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LACUNARITY FOR AMALGAMS

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ABSTRACT. Amalgams are spaces of functions on locally compact groups; the condition for membership of a function in an amalgam is a mixture of local and global conditions on the function. This paper concerns the extent to which the classical theory of thin sets in discrete abelian groups transfers to the context of amalgams on noncompact, locally compact, abelian groups. The classes of thin sets considered here are the $\Delta(p)$ sets and the p–Sidon sets. The transference is quite satisfactory for $\Delta(p)$ sets, while the results for p–Sidon sets are less complete. In both cases, the transference yields conclusions about pseudodilation of thin sets in discrete groups.

1. Introduction. Denote the unit circle by T and the real line by R. It is well known that if a function in $L^1(T)$ has its Fourier coefficients supported by a sufficiently thin set, then the function also belongs to the smaller spaces $L^p(T)$ for all indices p in the intervals $(1, \infty)$. The analogous statement also holds for functions in $L^1(R)$ with thinly supported Fourier transforms [**7**, Theorem 3], but this conclusion is less satisfying, because the spaces $L^p(R)$, for $1 < p < \infty$, are merely different from $L^1(R)$ rather than being smaller than the latter space. The point of the present paper is that replacing the L^p-spaces by amalgams based on them leads to more satisfactory results for certain types of thin sets in R and other nondiscrete, noncompact, locally compact, abelian groups. These results include new conclusions about lacunarity and L^p-spaces on such groups. Moreover, they also lead to new examples of thin sets in discrete abelian groups.

We will discuss the case of the group R in this section, and deal with the general situation in §2 and §3. Denote the integer lattice in R by Z. Let I be the half-open interval $[0, 1)$, and for each integer n, let

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$I_n = n + I$. Given a measurable function f on R and indices p and q in the interval $(0, \infty]$, let

$$(1) \qquad ||f||_{p.q} = (\sum_{n\in z}((\int_{I_n} ||f||^p)^{1/p})^q)^{1/q},$$

provided that p and q are both finite, and make the appropriate changes for infinite indices. In other words, the quantity $||f||_{p,q}$ is computed by taking the L^p-norm of f over each of the intervals I_n, thereby getting a sequence of extended-real numbers, and then taking the ℓ^q-norm of this sequence. Say that f belongs to the *amalgam* $(L^p, \ell^q)(R)$ if $||f||_{p,q}$ is finite. This amalgam becomes a Banach space with the norm $||\cdot||_{p,q}$ provided that the indices p and q are both at least 1, and functions that agree almost everywhere are identified. We require henceforth that $p \geq 1$ and $q \geq 1$. The amalgams $(L^p, \ell^p)(R)$ coincide isometrically with the spaces $L^p(R)$; since our goal in this paper is to discuss relations between amalgams, we will usually view the L^p-spaces as amalgams.

Given a subset E of Z, and a function f in $L^1(T)$, call f an *E-function* if the Fourier coefficients $\hat{f}(n)$ vanish for all integers n outside E. Given an index p in the interval $(1, \infty)$, call E a $\Lambda(p)$*set* if every E-function, which must a priori belong to $L^1(T)$, belongs to $L^p(T)$. Much is known, and much is still not known about $\Lambda(p)$ sets in discrete abelian groups like Z; see [**15**] and [**11**] and the references cited therein.

Similarly, we will call a measurable subset F of R a *Lambda*(p)*set* if every F-function belongs to $L^p(R)$. We initially define an *F-function* to be an element f, say, of $L^1(R)$ whose Fourier transform $\hat{f}$ vanishes off the set F; because $\hat{f}$ must be continuous in this case, the notion of F-function, and therefore the notion of $\Delta(p)$ subset of R, is mainly of interest when F has nonempty interior. In this paper, we will mostly consider subsets of R having the form $E + I$, where E is a subset of the lattice Z, and I is the interval $[0, 1)$ used above in the definition of amalgam. We will prove the following assertion in the next section.

