nabla) \to (M', \nabla')$ be a diffeomorphism which carries geodesics to geodesics; f is projective if one of the following conditions holds*

1) Ric_∇ *is symmetric and the equation*

$$\{\tilde{s}, s\} = 2(\omega_k\omega_j - \omega_{k,j})\frac{dx^j}{ds}\frac{dx^k}{ds} \tag{1.24}$$

has a solution.

2) Ric_∇ *is not symmetric, and the equation*

$$\{\tilde{s}, s\} = \{2\omega_j\omega_k - (\omega_{j,k} + \omega_{k,j})\}\frac{dx^j}{ds}\frac{dx^k}{ds} \tag{1.25}$$

has a solution.

PROOF. We prove 2). Let p, s be the projective and affine parameters of a ∇-geodesic γ and $\tilde{p}, \tilde{s}$ be that of the ∇'-geodesic $f \circ \gamma$. We have to show that $\{\tilde{p}, p\} = 0$.

In order to use the Schwartzian differential we have to symmetrise Ric_∇; and although there will be two different cases, e.g., $\mathrm{Ric}_{\nabla'}$, is also not symmetric and $\mathrm{Ric}_{\nabla'}$ is symmetric, both will give the same result.

We symmetrise Ric_∇ in (1.12) to obtain the expression

$$\hat{R}_{\beta\gamma}B^\beta_j B^\gamma_k = \hat{R}_{jk} + \frac{(n-1)}{2}(\omega_{j,k} + \omega_{k,j} - 2\omega_k\omega_k), \tag{1.26}$$

where $\hat{R}_{jk} = (R_{jk} + R_{kj})/2$ etc.

Multiplication by $\frac{dx^j}{ds}\frac{dx^k}{ds}$ leads to

(1.27)

$$\left(\hat{R}_{\beta\gamma}\frac{dy^\beta}{d\tilde{s}}\frac{dy^\gamma}{d\tilde{s}}\right)\left(\frac{d\tilde{s}}{ds}\right)^2 = (\hat{R}_{jk} + \frac{n-1}{2}(\omega_{j,k} + \omega_{k,j} - 2\omega_j\omega_k))\frac{dx^j}{ds}\frac{dx^k}{ds}.$$

The condition given by (1.25), when applied to (1.27), gives

$$-\{\tilde{s}, s\} = \frac{2}{n-1}\Big(R_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} - \hat{R}_{\beta\gamma}\frac{dy^\beta}{d\tilde{s}}\frac{dy^\gamma}{d\tilde{s}}\Big(\frac{d\tilde{s}}{ds}\Big)^2\Big).$$

This can be interpreted to say that (1.16) and (1.17) have solutions. Further in view of (1.21b) it implies $\{\tilde{p}, p\} = 0$, i.e., f is projective.

It is easy to check that Ric_∇ and $\mathrm{Ric}_{\nabla'}$ are simultaneously symmetric only if the 1-form ω is closed.

Hence we have

PROPOSITION 1.7. *A c^∞-map $f : (M, \nabla) \to (M', \nabla')$ is a projective diffeomorphism if and only if the assiciated 1-form ω is closed.*

COROLLARY 1.8. *If ∇ and ∇' are Riemannian connections, then the diffeomorphisms f which carries geodesics to geodesics is projective.*

REMARK. From Proposition (1.7) it is evident that if ω is a closed form, equations (1.24) and (1.25) are identical; moreover, the right sides of these equations are the negative of the right side of (1.23). Accordingly, retracing our steps from (1.23) to (1.15) we know that the solution exists; it is

$$\tilde{s} = c + \int e^{2\int \omega_j dx^j} ds.$$

From proposition (1.7) we also conclude that a diffeomorphism $f : (M, \nabla) \to (M', \nabla')$ which carries geodesics to geodesics is not projective if the 1-form associated with f is not closed.

Denoting the right side of (1.24) or (1.25) by $\phi(s)$ we conclude that f From proposition (1.7) we also conclude that a diffeomorphism $f : (M, \nabla) \to$ fails to be projective if

$$\{\tilde{s}, s\} = \phi(s) \tag{1.28}$$

fails to have a solution [**10**]. Evidently $f \in \tilde{\mathcal{P}}(M)$.

