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CONTINUA WITH A DENSE SET OF END POINTS

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ABSTRACT. The structure of metric continua with a dense set of end points is investigated. It is shown that a continuum has a dense set of end points if and only if it is either indecomposable or the union of two proper indecomposable subcontinua with connected intersection, each having a dense set of its end points lying outside the composant containing the intersection and such that the intersection is an end continuum in both subcontinua.

A continuum means a compact connected metric space. Throughout this paper X always denotes an arbitrary continuum, and $C(X)$ is the hyperspace of all nonempty subcontinua of X equipped with the Hausdorff distance denoted by dist (see [**5**; §42, II, p. 47] for the definition).

If $K \in C(X)$ and if for each $L, M \in C(X)$ with $K \subset L \cap M$ we have either $L \subset M$ or $M \subset L$, then K is called an end continuum in X. Note that X is an end continuum in itself. In particular, if $K = \{p\}$, then the point p is called an end point of X (see [**3**; p. 660, 661]). The set of all end points of X is denoted by $E(X)$. Observe that $K \in C(X)$ is an end continuum in X if and only if K is an end point of the decomposition space X/K of the monotone upper semi-continuous decomposition of X whose only nondegenerate element is K.

Note that if we restrict our considerations to proper subcontinua of a given continuum, then what we call "end continua" here are called "terminal continua" in [**4**; Definition 4, p.461] and "absolutely terminal continua" in [**2**; Definition 4.1, p.34]. The same concerns points.

PROPOSITION 1. *The set $E(X)$ is a G_δ-set*

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In fact, defining a function g from $C(X) \times C(X)$ into reals by

$$g(K, L) = \min\{\text{dist}(K, K \cup L), \text{dist}(L, K \cup L)\}$$

we see that g is continuous. Therefore the sets

$$F_n = \{x \in X : \text{ there are } K, L \in C(X) \text{ such that } x \in K \cap L \text{ and } g(K, L) \geq 1/n\}$$

are closed for each $n \in \{1, 2, ...\}$. The equality $X \setminus E(X) = \cup\{F_n : n \in \{1, 2, ...\}\}$ shows the conclusion.

This paper concerns continua X having a dense set $E(X)$ of end points. We begin with three simple observations. First, recall that each point of X is an end point, i.e., $X = E(X)$, if and only if X is hereditarily indecomposable. In particular, the pseudo–arc is an example of such a continuum ([**3**; Theorem 16, p. 662]). Second, note that if we replace a point of the pseudo–arc by a continuum, say an arc (in the sense that the continuum is a remainder of the complement of the point in the pseudo–arc; see [**1**; Theorem, p. 35]), then we get an indecomposable continuum having the set of end points as a dense proper subset. Third, the one–point union X of two pseudo–arcs P_1 and P_2 with $P_1 \cap P_2 = \{p\}$ also has the considered property: if C_1 and C_2 are composants of P_1 and P_2 respectively, both containing p, then $E(X) = (P_1 \setminus C_1) \cup (P_2 \setminus C_2)$.

PROPOSITION 2. *If* $\overline{E(X)} = X$, *then:*

(1) X *is unicoherent*;

(2) *if* $K \in C(X)$ *and* $\text{int}\, K \neq \emptyset$, *then* $X \setminus K$ *is connected;*

(3) *if* $K \in C(X)$, *then* $X \setminus K$ *has at most two components*;

(4) X *is irreducible*;

(5) *if* X *is decomposed into two proper subcontinua* A *and* B, *then* $\overline{X \setminus A}$ *and* $\overline{X \setminus B}$ *are closed connected domains in* B *and in* A *respectively, whose union is* X;

(6) X *contains at most two distinct proper closed connected domains*;

(7) *each closed connected domain properly contained in* X *is indecomposable*;

(8) *each closed connected domain contained in* X *has a dense set of its end points.*

PROOF 1. Suppose to the contrary that there are two proper subcontinua P and Q of X such that $X = P \cup Q$ and $p \cap Q = M \cup N$, where M and N are nonempty disjoint closed sets. Let sets U and V be open such that $M \subset U$, $N \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. There are components K_1 and K_2 of $P \cap \overline{U}$ and $P \cap \overline{V}$ intersecting M and N respectively. The unions $K_1 \cup Q$ and $K_2 \cup Q$ are continua. By the Janizewski theorem ([**5**], §47, III, Theorem 1, p. 172) we have $K_1 \cup Q \neq Q \neq K_2 \cup Q$. Further, $(K_1 \cup Q) \cap (K_2 \cup Q) = Q$. Therefore $Q \subset X \backslash E(X)$. Since int $Q \neq \emptyset$, we have a contradiction with the assumption $\overline{E(X)} = X$.