THEOREM 1. *Fix an index p in the interval $(1, \infty)$. Let E be a subset of Z, and let $F = E + I$. Then the following statements are equivalent.*

(i) *The set E is a $\Lambda(p)$ set.*

(ii) *The set F is a $\Lambda(p)$ set.*

(iii) *For each index q in the interval $[1, \infty]$, it is the case that if a function f belongs to the amalgam $(L^1, \ell^q)(R)$, and if $\hat{f}$ vanishes off F, then f also belongs to the amalgam $(L^p, \ell^q)(R)$.*

In part (iii) above, the object $\hat{f}$ and the condition that it vanish off F should in general be interpreted in the sense of distributions. To this end, note that, for fixed p, the amalgams $(L^p, \ell^q)(R)$ become larger as q *increases*, while, for fixed q, these amalgams become larger as p *decreases*. So, the smallest amalgam that we consider here is $(L^\infty, \ell^1)(R)$; the set of all continuous functions in this amalgam has received special attention under the name of *Wiener algebra* [**12**, 39.33]. The largest amalgam is $(L^1, \ell^\infty)(R)$; every function in this space can be regarded as a tempered distribution. Although we will not use this fact, we note that $(L^1, \ell^2)(R)$ is the largest amalgam with the property that every function in it has a Fourier transform that is also a function [**18**]. In the rest of this section we use the term *F-function* to mean any function in $(l^1, \ell^\infty)(R)$ whose Fourier transform is a distribution that annihilates all test functions supported by the complement of the set F.

For special sets F, conclusions that are stronger in some ways than assertion (iii) above can be deduced from the properties of the Littlewood-Paley square-function; see the discussion after the proof of Theorem 1 in §2. It was also proved in another way [**7**] for special sets F that if $f \in L^2(R)$, and if $\hat{f}$ vanishes off F, then $f \in L^p(R)$ for all p in the interval $(2, \infty)$; it then followed [**7**] by an argument [**15**, p.55] based on Holder's inequality that the latter conclusion also holds if $f \in L^r(R)$, for some r in the interval $[1, 2)$, and $\hat{f}$ vanishes off F.

On the other hand, it was not possible using the methods in [**7**] to deduce much from the assumption that f is an F-function in the space $L^r(R)$ for some index r in the interval $(2, \infty)$, the problem being that it does not follow from this assumption that $f \in L^2(R)$. Matters are simpler with assertion (iii) of the theorem in hand. The space $L^r(R)$ coincides with the amalgam $(L^r, \ell^r)(R)$, which is included in the larger amalgam $(L^1, \ell^r)(R)$; so every F-function f in $L^r(R)$ is an F-function in $(L^1, \ell^r)(R)$, and by the theorem $f \in (L^p, \ell^r)(R)$, which is a proper subamalgam of $L^p(R)$ when $p > r$. Moreover, although the notion of amalgam is not needed to state conditions (i) and (ii), this notion is needed in our proof that these conditions are equivalent, because the proof runs via statement (iii).

According to assertion (iii), the condition that $\hat{f}$ vanish off the set F forces f to belong locally to $L^p(R)$, which is locally a smaller space than $L^1(R)$. It is natural to ask if this restriction on the support of $\hat{f}$ also forces an improvement on the index q controlling the *global* size of f. We will see in the next section that no such improvement is possible if F has nonempty interior.

Next, recall that a subset E of Z is called a *p-Sidon set*, where $p \in [1,2)$, if every E-function f in $L^\infty(T)$ has the property that $\hat{f} \in \ell^p(Z)$; this definition also makes sense when $p \geq 2$, but is uninteresting in that case because $\hat{f} \in \ell^2(Z)$ for all f in $L^\infty(T)$. A set that is 1-Sidon is usually just called a *Sidon set*; the class of Sidon sets has been much studied [**15**]. See [**3**] for current knowledge about p-Sidon sets, with $p > 1$, in discrete abelian groups. We will prove the following statement in 3.

