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REMARKS ON GLOBAL BOUNDS OF SOLUTIONS OF PARABOLIC EQUATIONS IN DIVERGENCE FORM

TOMASZ DLOTKO

Introduction. We want to examine certain properties of solutions (understood here as the trajectories in a suitable Banach space) of nonlinear parabolic equations, which could be of use in the study, among other things, of the long time behaviour of these problems. We start with the proof of a variant of the maximum principle (cf. [**8**; Theorems 2 and 5], [**9**; Theorem 2.7] for the linear case) obtained here as a limiting case ($p \to \infty$) of the sub-exponential estimates of solutions in L^p. Next (§2) we prove a theorem concerning global boundedness of the spatial derivatives u_x, first in the one dimensional case ($n = 1$) for equations with bounded perturbation $f(t, x, u, u_x, u_{xx}, u_t)$, then (§3) by different method (and with stronger assumptions) for the general n-dimensional case. The results obtained in this work are linked up by the method of proofs developed under the stimulus of Theorem 3.1 of [**1**], first used in a different context by J. Moser in [**12**].

Preliminaries. Notation. The following standard notation is used:
(a) $\Omega \subset R^n$ is a bounded domain with a suitable smooth boundary;
(b) $R^+ = [0, \infty), D = R^+ \times \Omega, D_T = \{(t, x) : 0 \le t \le T, x \in \Omega\}$,
(c) $\langle \cdot, \cdot \rangle$ is the scalar product in R^n,
(d) $|\Omega|$ is the Lebesgue measure of Ω; and
(e) for $x \in R^n$ we write $u_x = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$; and
(f) use the usual notation for the L^p and Sobolev spaces. By convention, all sums are taken from 1 to n, and integrals with unspecified domain are taken over Ω.

The following easy lemma is required several times.

LEMMA 0. *Let* $y \in C^{\circ}(R^+)$, $x \in C^1(R^+)$, *let* $\alpha \ge 0, \beta, \delta, \lambda > 0$ *and*

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$\eta \in [0, 1)$ *be real constants. If* $0 \le \lambda x(t) \le y(t)$ *for* $t \in R^+$,

$$x'(t) \le \alpha - \beta y(t) + \delta y^\eta(t),$$

then $0 \le x(t) \le \max\{x(0), r_1/\lambda\}$, *where* r_1 *denotes the positive root of the equation* $\alpha - \beta r + \delta r^\eta = 0$.

PROOF. When $x(t) > r_1/\lambda$, then $y(t) > r_1$ and the right side of the differential ineqaulity is negative, hence x is decreasing.

1. We start with a version of the maximum principle for the nonlinear parabolic equation in divergence form considered in [**13**]. Our assumptions are different and so are the method of proof and our results. Consider the following Dirichlet problem

(1) $u_t = \sum a_i(t, x, u, u_x)_{x_i} + a(t, x, u, u_x)u + f(t, x)$,

(2) $u = 0$ on $\partial\Omega$, $u(0, x)$ is given, and assume that there exists a solution u with $u_t, \Delta u$ in $L^2(D_T)$, $u_{x_i} \in C^\circ(0, T; L^2(\Omega)), i = 1, \dots, n$ and $u \in C^1(0, T; L^\infty(\Omega))$ for all $T > 0$. Let $f(t, \cdot)$ be bounded in $L^\infty(\Omega)$ for $t \in R^+$; moreover, $a(t, x, u, u_x) \le a_o$, $a_0 \ge 0$ and

$$\sum a_i(t, x, u, u_x)u_{x_i} \ge \mu \sum u_{x_i}^2.$$

We then have

THEOREM 1. *The following estimate holds*

(3)

$$\forall_{0 \le t_o \le t} ||u(t, \cdot)||_{L^\infty(\Omega)} \le (||u(t_o, \cdot)||_{L^\infty(\Omega)} + M(t_o)(t - t_o)) \cdot \exp(a_o(t - t_o)),$$

where $M(t) = \sup_{s \ge t} ||f(s, \cdot)||_{L^\infty(\Omega)}$.