2. Let $K \in C(X)$ with int $K \neq \emptyset$ be given. Suppose $X \backslash K = M \cup N$, where M and N are nonempty mutually separated sets. Then $K \cup M$ and $K \cup N$ are continua and $(K \cup M) \cap (K \cup N) = K$. Thus $K \subset X \backslash E(X)$. But (int $K) \cap E(X) \neq \emptyset$, which is a contradiction.

3. Suppose to the contrary that there is $K \in C(X)$ such that $X \backslash K$ has more than two components. Then there are three mutually separated nonempty open sets U, V, and W such that $X \backslash K = U \cup V \cup W$ (see [**5**; §46, IV, Theorem 4, p. 143]). The set $K \cup U$ is a continuum ([**5**; §46, II, Theorem 4, p. 133]), and since U is open, we have int $(K \cup U) \neq \emptyset$. Thus $X \backslash (K \cup U)$ is connected by (2). On the other hand $X \backslash (K \cup U) = V \cup W$, where V and W are nonempty and separated, which is a contradiction.

4. Now (3) implies that X is not a triod, whence by (1) and the Sorgenfrey result ([**6**; Theorem 3.2, p. 456]; cf. [**2**; Theorem 2.12, p. 21]) we conclude that X is irreducible.

5. Since int $A \neq \emptyset$, the set $X \backslash A$ is connected by (2), and so is $\overline{X \backslash A}$. Further, $X \backslash A = \text{int}\,(X \backslash A) \subset \text{int}\,\overline{X \backslash A} \subset \overline{X \backslash A}$ implies $\overline{X \backslash A} = \overline{\text{int}\,\overline{X \backslash A}}$, which means that $\overline{X \backslash A}$ is a closed domain. The same holds for $\overline{X \backslash B}$. Note that $A \cap B$ is a continuum by (1), and it disconnects X, i.e., $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$, where $X \backslash A$ and $X \backslash B$ are nonempty, open and disjoint. Thus int $(A \cup B) = \emptyset$ by (2), whence $X = \overline{X \backslash (A \cup B)} = \overline{X \backslash A} \cup \overline{X \backslash B}$. So (5) is proved.

6. By (4) there are two points a and b in X such that X is irreducible between a and b. Let D be an arbitrary closed connected domain in X. Observe that D contains either a or b (or both), because otherwise $X \backslash D$ is a connected set (by (2)) containing both a and b, whence $\overline{X \backslash D}$ is a proper subcontinuum of X containing a and b contrary to irreducibility of X.

Suppose to the contrary that there are in X three distinct closed connected domains. So there are two of them, D_1 and D_2, containing the same point of irreducibility, say a. Since the family of all closed connected domains containing a is strictly monotone ([**5**; §48, III, Theorem 2, p. 195]), we may assume $D_1 \subset \operatorname{int} D_2$. Thus the sets D_1 and $\overline{X\backslash D_2}$ are disjoint, and the latter is a closed connected domain containing the point b ([**5**; §48, III, Theorem 5, p. 196]). Then the set $X\backslash(D_1 \cup \overline{X\backslash D_2})$ is open and connected ([**5**; §48, II, Theorem 4, p. 193]), and therefore its closure, D, is a closed connected domain containing neither a nor b, which is a contradiction.

7. Let a closed connected domain D be a proper subset of X. Then by (2) the set $\overline{X\backslash D}$ is a proper subcontinuum of X and we obviously have $X = D \cup \overline{X\backslash D}$. Suppose to the contrary that there are continua P and Q such that $D = P \cup Q$ and $P \neq D \neq Q$. Then at least one of them intersects $\overline{X\backslash D}$. Assume $Q \cap \overline{X\backslash D} \neq \emptyset$. If $P \cap \overline{X\backslash D} = \emptyset$, then $X\backslash(P \cup \overline{X\backslash D})$ is a nonempty (by connectedness of X) open subset of Q, whence $\operatorname{int} Q \neq \emptyset$, and by (2) the set $X\backslash Q$ is connected. On the other hand $X\backslash Q$ is the union of two nonempty disjoint open sets, namely $X\backslash D$ and $P\backslash Q = X\backslash(Q \cup \overline{X\backslash D})$. Thus $P \cap \overline{X\backslash D} \neq \emptyset \neq Q \cap \overline{X\backslash D}$. Hence $P \cup \overline{X\backslash D}$ and $Q \cup \overline{X\backslash D}$ are continua. Since $\overline{X\backslash D}$ has the nonempty interior, we have $\overline{X\backslash D} \cap E(X) \neq \emptyset$, and therefore one of the two continua contains the other. Assume $P \cup \overline{X\backslash D} \subset Q \cup \overline{X\backslash D}$. Then $X = P \cup Q \cup \overline{X\backslash D} = Q \cup \overline{X\backslash D}$, whence $\operatorname{int} D = X\backslash\overline{X\backslash D} \subset Q$ and thereby $D = \overline{\operatorname{int} D} \subset Q$, which is a contradiction.