THEOREM 2. *Fix an index p in the interval* $[1,2)$. *Let E be a subset of Z, and let $F = E + I$. Then the following statements are equivalent.*

(a) *The set E is a p-Sidon set.*

(b) *Every F-function f in the amalgam $(L^\infty, \ell^1)(R)$ has the property that $\hat{f} \in L^p(R)$.*

This theorem poses more questions than it answers. There is a Hausdorff-Young theorem for amalgams [**13**], which in particular asserts that if $f \in (L^2, \ell^1)(R)$, then $\hat{f} \in (L^\infty, \ell^2)(R)$. This suggests that the conclusion in part (b) above ought to be that $\hat{f} \in (L^\infty, \ell^p)(R)$. Similarly, it seems plausible that if E is a p-Sidon set, and if f is an F-function in $(L^\infty, \ell^q)(R)$ where q is some index in the interval $(1,2]$, then $\hat{f} \in (L^q, \ell^p)(R)$; here q denotes the index conjugate to q.

Finally, we will end §3 by applying our results about thin sets in nondiscrete groups to obtain interesting conclusions about pseudodilation of thin sets in *discrete* groups. For instance, given a subset E of Z, let "πE" be the subset of Z obtained by first forming the set πE in R, and then replacing each element of πE by the integer nearest to it. It follows easily from our work here that "πE" is a $\Delta(p)$ set if E is, and that the same is true for p-Sidonicity.

2. $\Lambda(p)$ sets. We define amalgams on a locally compact abelian group G using a disjoint cover of G by sets I_α obtained via the structure theorem [**12**; Theorem 24.30] for such groups. We summarize the procedure here, and refer to [**17**] or [**8**] for more details. We write all group operations additively. The main difficulties in the proofs of our theorems all arise when $G = R^N$ for some integer $N \geq 1$. Any reader who is only interested in this case can skip the paragraphs concerning other cases.

In general, the group G can be identified with a direct product $R^N \times J$, where H is a locally compact group with at least one compact open subgroup. Fix such a subgroup, say H, of J, and let $I_0 = [0,1)^N \times H$. Choose a subset M of J consisting of one representative from each coset of the subgroup H, and let $A = Z^N \times M$; then the sets I_α given by letting $I_\alpha = I_0 + \alpha$, for all α in A, form the desired cover of G. There is a similar cover $\{\hat{I}_\beta\}_{\beta \in B}$ of the dual group $\hat{G}$ of G. The results in this paper are only of interest when the groups G, and hence their duals $\hat{G}$, are noncompact and nondiscrete. We suppose in the rest of this paper that G and $\hat{G}$ have these properties; this means that each of the sets I_α and I_β is infinite, as are the index sets A and B. Amalgams on G are then defined as in formula (1) of the previous section by computing L^p-norms over each of the sets I_α, and then taking the ℓ^q-norm of the resulting function on the set A; the procedure on $\hat{G}$ is similar.

Amalgams in this setting can also be defined without using the structure theorem; see the papers cited in [**8**; §2] concerning this point. We need the structure theorem, however, to describe the thin sets that we will study, and we will use it in the proofs of our theorems. In R^N, the index set Z^N used to define amalgams is a group, and can be regarded as the dual group of T^N. Given a subset E of Z^N let $F = E + [0,1)^N$. Since Z^N is an abelian group, it makes sense to ask whether E is a $\Delta(p)$ set. We define the notions of *F-function* and *$\Delta(p)$ set* in R^N as we did for the real line. We can again use distributions to extend the notion of F-functions to the amalagam $(L^1, \ell^\infty)(G)$.

In general, it is only possible to choose the transversal set M in J to be a group if J is isomorphic to the direct product of H and J/H, and this is not always the case [**12**, §10]. Nevertheless, the index set A can be identified with a group by identifying each element m of the set M with the corresponding coset $[m] = m + H$ in the group J/H; similarly we identify each element $\alpha = \vec{n} \times m$ of the set A with the corresponding element $[\alpha] = \vec{n} \times [m]$ of the discrete group $Z^N \times (J/H)$, and we also denote this group by $[A]$. Identify the index set B in $\hat{G}$ with a discrete group $[B]$ in a similar way. Given a subset E of $[B]$, let F be the union of the sets I_β for which $[\beta] \in E$; call F a $\Delta(p)$ subset of G if every F-function in $L^1(G)$ also belongs to $L^p(G)$. Extend the notion of F-function to the amalgam $(L^1, \ell^\infty)(G)$ by interpreting $\hat{f}$ as a distribution, for instance as a *quasimeasure* [**10**], that is [**4**], as an object in the dual space of $A_c(\hat{G})$, the space of compactly-supported functions in the Fourier algebra of $\hat{G}$; alternatively, one can use a suitable amalgam [**2, 5**] as a test space for a theory of distributions on locally compact abelian groups.

