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THE STRONG FORM OF AHLFORS' LEMMA

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1. Introduction. Ahlfors [**1**] established an extension of Schwarz' lemma which plays an important role in geometric function theory. Ahlfors' lemma is valid for any hyperbolic Riemann surface; let us recall the result for the open unit disk D. If $\rho(z)|dz|$ is an ultrahyperbolic metric on D, then

$$\rho(z) \leq \lambda_D(z) = \frac{2}{1-|z|^2}. \tag{1}$$

Here $\lambda_D(z)|dz|$ is the hyperbolic metric on D normalized to have curvature -1. (In some references the curvature is taken to be -4; we will translate all such results to the context of curvature -1 without further comment.) The proof of (1) is astonishingly elementary; it relies on the fact that the Laplacian of a real-valued function is nonpositive at any point where the function has a relative maximum. Ahlfors did not show that equality in (1) at a single point implied $\rho = \lambda_D$ which would be the analog of the equality statement in Schwarz's lemma. Heins [**2**] introduced the class of SK metrics, which includes ultrahyperbolic metrics, and verified that (1) remains valid for SK metrics. In addition, he showed that equality at a single point implied $\rho = \lambda_D$. However, his proof of the equality statement is not as elementary as the proof of Ahlfors' lemma since it relies on an integral representation for a solution of the nonlinear partial differential equation $\Delta u = \exp(2u)$. In this paper we shall present an elementary proof of the equality statement for Ahlfors' lemma for SK metrics. Our proof is in the spirit of Ahlfors' derivation of (1) and is a modification of a method introduced by Hopf [**3**] for linear partial differential equations. A related proof was given by Jϕrgensen [**4**] in the special case of metrics with constant curvature -1.

2. Maximum principle. We prove a strong maximum principle for upper semicontinuous functions which satisfy the differential inequality

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$\Delta u \geq Ku$ in a generalized sense, where K is a positive constant.

DEFINITION. Suppose Ω is a region in C and $u : \Omega \to [-\infty, +\infty)$ is upper semicontinuous. The *generalized lower Laplacian* of u at any point a where $u(a) > -\infty$ is

$$\underline{\Delta} u(a) = \liminf_{r \to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta - u(a)\right).$$

If u is actually of class C^2 in a neighborhood of a, then it is straightforward to show that $\underline{\Delta} u(a) = \Delta u(a)$, the usual Laplacian of u at a. Also, if $u(a) > -\infty$ and u has a local maximum at a, then $\underline{\Delta} u(a) \leq 0$.

THEOREM 1. *Suppose Ω is a region in C, $u : \Omega \to [-\infty, +\infty)$ is upper semicontinuous and there is a positive constant K such that $\underline{\Delta} u(z) \geq Ku(z)$ at any point $z \in \Omega$ with $u(z) > -\infty$. If $\limsup_{z \to \xi} u(z) \leq 0$ for all $\xi \in \partial\Omega$, then either $u(z) < 0$ for all $z \in \Omega$ or else $u(z) = 0$ for all $z \in \Omega$.*

PROOF. We begin by showing that $u(z) \leq 0$ for all $z \in \Omega$. Set $M = \sup\{u(z) : z \in \Omega\}$. We wish to show $M \leq 0$. There is a sequence $\{z_n\}_{n=1}^{\infty}$ in Ω with $u(z_n) \to M$. There is no harm in assuming that $z_n \to a \in \mathrm{cl}(\Omega)$, the closure of Ω relative to the Riemann sphere. If $a \in \partial\Omega$, then

$$M = \lim_{n \to \infty} u(z_n) \leq \limsup_{z \to a} u(z) \leq 0.$$

Now, suppose $a \in \Omega$. Then

$$M = \lim_{n \to \infty} u(z_n) \leq \limsup_{z \to a} u(z) \leq u(a),$$

so that $u(a) = M$. If $u(a) = -\infty$, then $M = -\infty$. Otherwise, $u(a) > -\infty$ and u has a maximum value at a, so

$$0 \geq \underline{\Delta} u(a) \geq Ku(a).$$

Since $K > 0$, we obtain $M = u(a) \leq 0$. This proves that $u(z) \leq 0$ for all $z \in \Omega$.