PROOF. Clearly it is sufficient to consider the case $a \le 0$, otherwise we can use the transformation $v(t, x) = u(t, x)\exp(-a_o t)$ and study the equation for v (with $\tilde{a}(t, x, v, v_x) = a(t, x, u, u_x) - a_o \le 0$). Multiplying (1) by $u^{2^m - 1}(m \in N)$ and integrating the result over Ω we obtain

$$\int u_t u^{2^m - 1} dx = \int \sum a_i(t, x, u, u_x)_{x_i} u^{2^m - 1} dx + \int a(t, x, u, u_x) u^{2^m} dx + \int f(t, x) u^{2^m - 1} dx.$$

Integrating by parts and using Holder's inequality (with $p = 2^m(2^m - 1)^{-1}$, $q = 2^m$) we then get $(a \le 0)$:

$$2^{-m}\frac{d}{dt}\int_{\partial\Omega} u^{2^m}dx \le \int u^{2^m-1}\sum a_i(t,x,u,u_x)\cos(n,x_i)d\sigma$$
$$-(2^m-1)\int\sum a_i(t,x,u,u_x)u_{x_i}u^{2^m-2}dx$$
$$+\left(\int |f|^{2^m}dx\right)^{2^{-m}}\left(\int u^{2^m}dx\right)^{1-2^{-m}}.$$

Then, by the ellipticity condition (denoting $u^{2^{m-1}} = v$),

$$\frac{d}{dt}\int v^2dx \le -\mu(2^m-1)2^{2-m}\int\sum v_{x_i}^2dx$$
$$+M(t)2^m|\Omega|^{2^{-m}}\cdot\left(\int v^2dx\right)^{1-2^{-m}}.$$

This may be rewritten using a version of the Poincare inequality $(\lambda = \lambda(n,\Omega))$, i.e.,

$$\forall_{w\in H_o^1(\Omega)}\lambda\|w\|^2_{L^2(\Omega)} \le \|w_x\|^2_{L^2(\Omega)}, \tag{4}$$

in the following manner $(\alpha_m = (\mu\lambda)(2^m-1)2^{2-m}, \beta_m(t) = M(t)|\Omega|^{2^{-m}})$:

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \le -\alpha_m\|v\|^2_{L^2(\Omega)} + \beta_m(t)2^m(\|v\|^2_{L^2(\Omega)})^{1-2^{-m}} \tag{5}$$

A differential inequality of this kind is easily explicitly integratable:

$$\|v(t,\cdot)\|^2_{L^2(\Omega)} \le (\|v(t_o,\cdot)\|^{2^{1-m}}_{L^2(\Omega)}\exp(\alpha_m2^{-m}t_o)$$
$$+\int_{t_o}^t\beta_m(z)\exp(\alpha_m2^{-m}z)dz)^{2^m}\exp(-\alpha_mt).$$

The function β_m is decreasing, hence finally we arrive at the estimate

$$\begin{aligned}\|u(t,\cdot)\|_{L^{2^m}(\Omega)} \le(&\|u(t_o,\cdot)\|_{L^{2^m}(\Omega)}\exp(\alpha_m2^{-m}t_o)\\ &+\beta_m(t_o)(\exp(\alpha_m2^{-m}t)\\ &-\exp(\alpha_m2^{-m}t_o))\frac{2^m}{\alpha_m})\exp(-\alpha_m2^{-m}t).\end{aligned} \tag{6}$$

We are now interested in passing to the limit in (6) with m to ∞. Clearly (cf. [**15**; I.3, Theorem 1])

$$
\begin{aligned}
||u(t,\cdot)||_{L^\infty(\Omega)} &= \lim_{m\to\infty} ||u(t,\cdot)||_{L^{2^m}(\Omega)} \\
&\le \lim_{m\to\infty} (||u(t_o,\cdot)||_{L^{2^m}(\Omega)} \exp(-\alpha_m 2^{-m} t - t_0) \\
&\quad + \beta_m(t_o)\Big(1 - \exp(-\alpha_m 2^{-m}(t - t_o))\Big)\frac{2^m}{\alpha_m}\Big) \\
&= \begin{cases} ||u(t_o,\cdot)||_{L^\infty(\Omega)} + M(t_o)(t - t_o) & \text{for } t < \infty, \\ 0 & \text{for } t = \infty. \end{cases}
\end{aligned} \tag{7}
$$

REMARK 1. If $f = 0$ and $a_o = 0$ the estimate (7) is the usual version of the maximum principle.