THEOREM 1. *Fix an index p in the interval $(1,\infty)$. Let E be a subset of the group $[B]$, and let F be the corresponding subset of $\hat{G}$. Then the following statements are equivalent.*

(i) *The set E is a $\Lambda(p)$ set.*

(ii) *The set F is a $\Lambda(p)$ set.*

(iii) *For each index q in the interval $[1,\infty]$, it is the case that if an F-function belongs to the amalgam $(L^p,\ell^q)(G)$, then the F-function also belongs to the amalgam $(L^p,\ell^q)(G)$.*

PROOF. We will first show that statement (i) implies the following assertion.

(ii$'$) Every F-function in the amalgam $(L^1,\ell^1)(G)$ also belongs to the amalgam $(L^p,\ell^\infty)(G)$.

Since $L^p(G) \subset (L^p,\ell^\infty)(G)$, the assertion above is weaker than statement (ii), which is in turn obviously weaker than statement (iii). We will show, however, that (ii) $\Rightarrow$ (iii) for sets F defined as above; hence the three statements (ii), (ii$'$), and (iii) are equivalent for such sets F, and statement (i) implies all three of them. Finally, we will show that (ii$'$) $\Rightarrow$ (i). In some of the steps in the proof we will use one method to prove an implication in the case where $G + R^N$, then use a different method for the case where G has a compact open subgroup, and then deal with general groups G by combining the two methods.

LEMMA 1. *Conditions (i) implies condition (ii$'$).*

PROOF. Suppose first that $G+R^N$. Then A and B are just copies of Z^N in G and $\hat{G}$ respectively. We normalize the Fourier transform by declaring that

$$\hat{f}(y) = \int_{R^N} f(x)e^{-2\pi i x\cdot y}dx,$$

for all integrable functions f on R^N. It suffices to prove our conclusions for this normalization of the transform, because membership in amalgams is unaffected by dilation. This normalization has the advantage that each of the index subgroups A and B annihilates the other; that is $\gamma(x) = 1$ for all x in A and all γ in B. Moreover, the sets I_0 and $\hat{I}_0$ both have mass 1, and we identify them with copies of T^N, with dual groups B and A respectively. If an integrable function f vanishes off the set I_0, we can identify f with a function on T^N using the correspondence between I_0 and T^N given above. Then, for each element γ of the Fourier tranform, $\hat{f}(\gamma)$ as defined above coincides with Fourier coefficient $\hat{f}(\gamma)$ obtained by regarding f as a function on T^N.

Suppose that statement (i) holds, and let f be an f-function in the amalgam $(L^1, \ell^1)(G)$. For each index α in the set A, let f_α be the restriction of f to the set I_α, and let τ_α be the operator of translation by $-\alpha$; thus, $\tau_\alpha g(x) = g(\alpha + x)$ for all functions g on G and all elements x of G. Note that $\tau_\alpha$translates f_α to a function that vanishes off the set I_0. Since $\sum_{\alpha \in A} ||f_\alpha||_1 = ||f||_{1,1} < \infty$, the series $\sum_{\alpha \in A} \tau_\alpha f_\alpha$, regarded as a series of functions on I_0, converges in $L^1(I_0)$; denote the sum of the series by g_0. Regard the set I_0 as a copy of T^N with dual group B. Then, for all elements β of B, it is the case that

$$
\begin{aligned}
\hat{g}_0(\beta) &= \sum_{\alpha \in A} \int_{I_0} \tau_\alpha f_\alpha(x) \overline{\beta}(x) dx \\
&= \sum_{\alpha \in A} \int_{I_\alpha} f_\alpha(y) \overline{\beta}(y - \alpha) dy \\
&= \sum_{\alpha \in A} \int_{I_\alpha} f_\alpha(y) \overline{\beta}(y) dy \\
&= \int_G f(y) \overline{\beta}(y) dy = \hat{f}(\beta).
\end{aligned}
$$

Note that, in passing from the second line above to the third line, we used the normalization that each element β of the set β, regarded as a function on G, is identically equal to 1 on the set A. It follows from the calculation above, and the definition of the set F, that g_0, regarded as a function on the group I_0, is an E-function. Since E is a $\Lambda(p)$ set, the function g_0 must belong to $L^p(I_0)$; moreover [**15,** 5.3] there is a constant c, determined by the set E and the index p, and independent of f, so that $||g_0||_p \leq c||g_0||_1$. On the other hand, it is clear that $||g_0||_1 \leq ||f||_{1,1}$, whence $||g_0||_p \leq c||f||_{1,1}$.