These maps (which satisfy only condition (1) as already indicated are 'quasi-projective'. In §2 we shall see that any two quasi-projective maps can be composed to give another such map on suitable domains. In particular we shall show that $\tilde{\mathcal{P}}(M)$ is a group.

When f is projective we formulate the relation between Ricci tensors and the Schwartzian differential by using specific geodesics in M and M'.

For obvious reasons (§0) we are using $t, \tilde{t}$ to denote the affine parameters and S_ϕ to denote $\{\phi, t\}$ in the following paragraphs.

Suppose that the geodesics $\gamma = \gamma(t)$ in M with $\dot{\gamma}(0) = V$ is parametrised by the projective parameter p, then its image $\gamma' = f \circ \gamma \circ \tilde{t}^{-1}$ ($\tilde{t}$ being an affine parameter $t \to (f \circ \exp)(tV)$) can be parametrised by the projective parameter $\tilde{p}$, and our assumption on f implies that

$$S_{\tilde{p} \circ \tilde{t}} = S_p.$$

Using the chain rule for differentiation we can write $S_{\tilde{p}\circ\tilde{t}}$ as $(S_{\tilde{p}} \circ \tilde{t})(\frac{d\tilde{t}}{dt})^2 + S_{\tilde{t}}$. Since (1.16) in terms of the tangent vector V can be written as $S_p = (\frac{2}{(n-1)}) \operatorname{Ric}(V, V)$ and

$$(S_{\tilde{p}} \circ \tilde{t})\left(\frac{d\tilde{t}}{dt}\right)^2 = \frac{2}{n-1} (\operatorname{Ric}'(f_* V, f_* V)),$$

we obtain

$$\frac{2}{n-1} \operatorname{Ric}(V, V) = \frac{2}{n-1} \operatorname{Ric}'(f_* V, f_* V) + S_{\tilde{t}},$$

or, equivalently,

$$(f_* \operatorname{Ric}')(V, V) = \operatorname{Ric}(V, V) - \frac{n-1}{2} S_{\tilde{t}}. \tag{1.29}$$

Thus if f is a projective map, the difference between the Ric on M and the pullback of Ric$'$ on M' is given by the Schwartzian differential of affine parameters of geodesics on M and M'.

We next wish to examine the relation between harmonic and projective maps.

Suppose that M and M' are equidimensional and $f : (M, \nabla) \to (M', \nabla')$ is a projective map. For each fixed α, (B^α_{ij}) defines a symmetric matrix (B^α_{ij}). It is known that a diffeomorphic map f is harmonic if the trace of this matrix is zero for every α [**4**]. Hence we have

PROPOSITION 1.4. *A projective mapping between equidimensional manifolds can be harmonic if and only if M admits a coordinate cover* *such that, in every coordinate patch, the covector field induced by f satisfies*

$$\sum_i B_i^\alpha \omega_i = 0 \quad \alpha = 1, 2 \ldots n. \tag{1.30}$$

In other words, the ω_i's must be the solution of the system of equations given by (1.30). It is not hard to see, however, that there exists no non-trivial solution of this system of equations. Thus we conclude that projective and harmonic maps are incompatible as maps on a pair of manifolds, meaning, thereby, that f can either be projective or it can be harmonic.

2. In this section we list a few simple facts about projective mappings between equidimensional manifolds. Let $\{e_i'\}$ be an orthonormal frame field on M' induced under the mapping f by the frame field $\{e_i\}$ on M (i.e., for every i, $f_*(e_i) = \delta_i^k e_k'$), and let $\omega(e_k) = \omega_k$. Then (0.2) gives

$$\begin{aligned} \nabla'_{e_i'} - f\nabla_{e_i} e_j &= \omega_i f_*(e_j) + \omega_j f_*(e_i) \\ (\Gamma'^k_{ij} - \Gamma^k_{ij}) e_k' &= (\omega_i \delta_j^k + \omega_j \delta_i^k) e_k' \end{aligned} \tag{2.1}$$

$$\Gamma'^k_{ij} = \Gamma^k_{ij} + \omega_{/i} \delta_j^k + \omega_{/j} \delta_i^k. \tag{2.1a}$$

The relation above shows that the difference of the pullback of connection coefficients on M' and the connection coefficients on M is a linear combination of components of a co-vector field on M.

On the other hand these equations lead to relations between their curvature tensors, e.g., equation (1.11) or (1.29). We use these relations to prove the following proposition and some results in §4.