8. Let D be a closed connected domain in X. To prove $\overline{E(D)} = D$ observe first that $E(X) \cap \operatorname{int} D \subset E(D)$. Since the set $E(X) \cap \operatorname{int} D$ is dense in $\operatorname{int} D$ and $\operatorname{int} D$ is dense in D, hence $E(X) \cap \operatorname{int} D$ is a dense subset of D, and so the needed equality follows from the inclusion.

STATEMENT 3. *The following conditions are equivalent:*

(9) *X is indecomposable and $\overline{E(X)} = X$;*

(10) *for each composant C of X we have $\overline{E(X)\backslash C} = X$;*

(11) *there is a composant C of X such that $\overline{E(X)\backslash C} = X$.*

If (9) is assumed, then, since C is a boundary F_σ-set in X and $E(X)$ is a dense G_δ-set in X (by Proposition 1) we have (10) by the Baire category theorem. The implication from (10) to (11) is trivial. Finally (11) implies that $X = \overline{E(X)\backslash C} \subset \overline{X\backslash C} \subset X$, thus the composant C is a boundary subset of X and, consequently, X is indecomposable ([**5**; §48, VI, Theorem 8, p. 213]). Further, $X = \overline{E(X)\backslash C} \subset \overline{E(X)} \subset X$, whence $\overline{E(X)} = X$ and so (9) holds.

It is obvious that an end point of a continuum is an end point of a subcontinnum containing the point. The next proposition shows that in certain circumstances the inverse also is true.

PROPOSITION 4. *Let X be the union of two proper indecomposable subcontinua X_1 and X_2 whose intersection $X_1 \cap X_2$ is connected and is an end continuum in both X_1 and X_2. Let C_1 and C_2 denote composants of X_1 and X_2 respectively, containing $X_1 \cap X_2$. Then*

$$(E(X_1)\backslash C_1) \cup (E(X_2\backslash C_2) \subset E(X).$$

PROOF. By the symmetry of assumptions it is enough to show $E(X_1)\backslash C_1 \subset E(X)$ only. So take a point $p \in E(X_1)\backslash C_1$ and let $L \in C(X)$ contain p. We claim that

(*) if $L\backslash X_1 \neq \emptyset$, then $X_1 \subset L$ and $L \cap X_2$ is connected.

In fact, since X_1 and X_2 are proper subcontinua of X, their intersection separates X between $X_1\backslash X_2$ and $X_2\backslash X_1$. Thus $L \cap X_1 \cap X_2 \neq \emptyset$, and thereby $L \cup (X_1 \cap X_2)$ is a continuum. Now $L\backslash(X_1 \cap X_2) = (L\backslash X_1) \cup (L\backslash X_2)$ and the sets $L\backslash X_1$ and $L\backslash X_2$ are both nonempty (the former just by the assumption; the latter since $p \in L \cap (E(X_1)\backslash C_1) \subset L \cap (X_1\backslash(X_1 \cap X_2)) \subset L\backslash X_2$) and mutually separated. Thus the unions $(L\backslash X_1)\cup(X_1\cap X_2)$ and $(L\backslash X_2)\cup(X_1\cap X_2)$ are continua ([**5**; §46, II, Theorem 4, p. 133]), the latter of which lies in X_1 and joins p with $X_1 \cap X_2$. Since p is out of C_1, the continuum is X_1, i.e., $(L\backslash X_2)\cup(X_1\cap X_2) = X_1$. Thus $X_1\backslash(X_1 \cap X_2) = X_1\backslash X_2 \subset L$, whence $\overline{X_1\backslash X_2} \subset L$. Since $X_1 \cap X_2 \subset C_1$ and C_1 is a boundary subset of the indecomposable continuum X_1 ([**5**; §48, VI, Theorem 6, p. 212]), we have $\overline{X_1\backslash X_2} = X_1$ and so the inclusion $X_1 \subset L$ follows. Now we see that the continnum $(L\backslash X_1) \cup (X_1 \cap X_2)$ equals $L \cap X_2$ (because $X_1 \cap X_2 \subset X_1 \subset L$), and so (*) is proved.