Next, we show that either $u(z) < 0$ for all $z \in \Omega$ or $u(z) = 0$ for all $z \in \Omega$. The set $A = \{z \in \Omega : u(z) < 0\}$ is open because u is upper semicontinuous. It suffices to show that $A = \Omega$ if $A \neq \emptyset$. For each $a \in A$ we will show that $\{z : |z - a| < \delta\} \subset A$, where δ is the distance from a to $\partial\Omega$. From this it readily follows that $A = \Omega$ when $A \neq \emptyset$. Note that $Z = \{z \in \Omega : u(z) = 0\}$ is a closed set in Ω. If the distance from the point a to the set Z is at least δ, then we are done. Otherwise, there exists $R \in (0, \delta)$ such that $u(z) < 0$ for $|z - a| < R$ and there exists a point $z_0 \in Z$ with $|z_0 - a| = R$. Now, we construct an auxiliary function. Set

$$v(z) = \exp(-\alpha|z - a|^2) - \exp(-\alpha R^2),$$

where $\alpha > 0$ is to be specified. Note that $v(z) > 0$ for $|z - a| < R, v(z) = 0$ for $|z - a| = R$ and $v(z) < 0$ for $|z - a| > R$. Now,

$$\Delta v(z) = (4\alpha^2|z - a|^2 - 4\alpha)\exp(-\alpha|z - a|^2)$$

so that

$$\Delta v(z) - Kv(z) \geq (4\alpha^2|z - a|^2 - 4\alpha - K)\exp(-\alpha|z - a|^2).$$

For $R/2 \leq |z - a| \leq R$ we have

$$\Delta v(z) - Kv(z) \geq (\alpha^2 R^2 - 4\alpha - K)\exp(-\alpha|z - a|^2).$$

We take $\alpha > 0$ large enough to insure that

$$\alpha^2 R^2 - 4\alpha - K \geq 0. \tag{2}$$

Set $w = u + \varepsilon v$, where $\varepsilon > 0$ is to be determined. Because $u(z) \leq 0$ in Ω and $v(z) = 0$ on $|z - a| = R$, we obtain $\limsup_{z \to \xi} w(z) \leq 0$ for all ξ with $|\xi - a| = R$. Since the upper semicontinuous function u is negative on the compact set $\{z : |z - a| = R/2\}$, it attains a negative maximum on this set. Thus, we can take $\varepsilon > 0$ so small that $w(z) \leq 0$ on $|z - a| = R/2$. From the fact that v is of class C^2, we obtain

$$\begin{aligned} \underline{\Delta} w(z) &= \underline{\Delta} u(z) + \varepsilon \Delta v(z) \\ &\geq Ku(z) + \varepsilon K v(z) = Kw(z) \end{aligned}$$

at any point z with $w(z) > -\infty$. The first part of the proof yields $w(z) \leq 0$ for $R/2 \leq |z - a| < R$. Because $u(z) \leq 0$ in Ω and $v(z) < 0$

in $|z-a|>R$, we conclude that $w(z)\le 0$ in a full neighborhood of any point on the circle $|z-a|=R$ which lies in Ω. In particular, this holds at z_0 so w attains a local maximum at z_0. Thus,

$$\underline{\Delta}u(z_0)+\varepsilon\Delta v(z_0)=\underline{\Delta}w(z_0)\le 0,$$

so

$$Ku(z_0)\le\underline{\Delta}u(z_0)\le-\varepsilon\Delta v(z_0)=-\varepsilon(4\alpha^2R^2-4\alpha)\exp(-\alpha R^2).$$

Since (2) implies $4\alpha^2R^2-4\alpha>0$, we obtain $u(z_0)<0$, a contradiction. Consequently, we must have $u(z)<0$ for $|z-a|<\delta$ and the proof is complete.