PROPOSITION 2.1. *Let $f_i : (M_{i-1}, \nabla_{i-1}) \to (M_i, \nabla_i) (i = 1, 2, \ldots)$ be quasi-projective/projective mappings; then $f_2 \circ f_1 : (M_0, \nabla_0) \to (M_2, \nabla_2)$ is quasi-projective/projective. We denote $f_2 \circ f_1$ by F.*

PROOF. Condition (1) of our definition can be interpreted as follows: Given $f_1 : (M_0, \nabla_0) \to (M_1, \nabla_1)$ for each γ_0-geodesic of ∇_0, $f_1 \circ \gamma_0 f_1$ is a reparametrisation of a geodesic of ∇_1, hence there exists a strictly increasing C^∞-function h_1 on an open interval such that $f_1 \circ \gamma_0 \circ h_1$ is a ∇_1-geodesic. Similarly, $f_2 : (M_1, \nabla_1) \to (M_2, \nabla_2)$ implies that, for every ∇_1-geodesic γ_1, there must exist a C^∞-function h_2 on an open interval such that $f_2 \circ \gamma_1 \circ h_2$ is a ∇_2-geodesic. In particular, writing $f_1 \circ \gamma_0 \circ h_1$ for γ_1, we note that $(f_2 \circ f_1) \circ \gamma_0 \circ (h_1 \circ h_2)$ is a ∇_2-geodesic for suitable choice of open intervals. Since $h_1 \circ h_2$ is a C^∞-function on an open interval, the ∇_2-geodesic obtained in this manner is a reparametrisation of a ∇_0-geodesic γ_0 under the map F. Hence, by definition, F is quasi-projective.

To prove that $f_2 \circ f_1$ is projective we use (1.29) for the mappings f_1 and f_2 to write

$$\mathrm{Ric}_1(f_{1*}\dot\gamma_0, f_{1*}\dot\gamma_0) = \mathrm{Ric}_0(\dot\gamma_0, \dot\gamma_0) - \frac{n-1}{2} S_{\phi_1} \tag{2.2}$$

$$\mathrm{Ric}_2(f_{2*}\dot\gamma_1, f_{2*}\dot\gamma_1) = \mathrm{Ric}_1(\dot\gamma_1, \dot\gamma_1) - \frac{n-1}{2} S_{\phi_2} \tag{2.3}$$

where we have denoted h_1^{-1} and h_2^{-1} by ϕ_1 and ϕ_2. From $\gamma_1 = f_1 \circ \gamma_0 \circ \phi_1^{-1}$ it is easy to check that

$$\dot\gamma_1 = f_{1*}\left(\frac{\dot\gamma_0}{D\phi_1}\right) \circ \phi_1^{-1}, \quad D_{\phi_1} = \frac{d}{dt}\phi_1$$

and

$$\mathrm{Ric}_1(\dot\gamma_1, \dot\gamma_1) = (\mathrm{Ric}_1(f_{1*}\dot\gamma_0, f_{1*}\dot\gamma_0))\frac{1}{(D\phi_1)^2} \circ \phi_1^{-1}.$$

Similarly,

$$\mathrm{Ric}_2(f_{2*}\dot\gamma_1, f_{2*}\dot\gamma_1) = (\mathrm{Ric}_2(f_{2*}f_{1*}\dot\gamma_0, f_{2*}f_{1*}\dot\gamma_0))\frac{1}{(D\phi_1)^2} \circ \phi_1^{-1}.$$

Substituting these in (2.3) and using (2.2), we have

$$\frac{1}{(D\phi_1)^2}(\mathrm{Ric}_2(F_*\dot\gamma_0, F_*\dot\gamma_0)) = \left(\mathrm{Ric}_0(\dot\gamma_0, \dot\gamma_0) - \frac{n-1}{2} S_{\phi_1}\right)\frac{1}{(D\phi_1)^2} - \left(\frac{n-1}{s} S_{\phi_2}\right) \circ \phi_1$$

or

$$\mathrm{Ric}_2(F_*\dot\gamma_0, F_*\dot\gamma_0) = \mathrm{Ric}_0(\dot\gamma_0, \dot\gamma_0) - \frac{n-1}{2}(S_{\phi_1} + (D\phi_1)^2(S_{\phi_2} \circ \phi_1)). \tag{2.4}$$

In view of the chain rule for Schwartzian derivative, the second term on the right side equals

$$-\frac{n-1}{2}S_{\phi_2\circ\phi_1}. \tag{2.5}$$

Consequently, (2.4) is analogous to (2.2); the Ricci tensor is now evaluated for the tangent vector of the reparametrised geodesic $f_2 \circ f_1 \circ \gamma_0 \circ h_1 \circ h_2 \equiv F \circ \gamma_0 \circ \phi_1^{-1} \circ \phi_2^{-1}$. Since this is the condition that the projective parameter be preserved under F, we have proved the result.