REMARK 2. Under the more restrictive hypothesis $a_i = a^o(t,x,u)u_{x_i}$ (with a^o strictly positive) our proof remains valid also for the third boundary problem (for $u_{x_i} i = 1,\dots n$, continuous). Let $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$ with $|\Gamma_1| > 0$ (the $(n-1)$-dimensional measure), and consider (1) with the boundary condition

$$
\begin{cases} u = 0, & \text{on } \Gamma_1 \times R^+, \\ p(t,x)\frac{\partial u}{\partial n} + q(t,x)u = 0, & \text{on } \Gamma_2 \times R^+, \end{cases} \tag{2'}
$$

$p, q \ge 0, p + q = 1$. Under such assumptions (c.f. [**11**; Theorem 3.6.4, p.82]) there exists a constant μ depending on n, Γ_1, and $\partial\Omega$ (which is assumed to be Lipschitz, see [**11**], such that

$$
\forall_{v\in C^1(\overline{\Omega})} \mu \int v^2 dx \le \int |v_x|^2 dx.
$$

From this estimate, and using (2′) for $v = u^{2^m-1}$, $v = 0$ on $\Gamma_1 \times R^+$, we have

$$
p(t,x)\frac{\partial v}{\partial n} = (2^m - 1)(-q(t,x))u^{2^m-1} = -(2^m - 1)q(t,x)v
$$

on $\Gamma_2 \times R^+$. After normalization,

$$
\frac{p(t,x)}{1 + (2^m - 2)q(t,x)}\frac{\partial v}{\partial n} + \frac{q(t,x)}{1 + (2^m - 2)q(t,x)}v = 0.
$$

The constant μ is the same for both powers of u. Also the boundary integral is non-positive:

$$\int_{\partial\Omega} u^{2^m-1} \sum a^o(t,x,u)u_{x_i} \cos(n,x_i)d\delta$$
$$= \int_{\Gamma_1} a^o u^{2^m-1} \frac{\partial u}{\partial n} d\delta + \int_{\Gamma_2} a^o \Big(-\frac{p(t,x)}{q(t,x)}\Big)^{2^m-1} \Big(\frac{\partial u}{\partial n}\Big)^{2^m} d\delta \le 0.$$

The rest of the proof remains unchanged.

REMARK 3. If instead of the component $a(t,x,u,u_x)u$, we take $b(t,x,u,u_x)$ with $\frac{\partial b}{\partial u} \le 0$, $b(t,x,0,u_x) = 0$, then

$$\begin{aligned} b(t,x,u,u_x)u^{2^m-1} &= (b(t,x,u,u_x) - b(t,x,0,u_x))u^{2^m-1} \\ &= \frac{\partial b}{\partial u}(t,x,\tilde{u},u_x)u^{2^m} \le 0, \end{aligned}$$

and the proof remains valid.

2. We deal now with the problem of global in t boundedness of spatial derivative of solutions of the problem

(8) $u_t = u_{xx} + f(t,x,u,u_x,u_{xx},u_t)$,

(9) $u_x = 0$ on $\partial\Omega$, given $u(0,x)$,

$x \in \Omega = (\alpha,\beta) \subset R$, with bounded perturbation $f, |f| \le M$. Assume there exists a classical solution u of this problem with $u_{xx}, u_{tx} \in L^2(D_T)$ for all $T > 0$. Then, clearly (compare [**10**]),

$$\frac{d}{dt}\int u_x^{2k} dx = 2\int (u_x^k)_t(u_z^k)dx = 2k\int u_{tx}u_x^{2k-1}dx.$$

We have the following theorem.