Similarly, for each element γ of the set I_0, the series $\sum_{x \in A} \tau_\alpha \ (\overline{\gamma} \cdot f_\alpha)$ converges in $L^1(I_0)$ to an E-function g_γ, with $||g_\gamma||_1 \leq ||f||_{1,1}$. As above, $||g_\gamma||_p \leq c||f||_{1,1}$. The map $\gamma \to g_\gamma$ is continuous from the set $\hat{I}_0$ to the closed subspace of all E-functions in $L^1(I_0)$, and, since E is a $\Lambda(p)$ set, this map is continuous into this subspace endowed with the L^p-norm. Hence the integral $\int_{I_0} \gamma \cdot g_\gamma dy$ converges in $L^p(I_0)$ to a function g with $||g||_p \leq c||f||_{1,1}$. Now $\int_{I_0} \gamma \cdot \tau_0(\overline{\gamma} \cdot f_0) dy = f_0$, because

$\gamma\cdot\overline{\gamma}=1$ and the set $\hat{I}_0$ has mass 1. If $\alpha\neq 0$ however, then the integral $\int_{I_0}\gamma\cdot\tau_\alpha(\overline{\gamma}\cdot f_\alpha)d\gamma$ is the zero function, because $\gamma\cdot\tau_\alpha\overline{\gamma}=\overline{\gamma}(\alpha)$, and the integral, with respect to γ, of this quantity over $\hat{I}_0$ is 0. So $g=f_0$, whence $||f_0||_p\leq c||f||_{1,1}$. The same argument with f replaced by $\tau_\alpha f$ yields that $||f_\alpha||_p\leq c||f||_{1,1}$ also. Thus assertion (ii$'$) holds in this case.

Consider next the other extreme, that is the case where $G=J$, a group with a compact open subgroup H. Denote the annihilator [**12**, §23.23] of H in $\hat{J}$ by $H^\perp$. Since $H^\perp$ is a compact open subgroup of $\hat{J}$, we may let $I_0=H$ and $\hat{I}_0=H^\perp$ in defining amalgams on G and $\hat{G}$. Suppose again that f is an F-function, and observe that in this case the function f_0 is just the product $f\cdot 1_H$ of f with the indicator function of H. So $\hat{F}_0$ is the convolution of $\hat{f}$ and $(1_H)\hat{}$. Now if the Haar measure on G is suitably normalized, then $(1_H)\hat{}=1_{H^\perp}$. Therefore f_0 is also an F-function, and in fact its transform is constant on each coset of H^q. This forces f, regarded as a function on the compact group H, to be an E-function. Since E is a $\Lambda(p)$ set, there is again a constant c so that

$$||f_0||_p\leq c||f_0||_1\leq ||f||_{1,1}.$$

Condition (ii$'$) now follows by applying the same argument to translates of f.

Finally, suppose that $G=R^N\times J$, where $N>0$, and J is nontrivial and has a compact open subgroup. If J itself is compact, then we can choose I_0 to be the product set $[0,1)^N\times J$. Then the index set A is just $Z^N\times\{e\}$, where e denotes the identity element in the group J, and the index set B can be taken to be $Z^N\times\hat{J}$. With these conventions, the index sets A and B annihilate each other, and we can apply the method used earlier in the case where $G=R^N$.