COROLLARY 2.2. *Let* $f_i : (M_{i-1}, \nabla_{i-1}) \to (M_i, \nabla_i)(i = 1, 2)$ *be quasi-projective diffeomorphisms on* M. *Then* $f_2 \circ f_1 : (M_0, \nabla_0) \to (M_2, \nabla_2)$ *is quasi-projective.*

This shows that if $f_1, f_2 \in \tilde{P}(M)$, then $f_2 \circ f_1$ as well as $f_1 \circ f_2$ also belongs to $\tilde{p}(M)$.

3. Let M be an oriented compact manifold without boundary and let ϕ denote a section ($\varepsilon\Gamma(E)$) of an arbitrary bundle $\pi : E \to M$. Consider an action

$$I(\phi) = \int_M L(\phi) * 1. \tag{3.1}$$

An extremal of (3.1) is obtained by varying the sections ϕ, and ϕ is an extremal if it satisfies the Euler-Lagrange equation written notationally as

$$\varepsilon L(\phi) = 0. \tag{3.2}$$

In this section we use the bundle of affine connections to form our Lagrangian for the following reason. Projective maps between two manifolds essentially relate to their symmetric affine connections quite independently of their metric, hence (as we shall soon see) a connection dependent Lagrangian on a manifold M is an equally appropriate object of study. Moreover, since a projective map can be expressed locally by means of equation (2.1), one can interpret a result based on the Lagrangian $L(\Gamma^i_{jk})$ as a result on the projective maps.

Let $\hat{P}(M)$ denote the collection of all projective maps that can be defined from M to a fixed manifold (M', ∇'); we call a projective map

f on M an *extremal* (or a critical point of $\hat{\mathcal{P}}$) if the connection Γ on the right side of (2.1a) is the solution of the Euler-Lagrange equation $\varepsilon L(\Gamma^i_{jk}) = 0$.

Since the projective map also implies the relation (0.8) we can write the Lagrangian as a function of the Ricci-tensor and the convector field. Thus the connection dependence of L turns out to be via the Ricci tensor. It is in this case that we have the following result.

PROPOSITION 3.1. *Given a fixed (M', ∇', g') let $\hat{\mathcal{P}}(M)$ be the collection of projective maps $f : (M, \nabla) \to (M', \nabla')$ and let $L(f^* \; Ric')$ be the Lagrangian (computed with respect to $f^* g'$) on M associated to the collection $\hat{\mathcal{P}}(M)$. Then the critical point f of $\hat{\mathcal{P}}(M)$ determines the Levi-Civita connection whose metric is given by $\frac{(\partial L)}{\partial \; Ric}$ provided the covector field induced by f is parallel.*

PROOF. We use equation (0.8) to write the Lagrangian $L(f^* \; \mathrm{Ric}')$. In view of equation (1.12) the local expression for the Lagrangian is

$$L(f^* \; \mathrm{Ric}') \equiv L(R_{ij}, \omega_i, \omega_{i,j}), \tag{3.3}$$

where $R_{ij} \equiv (R^h_{ijk})_{h=k}$ equals

$$R_{ij} = \frac{\partial \Gamma^\ell_{ij}}{\partial x^\ell} - \frac{\partial \Gamma^\ell_{i\ell}}{\partial x^j} + \Gamma^\ell_{ij}\Gamma^m_{\ell m} - \Gamma^\ell_{im}\Gamma^m_{\ell j}. \tag{3.4}$$

Using the symmetry of Γ^h_{ij} which implies $\Gamma^h_{ij} = \frac{1}{2}(\Gamma^h_{ij} + \Gamma^h_{ji})$, and denoting $\partial \Gamma^h_{ij}/\partial x^k$ by Γ^h_{ijk}, we can write the expression for R_{ij} as

$$R_{ij} = (\Gamma^h_{k\ell m} + \Gamma^h_{nm}\Gamma^n_{k\ell})(\delta^m_h \delta^k_i \delta^\ell_j - \delta^\ell_h \delta^k_i \delta^m_j). \tag{3.5}$$