THEOREM 2. *The derivative u_x is bounded globally in time in both $L^p(\Omega), 2 \le p < \infty$, by constants depending only on Ω, M, and on $||u_x(0,\cdot)||_{L^p(\Omega)}$, respectively.*

PROOF. Multiplying (8) by $(u_x^p)_x, p = 1, 2, 3\dots$, and integrating over Ω, we get

$$\int u_t(u_x^p)_x dx = \int u_{xx}(u_x^p)_x dx + \int f(u_x^p)_x dx.$$

Transforming the components, i.e.,

$$\begin{aligned}\int u_t(u_x^p)_x dx &= u_t u_x^p|_{x=\alpha,\beta} - \int u_{tx}u_x^p dx \\ &= -\frac{1}{p+1}\frac{d}{dt}\int u_x^{p+1}dx, \\ \int u_{xx}(u_x^p)_x dx &= p\int (u_{xx})^2 u_x^{p-1}dx \\ &= \frac{4p}{(p+1)^2}\int [(u_x^{\frac{p+1}{2}})_x]^2 dx,\end{aligned}$$

and using the Holder inequality

$$\begin{aligned}\int f(u_z^p)_x dx = p\int f u_{xx}(u_x)^{p-1}dx &= \frac{2p}{p+1}\int f u_x^{\frac{p-1}{2}}(u_x^{\frac{p+1}{2}})_x dx \\ &\le \frac{2p}{p+1}\Big(\int |f|^2|u_x|^{p-1}dx\Big)^{\frac{1}{2}}\Big(\int [(u_x^{\frac{p+1}{2}})_x]^2 dx\Big)^{\frac{1}{2}},\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}\int u_x^{p+1}dx \le &-\frac{4p}{p+1}\int [(u_x^{\frac{p+1}{2}})_x]^2 dx \\ &+ 2pM\Big(\int |u_x|^{p-1}dx\Big)^{\frac{1}{2}}\Big(\int [(u_x^{\frac{p+1}{2}})_x]^2 dx\Big)^{\frac{1}{2}},\end{aligned} \tag{10}$$

remembering that $|f| \le M$. Inequality (10) allows us to estimate inductively the norms of u_x in $L^{p+1}, p = 1, 3, 5, \dots$. Clearly, (10) with $p = 1$ has the form

$$\frac{d}{dt}||u_x||^2_{L^2(\Omega)} \le -2||u_{xx}||^2_{L^2(\Omega)} + 2M|\Omega|^{\frac{1}{2}}||u_{xx}||_{L^2(\Omega)}.$$

Hence, by (4) and Lemma 0, we get the estimate

$$\int u_x^2(t,x)dx \le \max\Big(\int u_x^2(0,x)dx; \frac{M}{\lambda}|\Omega|^{\frac{1}{2}}\Big) = m_1. \tag{11}$$

Having the estimate of $\int u_x^{p-1}dx \le m_{p-1}$ (p-odd number), from (10) we conclude that

$$\frac{d}{dt}\int u_x^{p+1}dx \le -\frac{4p}{p+1}\int ((u_x^{\frac{p+1}{2}})_x)^2dx + 2pM(m_{p-1})^{\frac{1}{2}}\Big(\int ((u_x^{\frac{p+1}{2}})_x)^2dx\Big)^{\frac{1}{2}}. \tag{12}$$

Hence, via (4) with $w = u_x^{\frac{p+1}{2}}$ and Lemma 0, we get the estimate

$$\int u_x^{p+1}(t,x)dx \le \max\Big\{\int u_x^{p+1}(0,x)dx; \Big(\frac{M(p+1)}{2}\Big)^2\frac{m_{p-1}}{\lambda}\Big\} = m_{p+1}, \tag{13}$$

which together with (11), allows us to estimate globally in time u_x's norms in both $L^p(\Omega)$ (the p root is increasing):

$$|u_x(t,\cdot)||_{L^p(\Omega)} \le \max\Big\{||u_x(0,\cdot)||_{L^p(\Omega)}; \Big(\frac{M^2p^2}{4\lambda}m_{p-2}\Big)^{\frac{1}{p}}\Big\}, \tag{14}$$

for even p. For odd $p = 2k+1$ we will use

$$||v||_{L^{2k+1}_{(\Omega)}} \le \Big(\Big(\int v^{2k}dx\Big)^{\frac{1}{2}}\Big(\int v^{2k+2}dx\Big)^{\frac{1}{2}}\Big)^{\frac{1}{2k+1}}.$$

The proof is finished.