Matters thus reduce to the case where $N>0$, and J is not compact, but J has a compact open subgroup H. In this case, let I_0 be the set $[0,1)^N\times H$, and let $\hat{I}_0=[0,1)\times H^\perp$. Let k be the indicator function of the set $R^N\times H$; then k is the inverse transform of a measure that is supported by the compact subgroup $\{0\}\times H^\perp$. Given an F-function f, let $h=f\cdot k$, and regard h as a function on the group $R^N\times H$. Define amalgams on this smaller group using the index sets $Z^N\times\{e\}$, and $Z^N\times\hat{H}$ as in the previous paragraph. The original $\Lambda(p)$ set E is a subset of the group $Z^N\times\hat{H}$. Let $F^\sim$ be the set obtained by fattening E in the group $R^N\times\hat{H}$. Then it follows from the properties listed above for the functions f, k, and h that h, regarded as a function on the group

$R^N \times H$, is an $F^{\sim}$-function. We have already seen that condition (ii′) holds for this group. So,

$$||h_0||_p \le c||h||_{1,1}.$$

But $||f_0||_p = ||h_0||_p$, and $||h||_{1,1} \le ||f||_{1,1}$. Hence $||f_0||_p \le c||f||_{1,1}$, and it follows as before that $f \in (L^p, \ell^\infty)(G)$. This completes the proof of Lemma 1.

LEMMA 2. *Condition (ii′) implies condition (iii).*

PROOF. Suppose first that $G = R^N$. Fix an index q in the interval $[1, \infty]$, and an F-function f in the amalgam $(L^1, \ell^q)(G)$. The first step in showing that f belongs to the smaller amalgam $(L^p, \ell^q)(G)$ is to split f as a sum of finitely-many functions in $(L^1, \ell^q)(G)$ with the property that each of the summands has its Fourier transform supported by a translate of a set of the form $E + K$, where K is a compact subset of the interior of $\hat{I}_0$. When $N = 1$, such a splitting can be accomplished as follows. Let ϕ be the function in the interval $[0, 1]$ that is equal to 0 at the points $0, 2/3$. and 1, equal to 1 at the point $1/3$, and linear in the closed intervals of length $1/3$ with these endpoints. Identify the set $\hat{I}_0$ with the circle group T, and regard $\hat{I}_0$ as the dual of the group Z. Then by direct calculation, or by the fact that ϕ is the translate by $1/3$ of a positive-definite function, the inverse transform ϕ has norm 1 in $\ell^1(Z)$. Regard Z as a subset of the group R, and let μ_1 be the discrete measure supported by Z with $\mu_1(\{n\}) = \check{\phi}(n)$ for all n. Then $\hat{\mu_1}$ is equal to ϕ extended to be periodic with period 1 on all of R. Let μ_2 and μ_3 be the measures obtained from μ_1 by multiplying each mass $\mu_1(\{n\})$ by $e^{2\pi in/3}$ and by $e^{4\pi in/3}$ respectively. Then $\hat{\mu}_2$ and $\hat{\mu}_3$ are obtained by translating $\hat{\mu}_1$ by $1/3$ and $2/3$ respectively. It follows that the sum of the transforms of these three measures is identically equal to 1. Let g_1, g_2, and g_3 be the convolutions of f with the three discrete measures defined above. Then the sum of these three functions is f, and each of them has its Fourier transform supported by some translate of the set $E + [1/6, 5/6]$. Similarly, when $N > 1$, the F-function f can be split into 3^N pieces with the desired property.

It suffices to show that $f \in (L^p, \ell^Q)(G)$ when f is one these pieces, that is, when there is a compact subset k of the interior of $\hat{I}_0$ so that $\hat{f}$ is supported by a translate of the set $E + K$. Since membership of f in various amalgams is unaffected by translation of $\hat{f}$, it is enough to deal with the case where $\hat{f}$ is supported by $E + K$. In this case, choose a nonempty, open subset, V say, of $\hat{G}$ so that $K+V \subset \hat{I}_0$. As in [**8**, p. 132] there is a continuous function h supported by the set V so that $|\check{h}| \geq 1$ everywhere in the set I_0, and so that $\check{h} \in (L^\infty, \ell^1)(G)$. For each element α in the index group A, let h_α be the product of h with the character α^{-1}; then $|(h_\alpha)\check{}| > 1$ everywhere on the set I_α. Let $k_\alpha = f \cdot (h_\alpha)$, then $|k_\alpha| \geq |g_\alpha|$ on the set I_α. Moreover, $k_\alpha \in (L^1, \ell^1)(G)$, because $(h_\alpha) \in (L^\infty, \ell^1)(G)$ and $f \in (L^1, \ell^q)(G)$. Recall that $\hat{f}$ is supported by $E + K$ and $((h_\alpha)\check{})\hat{}$ by V; so $\hat{k}_\alpha$ is supported by $E + K + V$, which is a subset of $E + \hat{I}_0$. To summarize, k_α is an F-function in the amalgam $(L^1, \ell^1)(G)$, and $|k_\alpha| \geq |f_\alpha|$ on the set I_α.