Now let us come back to the point p, and take two continua $L, M \in C(X)$ such that $p \in L \cap M$. If $L \cup M \subset X_1$, then either $L \subset M$ or $M \subset L$ since $p \in E(X_1)$. If $L\backslash X_1 \neq \emptyset$ and $M \subset X_1$, then by (*) we have $M \subset L$. It remains to consider the case when $L\backslash X_1 \neq \emptyset$ and $M\backslash X_1 \neq \emptyset$. By (*) we have $X_1 \subset L \cap M$ and the intersections $L \cap X_2$ and $M \cap X_2$ are both subcontinua of X_2 that contain $X_1 \cap X_2$. Since $X_1 \cap X_2$ is an end continuum in X_2 we have either $L \cap X_2 \subset M \cap X_2$ or $M \cap X_2 \subset L \cap X_2$. Since $X_1 \subset L \cap M$, we see that $L = X_1 \cup (L \cap X_2)$

and $M = X_1 \cup (M \cap X_2)$, whence either $L \subset M$ or $M \subset L$. Therefore $p \in E(X)$ and the proof is complete.

In the next proposition a sufficient condition is presented for density of the set of end points of a continuum.

PROPOSITION 5. *If X is the union of two proper indecomposable subcontinua X_1 and X_2, each having a dense set of its end points, such that the intersection $X_1 \cap X_2$ is connected and is an end continuum in both X_1 and X_2, then $\overline{E(X)} = X$.*

PROOF. Let C_1 and C_2 have the same meaning as in Proposition 4. Applying Statement 3 to X_1 and to X_2 separately, we get

$$\overline{E(X_1)\backslash C_1} = X_1 \text{ and } \overline{E(X_2)\backslash C_2} = X_2,$$

whence by Proposition 4 we obtain $X = X_1 \cup X_2 \subset \overline{E(X)} \subset X$, and thereby the conclusion holds.

Combining Propositions 2 and 5 we have

THEOREM 6. *X is a decomposable continuum with $\overline{E(X)} = X$ if and only if X is the union of two proper indecomposable continua X_1 and X_2 with dense sets of their end points and such that $X_1 \cap X_2$ is an end continuum in both X_1 and X_2.*

Further information about how $E(X)$ is situated in X, when conditions considered above are satisfied, is contained in a proposition below.

PROPOSITION 7. *If X is the union of two proper subcontinua X_1 and X_2 whose intersection $X_1 \cap X_2$ is connected, and if C_1 and C_2 denote the composants of X_1 and X_2 respectively, containing $X_1 \cap X_2$, then $E(X) \subset X\backslash(C_1 \cup C_2)$.*

PROOF. Pick a point $p \in C_1 \cup C_2$ and assume $p \in C_1$. By the definition of a composant there are continua P_1, $Q_1 \in C(X_1)$ and $P_2, Q_2 \in C(X_2)$ such that $\{p\} \cup (X_1 \cap X_2) \subset P_1 \subset Q_1 \neq P_1$ and $X_1 \cap X_2 \subset P_2 \subset Q_2 \neq P_2$ and $P_1\backslash(X_1 \cap X_2) \neq \emptyset \neq P_2\backslash(X_1 \cap X_2)$. So the continua $P_1 \cup Q_2$ and $P_2 \cup Q_1$ both contain p, and we have $(P_1 \cup Q_2)\backslash(P_2 \cup Q_1) \neq \emptyset \neq (P_2 \cup Q_1)\backslash(P_1 \cup Q_2)$. Thus p is not an end point of X.

REMARK 8. As it was said in the beginning, metrizability of the continuum X has been assumed in the whole paper and it was essentially exploited in the presented proofs of some results–see e.g. Proposition 1 and the proof of Statement 3, where the Baire category theorem has been used. If we however replace these "metric" arguments by a condition stated in Proposition 7, then, arguing as above with necessary changes, we are able to show in the nonmetric setting the following result that is slightly weaker that Theorem 6 for the metric case.

THEOREM 9. *A decomposable Hausdorff (not necessarily metric) continuum has a dense set of end points if and only if it is the union of two proper indecomposable subcontinua with connected intersection, each having a dense set of its end points lying outside the composant containing the intersection and such that the intersection is an end continuum in both subcontinua.*

PROBLEM 10. Does there exist a Hausdorff (nonmetric) indecomposable continuum having a dense set of end points and exactly one composant?

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