REMARK 4. Observe that the simple exemplary problem $u_t = u_{xx}+1, u_x = 0$ on $\partial\Omega$, $u(0,x)$ given, satisfies all assumptions of Theorem 2. Even though its solution u is clearly unbounded, u_x remains bounded (by $||u_x(0,\cdot)||_{C^0(\overline{\Omega})}$) for all time. Note also that the usual way to get the estimate of u_x for the nonlinear problems is to find first the bound for u (compare [**9**, **11**, **6**]), and then for u_x with the use of one of several known methods (following [**9**;Theorem, 5.1 Chapter VI., §5], [**6**], or by using the variation of constants formula and studying it from the analytical point of view - through fractional powers of elliptic operators - [**7**, p.24]).

REMARK 5. Similar to our Theorem 2, compare Theorems 5.1 and 5.2 of [**3**] concerning the Cuachy problem (also for $n = 1$).

3. We are interested now in the result analogous to Theorem 2 for the n-dimensional case:

$$u_t = \Delta u + f(t, x, H(u)(t)), \tag{15}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \text{ given } u(0, x), \tag{16}$$

where $H : L^\infty(\Omega) \to R$ is a continuous and bounded functional. For example,

$$H(u)(t) = \int u^k(t, x)dx, \ \tilde{H}(u)(t) = \text{ess. } \sup_\Omega |u(t, x)|.$$

Assume $f, \frac{\partial f}{\partial x_i}, i = 1, \dots, n$, to be bounded in $L^\infty(\Omega)$ independently on t and let there exist a classical solution u of (15)-(16) with $u_{x_i x_j x_k}$ in $L^2(D_T)$, $u_{x_i x_j}$ in $C^\circ(0, T; L^\infty(\Omega))$ for all $T > 0$. We need also the following version (see [**9**; Remark 2.1, Chapter II., §2]) of the Nirenberg-Gagliardo interpolation inequality: for every $v \in W^{1,2}(\Omega)$ with zero average $|\Omega|^{-1} \int v(x)dx$,

$$||v||_{L^2(\Omega)} \leq c||v||^{\Theta}_{W^{1,2}(\Omega)} ||v||^{1-\Theta}_{L^1(\Omega)}, \tag{17}$$

with $\Theta = \frac{n}{(n+2)}$, $c = c(n, \Theta, \Omega)$. With the help of Young's inequality (with $m = \Theta^{-1}$, see [**9**]), from (17), we can get (as in [**1**, p. 209]) the estimate (valid for any $\varepsilon \in (0, 1)$)

$$||v||^2_{L^2(\Omega)} \leq \varepsilon||v_x||^2_{L^2(\Omega)} + C_\varepsilon||v||^2_{L^1(\Omega)} \tag{18}$$

$C_\varepsilon =$ const $\varepsilon^{-\frac{n}{2}}$. We are now able to show

THEOREM 3. *Let the solution u of* (15), (16) *be globally bounded* (*by* M) *in $L^\infty(\Omega)$ for $t \geq 0$. Then*

$$\max_\Omega |u_x| \leq \text{ const }, \quad \text{osc } (u_x, \Omega_\delta) \leq \text{ const } \delta^\alpha$$

(*where $\Omega_\delta = \Omega \bigcap B_\delta, B_\delta \subset R^n$ a ball will radius δ, "osc" as in* [**9** *Chapter* V., §7), *where the constants and α depend only on M, $\partial\Omega$, the global* *bound N of $\frac{\partial f}{\partial x_i}$ and the global bound of $f(t,x,H(u)(t))$.*

PROOF. Since our solution is not sufficiently smooth for further calculations, instead of u we must study its Steklov average $u_h(t,x) = (\frac{1}{h})\int_t^{t+h} u(z,x)dz$. Note that the global $L^\infty(\Omega)$ bounded for a function remains valid also for its Steklov average. If we fix $h > 0$, take the average of both components in (15), differentiate with respect to x_i, multiply by $[(\Delta(u_h))^{2^p-1})]_{x_i}|p = 1,2,\ldots$, sum over i and integrate over Ω, we get ($u_{th} = u_{ht}$)

$$\begin{aligned}&\int \sum u_{htx_i}[(\Delta(u_h))^{2^p-1}]_{x_i}dx\\ &= \int \sum [\Delta(u_h)]_{x_i}[(\Delta(u_h))^{2^p-1}]_{x_i}dx + \int f_{hx_i}[(\Delta u_h))^{2^p-1}]_{x_i}dx.\end{aligned}$$