3. The Strong Form of Ahlfors' Lemma. Heins [**2**] introduced the notion of an SK metric. A conformal metric $\rho(z)|dz|$ on a region Ω is called an SK metric provided $\rho:\Omega\to[0,+\infty)$ is upper semicontinuous and $\underline{\Delta}\log\rho(a)\ge\rho^2(a)$ at each point $a\in\Omega$ such that $\rho(a)>0$. Recall that if ρ is of class C^2 in a neighborhood of a and $\rho(a)>0$, then

$$\kappa(a,\rho)=-\frac{\Delta\log\rho(a)}{\rho^2(a)}$$

is the (Gaussian) curvature of $\rho(z)|dz|$ at a. Thus, an SK metric is a conformal metric with generalized curvature at most –1 at each point where it does not vanish. A region Ω is hyperbolic if $C\backslash\Omega$ contains at least two points. The hyperbolic metric $\lambda_\Omega(z)|dz|$ on Ω is the unique metric on Ω such that $\lambda_D(z)|dz|=\lambda_\Omega(f(z))|f'(z)|$, where $f:D\to\Omega$ is any holomorphic universal covering projection. The hyperbolic metric has constant curvature –1.

THEOREM 2. *Let Ω be a hyperbolic region in C and $\lambda_\Omega(z)|dz|$ the hyperbolic metric on Ω. If $\rho(z)|dz|$ is an SK metric on Ω, then either $\rho(z)<\lambda_\Omega(z)$ for all $z\in\Omega$ or else $\rho(z)=\lambda_\Omega(z)$ for all $z\in\Omega$.*

PROOF. Ahlfors' lemma, as refined by Heins [**2**], gives $\rho(z)\le\lambda_\Omega(z)$ for all $z\in\Omega$. Because Ω is connected, it is sufficient to show that each point of Ω has an open neighborhood such that either $\rho(z)=\lambda_\Omega(z)$ in this neighborhood or $\rho(z) < \lambda_\Omega(z)$ in this neighborhood. Fix $a \in \Omega$ and take $r > 0$ such that $\{z : |z-a| \leq r\} \subset \Omega$. Set $B = \{z : |z-a| < r\}$. There exists $M > 0$ such that $\rho(z) \leq \lambda_\Omega(z) \leq M$ for all $z \in B$. Now, $u = \log(\rho/\lambda_\Omega)$ is upper semicontinuous on $B, u(z) \leq 0$ for $z \in B$ and at any point $z \in B$ where $u(z) > -\infty$; that is, where $\rho(z) > 0$, we have

$$\begin{aligned} \underline{\Delta} u(z) &= \underline{\Delta} \log \rho(z) - \Delta \log \lambda_\Omega(z) \\ &\geq \rho^2(z) - \lambda_\Omega^2(z) \\ &\geq 2M(\rho(z) - \lambda_\Omega(z)). \end{aligned}$$

Here we have used the facts that $\rho(z) - \lambda_\Omega(z) \leq 0$ and $\rho(z) + \lambda_\Omega(z) \leq 2M$ for $z \in B$. For $0 < s \leq t \leq M$ we have the elementary inequality $M \log(\frac{t}{s}) \geq t - s$. Thus $M \log \frac{\lambda_\Omega(z)}{\rho(z)} \geq \lambda_\Omega(z) - \rho(z)$, and so $\underline{\Delta} u(z) \geq 2M^2 u(z)$ for $z \in B$ and $u(z) > -\infty$. Theorem 1 implies that either $\rho(z) < \lambda_\Omega(z)$ for $z \in B$ or else $\rho(z) = \lambda_\Omega(z)$ for $z \in B$.

Added in proof. Recently, H.L Royden [The Ahlfors–Schwarz lemma: the case of equality, J. Analyse Math. 46 (1986), 261-2701] established the sharp form of Ahlfors' lemma by a different method.

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