Accordingly, the Euler-Lagrange equation

$$\frac{\partial}{\partial x^m}\Big(\frac{\partial L}{\partial \Gamma^h_{k\ell m}}\Big) = \frac{\partial L}{\partial \Gamma^h_{k\ell}} \tag{3.6}$$

can be simplified by writing

$$\begin{aligned} \frac{\partial L}{\partial \Gamma^h_{k\ell m}} &= \frac{1}{2}\frac{\partial L}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial \Gamma^h_{k\ell m}} \\ &= p^{ij}(\delta^m_h \delta^k_i \delta^\ell_j - \delta^\ell_h \delta^k_i \delta^m_j) \\ &= \delta^m_h p^{k\ell} - \delta^\ell_h p^{km} \end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial \Gamma^h_{k\ell}} &= \frac{1}{2} p^{ij} \frac{\partial R_{ij}}{\partial \Gamma^h_{k\ell}} \\
&= p^{ij} \frac{\partial}{\partial \Gamma^h_{k\ell}} (\Gamma^h_{nm} \Gamma^n_{k\ell} \{\delta^m_h \delta^k_i \delta^\ell_j - \delta^\ell_h \delta^k_i \delta^m_j\}),
\end{aligned} \tag{3.8}
$$

where we have denoted $\frac{\partial L}{\partial R_{ij}}$ as P^{ij} and have used the fact that ω_i is parallel. A careful matching of indices in

$$\frac{\partial}{\partial x^m} (\delta^m_h P^{k\ell} - \delta^\ell_h P^{km})$$

with (3.8) shows that (3.6) is indeed the covariant derivative

$$P^{k\ell}_{,h} = 0. \tag{3.9}$$

If now we interpret the tensor density $\frac{\partial L}{\partial R_{ij}} = P^{ij}$ as a contravariant density of the metric tensor defined as

$$p^{ij} = \lambda \sqrt{g} g^{ij}, \tag{3.10}$$

for some constant λ, then (3.7) is the metricity condition

$$g_{k\ell,h} = 0 \tag{3.11}$$

for the connection Γ^i_{jk}, showing that this is a Levi-Civita connection.

The following proposition based on Hamilton [**8**] and DeTurck's [**3**] findings leads to a similar result. The hypothesis in this case, though, is more stringent.

Let $f : M_n \to M'_n$ be an arbitrary C^∞-map and let $\mathcal{A}_s$ and $\mathcal{A}'_s$ denote the spaces of affine symmetric connections on M, M'. Consider a map $\phi : f^{-1}(\mathcal{A}'_s) \otimes \mathcal{A}_s \otimes \wedge^1(M) \to S^2(T^*M \otimes T^*M)$ defined as

$$\phi(\nabla', \nabla, \omega) = (f^* \operatorname{Ric}' - d\omega - (n-1) \ \ \omega)_{\text{sym}}. \tag{3.12}$$

Suppose that ϕ is non-zero at every point $p \in M$, then we can prove

PROPOSITION 3.2. *Given $f : M_n \to M'_n$ and a map ϕ defined above, there exists a unique metric g on M with respect to which f is quasi-projective provided M is simply connected.*

PROOF. We use the local expression (see (1.14)) for the right side of (3.10) to write

$$S_{ij} = {'R_{(ij)}} - \left(\frac{n-1}{2}\right)\{(\omega_{i,j} + \omega_{j,i}) - 2\omega_i\omega_j\}, \tag{3.13}$$

where $'R_{(\alpha\beta)} B_i^\alpha B_j^\beta$ is denoted $'R_{(ij)}$ and the covariant derivative $\omega_{i,j}$ of ω is computed with respect to $\nabla \in \mathcal{A}_s$.

On account of our assumption on ϕ, $S_{ij}(p) \equiv S(p)$ is not zero, i.e., $S^{-1}(p)$ is defined. Thus in view of the result: "Given a symmetric C^∞-tensor field S_{ij} in a neighborhood of a point p on a manifold, if $S^{-1}(p)$ exists, then there is a Riemannian metric g such that $\mathrm{Ric}(g) = S$ in a neighborhood of p" [**3**], it follows that there is a unique metric g in the neighborhood of p, the Ricci curvature pertaining to which equals S_{ij}. But this is the requirement that f be a quasi-projective map in the neighborhood of p. As M is simply-connected, we have our result.