Integrating by parts and using the Holder inequality, this gives

$$\begin{aligned}&\int_{\partial\Omega}(\Delta(u_h))^{2^p-1}\Big(\sum u_{x_i}\cos(n,x_i)\Big)_{ht}d\delta\\ &-\int(\Delta(u_h))_t(\Delta(u_h))^{2^p-1}dx = -\frac{2^p-1}{2^{2p-2}}\int\sum((\Delta(u_h))^{2^{p-1}})^2_{x_i}dx\\ &+\frac{2^p-1}{2^{p-1}}\Big(\int\sum f_{hx_i}(\Delta(u_h))^{2^p-2}dx\Big)^{\frac12}\Big(\int\sum((\Delta(u_h))^{2^{p-1}})^2_{x_i}dx\Big)^{\frac12}.\end{aligned}$$

Then from (16) and the boundedness of f_{hx_i} it follows that

$$\begin{aligned}&\frac{d}{dt}\int(\Delta u_h)^{2^p}dx \le -\frac{2^p-1}{2^{p-2}}\int\sum((\Delta u_h)^{2^{p-1}})^2_{x_i}dx\\ &+2(2^p-1)N\Big(n\int(\Delta u_h)^{2^p-2}dx\Big)^{\frac12}\Big(\int\sum((\Delta u_h)^{2^{p-1}})^2_{x_i}dx\Big)^{\frac12}.\end{aligned}\tag{19}$$

To obtain the global bound for $\int \Delta u_h^2 dx$ (the first step of induction $p=1$), we need (see [**11**, p. 83, Theorem 3.6.5]) the Poincare inequality ($\partial\Omega$-Lipschitz boundary)

$$\forall_{v\in W^{1,q}(\Omega)} c\int|v(x)-|\Omega|^{-1}\int v(y)dy|^q dx \le \int\sum|\frac{\partial v}{\partial x_i}|^q dx,\tag{20}$$

$c = c(n,q,\Omega)$, with $q=2$. This can be used to bound Δu_h^2, noting that in the presence of (16),

$$\int \Delta u_h(t,y)dy = 0.$$

Hence from (19) with $p = 1$ and Lemma 0, it follows that

$$\int (\Delta u_h)^2(t,x)dx \le \max\left\{ \int (\Delta u_h)^2(0,x)dx; \frac{N^2 n|\Omega|}{c} \right\}. \tag{21}$$

Denote the inductive bound of $\int \Delta u_h^{2^p} dx, p = 1, 2 \dots,$ by m_p; also let $w_p = \Delta u_h^{2^p}$. It follows from (19), the simple inequalities $(\Delta u_h)^{2^p-2} \le (\Delta u_h)^{2^p} + 1, \sqrt{a+b+c} \le \sqrt{a} + \sqrt{b} + \sqrt{c},\ a, b, c \ge 0$, and (18) that
(22)

$$\begin{aligned}\frac{d}{dt}||w_{p-1}||^2_{L^2(\Omega)} &\le \left(-\frac{2^p-1}{2^{p-2}} + 2(2^p-1)N\sqrt{n\varepsilon}\right)||(w_{p-1})_x||^2_{L^2(\Omega)} \\ &+ 2(2^p-1)N\sqrt{n}||(w_{p-1})_x||_{L^2(\Omega)}(\sqrt{C_\varepsilon}||w_{p-1}||_{L^1(\Omega)} + \sqrt{|\Omega|}).\end{aligned}$$

Choose $\varepsilon(p) = \varepsilon_p(\varepsilon(0,1))$ such that the first bracket above is less than or equal to –1, then by Lemma 0 and (20) it follows (for explicit w_{p-1}) that

$$\begin{aligned}\int \Delta u_h^{2^p}(t,x)dx \le \max\{&\int \Delta u_h^{2^p}(0,x)dx; \\ &c^{-1}(2(2^p-1)N\sqrt{n}(\sqrt{C_{\varepsilon_p}}m_{p-1} + \sqrt{|\Omega|}))^2\} \\ &=: m_p,\ p = 2, 3, \dots.\end{aligned} \tag{23}$$

Now h in (23) may be omitted, so we have the uniform bound for $\int \Delta u^{2^p} dx$ for all p. This, together with the boundedness of f, is, through (15), equivalent to the global estimates

$$\int u_t^{2^p} dx \le \text{ const },\ p = 1, 2, \dots, \tag{24}$$

and clearly gives global boundedness of u_t in both $L^k(\Omega)$. Our parabolic equation with fixed arbitrary $t > 0$ will now be considered as the elliptic problem

$$\Delta u + (f(t,x,H(u)(t)) - u_t) = 0,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

with the bracket bounded in $L^k(\Omega)$ independently on t. This, via the results of Chapter V, §7 of **[9]** ensures global in t estimates

$$\max_{\Omega} |u_x| \le \text{ const },\ \text{osc }(u_x, \Omega_\delta) \le \text{ cont } \delta^\alpha$$

and finishes our proof.

REMARK 6. By Theorem 3 and results of Chapter V, §7 of [**9**], one can get estimates for Holder norms of u_{x_i}; $i = 1, \dots, n$ (see Theorem 7.2 in [**9**]).

REMARK 7. See also Appendix B in [**4**].

REMARK 8. In several papers (see [**1, 3, 4, 5, 6, 11**]), generalizations of the reaction-diffusion problem

$$u_t = \Delta u + f(u), \tag{25}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u(0,x) \text{ given} \tag{26}$$

are considered. Often (cf. [**1,3,4,5,6,13,14**]) the global in t boundedness of its solution is shown. Having such $L^\infty(C^\circ)$ global boundedness, theorems of the proposed type lead (via Sobolev Imbedding Theorems, for example) to the compactness results for solutions considered as trajectories in suitable Banach spaces (compare [**5**, §5], [**6**, Remark 3]).

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REFERENCES

1. N.D. Alikakos, *An application of the invariance principle to reaction-diffusion equations,* J. Differential Equations **33** (1979), 201-225.
2. ———, and R. Rostamian, *Gradient estimates for degenerate diffusion equations,* Math. Ann. 259 (1982), 53-70.
3. K.N. Chueh, C.C. Conley, and J.A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations,* Indiana Univ. Math. J. **26** (1977), 373-392.
4. E. Conway, D. Hoff, and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations,* SIAM J. Appl. Math. **35** (1978), 1-16.
5. T. Dlotko, *Stability of non-linear parabolic equations with dominant main part,* Demonstratio Math. **19** (1986), 123-137.

6. ———, *Global solutions of reaction-diffusion equations,* submitted to Funkcial. Ekvac.

7. D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Springer Lecture Notes **840**, Berlin, 1981.

8. A.M. Ilin, A.S. Kalashnikov and O.A. Olejnik, *Linear second order parabolic equations,* Uspehi Mat. Nauk **17** (1962), 3-146.

9. O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural tseva, *Linear and Quasilinear Equations of Parabolic Type,* Nauka, Moscow, 1967.

10. J.L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites non Lineaires,* Dunod, Paris, 1969.

11. C.B. Morrey, *Multiple Integrals in the Calculus of Variations,* Springer, New York, 1966.

12. J. Moser, *A Harnack inequality for parabolic differential equations,* Comm. Pure Appl. Math. **17** (1964), 101-134.

13. M.M. Tang, *Asymptotic stability of some quasilinear parabolic equations in divergence form. L^∞ results,* J.Math. Anal. Appl. **52** (1977), 368-381.

14. M. Wiegner, *On the asymptotic behaviour of solutions of nonlinear parabolic equations,* Math. Z. **188** (1984) 3-22.

15. K. Yosida, *Functional Analysis,* Springer, Berlin, 1965.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, 40-007 KATOWICE, POLAND.