4. In this section we examine the relation between projective maps and topological invariants of the manifold. Our first result deals with the non-existence of a projective map.

PROPOSITION 4.1. *Let M and M' be two manifolds such that $Ric_M > 0$ and $Ric_{M'} < 0$. If M is compact, orientable and without boundary then there does not exists a map $f : M \to M'$ which is projective or quasi-projective.*

PROOF. Suppose that there exists a map $f : M \to M'$ which is at least quasi-projective; then, using the local description in symmetrised form (1.16), we have

$$'R_{\alpha\beta} B_i^\alpha B_j^\beta = R_{ij} - \frac{n-1}{2}(\omega_{i,j} + \omega_{j,i}) + (n-1)\omega_i\omega_j. \tag{4.1}$$

Now, $\mathrm{Ric}_{M'} < 0$ implies $'R_{\alpha\beta} B_i^\alpha B_j^\beta < 0$, and $\mathrm{Ric}_M > 0$ implies that $R_{ij} > 0$.

Hence, integrating the equation with respect to the metric on M and noticing that the term-$(\omega_{i,j} + \omega_{j,i})$ makes no contribution to the integral, we obtain a negative term on the left side equated to a positive term on the right side. This shows that our assumption on f is not valid, i.e., there exists no projective map $f : M \to M'$.

COROLLARY 4.2. *If M and M' are Einstein manifolds with scalar curvatures $S > 0$ and $S' < 0$, and if M is compact orientable and without boundary, then there exists no projective/quasi-projective map between M and M'.*

PROOF. We write $R_{ij} = (\frac{S}{n})g_{ij}$ and $'R_{\alpha\beta} = (\frac{'S}{n})g'_{\alpha\beta}$ in (4.1), where we have assumed that there exists a map $f : M \to M'$ which is at least quasi-projective. Suppose that the pullback metric f^*g' is conformal to g with $\lambda(x)$ as the scalar of conformality, then $g_{\alpha\beta}B_i^\alpha B_j^\beta$ equals $\lambda(x)g_{ij}$. (We note that conformality is not required for the result.) The integration over M gives the same conclusion as we had in Proposition (4.1).

EXAMPLE 4.3. Let M be the $S^2 \subset R^3$ given by (Cos$u$Cos$v$, Sin$u$Cos$v$, Sin$v$) and M' be the hyperboloid of revolution defined as (Cos$u$Cosh$v$, Sin$u$Cosh$v$, Sinhv). It is easy to check that the metric tensor $g_{ij} = 0$, for $i \neq j$, in both cases and that the Ricci tensor for M and M' is given by the relations

$$\begin{aligned} R_{11} &= \frac{S}{2}g_{11} = \Big(\frac{1}{2} + \mathrm{Sin}^2 v\Big)\mathrm{Cos}^2 v \\ R_{22} &= \frac{S}{2}g_{22} = \Big(\frac{1}{2} + \mathrm{Sin}^2 v\Big) \\ R'_{11} &= \frac{S'}{2}g_{11} = \frac{1}{2}(1 + \mathrm{Cosh}2v)\mathrm{Sech}^2 2v \\ R'_{22} &= \frac{S'}{2}g'_{22} = -\frac{1}{2}(1 + \mathrm{Sech}2v)\mathrm{Sech}^2 v. \end{aligned}$$

Thus $\mathrm{Ric}_M > 0$ for all values of v and $\mathrm{Ric}'_{M'} < 0$ for all values of v. The parameter v in the first case has a finite domain but in the second case it has an infinite domain. It is known that there is no C^∞-map f which can carry geodesics of M which are great circles to geodesics of M' which are hyperbolas.

PROPOSITION 4.4. *Let $f : (M, \nabla) \to (M', \nabla')$ be a quasi-projective map and suppose that $Ric_{M'}$ is parallel with respect to ∇', then Ric_M is parallel with respect to ∇ if and only if the second order covariant tensor $(d\omega + (n-1)\ \omega)$ induced by f on M and $df \otimes df$ on M' are covariantly constant.*

PROOF. Let M and M' be covered by suitable charts; then, using the covariant constancy of $df \otimes df$, we write the covariant derivative of (1.12) as

$$(4.2)\quad 'R_{\alpha\beta,\gamma}B_i^\alpha B_j^\beta B_k^\gamma = R_{ij,k} + ((\omega_{i,j} - \omega_{j,i}) + (n-1) \times (\omega_{i,j} - \omega_i\omega_j))_k.$$

The Ricci-curvature on M' is parallel and it will be parallel on M if and only if the second term on the right side is zero.

We can also prove the following result.

PROPOSITION 4.5. *Let $f : (M, \nabla) \to (M', \nabla')$ be a quasi-projective map such that the tensor $(d\omega + (n-1)\ \omega)$ and $df \otimes df$ induced by f are covariantly constant on M. If M is a Riemannian manifold with parallel Ricci-tensor and if $C_0(M) \neq I_0(M)$, then if M' is complete it must be isometric to a sphere.*

PROOF. Nagano and Yano have proved that a complete Riemannian manifold whose groups of conformal transformations and isometries are distinct is isometric to a sphere if its Ricci tensor is parallel [**14**].

Since f_* has constant rank and $(d\omega + (n-1)\omega)$ is covariantly constant, equation (4.2) implies $'R_{\alpha\beta,\gamma} = 0$, i.e., the Ricci-tensor on M' is parallel. Using a local chart for the mapping f, it can be shown that if the (connected) groups of conformal transformations and that of isometries of M are distinct, then those of M' are distinct as well. Hence M' satisfies the criteria required for it to be isometric to a sphere.

Using the hypothesis of the above proposition we can also state

COROLLARY 4.6. *If M is complete, f maps a sphere to a sphere.*

5. In this section we establish the conditions under which quasi-projective (projective) mappings defined on space-time manifolds are volume decreasing (distance-decreasing).

We choose time-oriented space-time Lorentz manifolds (M, ∇, g), (M', ∇', g'), i.e., manifolds whose metrics are locally reducible to $(-1, +1, +1, +1)$ and whose time-like vectors can be labelled as future-pointing or past-pointing.

We assume that the mapping $f : (M, \nabla) \to (M', \nabla')$, which is quasi-projective or projective is also orientation preserving. It can be shown that under suitable conditions a space-like 3-surface H imbedded in M

can be mapped to a space-like 3-surface $f(H)$ imbedded in M'. We denote $f(H)$ as H' [**14**].

The Einstein field equations on M and M' in the presence of matter can be written as

$$\mathrm{Ric}(X,Y) - \frac{S}{2}g(X,Y) = 8\pi T(X,Y) \tag{5.1}$$

$$\mathrm{Ric}'(X',Y') - \frac{S'}{2}g'(X',Y') = 8\pi T'(X',Y'), \tag{5.2}$$

where T and T' stand for energy momentum tensors on M and M'.

Since space-like surfaces carry Riemannian structures we can prove the following two results.

PROPOSITION 5.1. *Let $f : M \to M'$ be a projective mapping. Let H be a non-compact complete space-like 3-surface embedded in M such that the energy momentum tensor T restricted to H is bounded below by a constant $-A$ ($A > 0$) and scalar curvature S is positive on H. If T' is bounded above by a constant $-B$ ($B > 0$) on H' and S' is negative there, then f_H is distance decreasing up to a constant $(A/B)^{\frac{1}{2}}$.*

PROPOSITION 5.2. *Let $f : M \to M'$ be a quasi-projective mapping other conditions remaining the same as in Proposition (5.1); then f_H is a volume decreasing mapping up to a constant $(A/B)^{\frac{3}{2}}$.*

PROOF OF 5.1. Since the scalar curvature S is positive on H, equation (5.1) implies that Ric cannot be less than $8\pi T$ at any point of H; this means that Ric can be considered as bounded below by a constant $-A/8\pi$. Similarly, since the scalar curvature S' is negative on H', the bound on T' can be used to say that the Ric$'$ on H' is bounded above by a constant $-B/8\pi$. Thus, bounds on Ricci curvatures on H and H' are the same as in Theorem (A), except for the constant $\frac{1}{8}\pi$. Hence the map F is distance-decreasing up to a constant $(A/B)^{\frac{1}{2}}$, as is already known according to Theorem (A).

A proof of proposition (5.2) follows by repeating the above argument and using the result of theorem (B) for $n = 3$.

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