Our goal is to show that the function $\alpha \to ||f_\alpha||_p$ belongs to $\ell^q(A)$. We claim that it suffices to show that the function $\alpha \to ||k_\alpha||_1$ belongs to $\ell^q(A)$. To see this, observe first that the space of all F-functions in $(L^1, \ell^1)(G)$ is a closed subspace of $(L^1, \ell^1)(G)$, and that the space of all F-functions in $(L^p, \ell^\infty)(G)$ is a closed subspace of $(L^p, \ell^\infty)(G)$. Moreover the former subspace imbeds into the latter subspace, by assertion (ii′). Since convergence in either of the amalgams $(L^1, \ell^1)(G)$ or $(L^p, \ell^\infty)(G)$ implies convergence in $(L^1, \ell^\infty)(G)$, this imbedding has a closed graph, and is therefore continuous. Thus, there is a constant c so that $||g||_{p,\infty} \leq c||g||_{1,1}$ for all F-functions g. In particular, this is so with the function g replaced by k_α; on the other hand, $|f_\alpha| \leq |k_\alpha|$ on the set I_α. Thus,

$$||f_\alpha||_p \leq ||k_\alpha||_p \leq c||k_\alpha||_1,$$

and our claim is proved.

Define functions b and d on the group A by letting $b(\alpha) = ||f_\alpha||_1$, and $d(x) = ||k_\alpha||_1$ for all α in A. Also let $c(\alpha)$ be the supremum of the restriction of the function $\check{h}$ to the set I_α. Consider, too, for each element α' of A, the supremum of the restriction of $(h_\alpha)\check{}$ to the set $I_{\alpha'}$; since $(h_\alpha)\check{}$ is simply h translated by α, the latter supremum is just $c(\alpha' - \alpha)$. Now,

$$\begin{aligned} d(\alpha) = ||k_\alpha||_1 &= \sum_{\alpha' \in A} \int_{I_\alpha} |k_\alpha| \\ &\leq \sum_{\alpha' \in A} b(((\alpha')c(\alpha' - \alpha). \end{aligned}$$

In other words, the nonnegative function d is majorized by the convolution, on the group A, of the function b and the flipped function:

$\alpha \to c(-\alpha)$. By Young's inequality for convolution,

$$||d||_q \leq ||b||_q ||c||_1.$$

Our assumptions that $f \in (L^1, \ell^q)(G)$ and that $\check{h} \in (L^\infty, \ell^1)(G)$ guarantee that the two norms on the right above are finite, and hence that $d \in \ell^q(A)$ as required.

Suppose next that G is isomorphic to $R^N \times J$, where J has a compact subgroup, say H. Let $I_0 = [0,1) \times H$. Since H is compact, the procedure used when $G = R^N$ can be applied in the general case to split any F-function into 3^N pieces with the properties specified above. The only obstruction to transferring the rest of the argument for R^N to the general context is that the index set A might not be a group. The functions b, c, and d on the set A can be transferred to the quotient group [A], however, and then matters reduce as before to Young's inequality for convolution on the group [A], see [**8**, pp. 132-133] for a special case of this.

LEMMA 3. *Condition (ii′) implies condition (i).*

PROOF. Suppose that the set F satisfies condition (ii′), and recall that, as in the proof of the implication (ii′)$\Rightarrow$ (iii), there must then be a constant c so that $||g||_{p,\infty} \leq c||g||_{1,1}$ for all F-function g. Again consider first the case where $G = R^N$. Let f be an E-polynomial, in other words an E-function for which the support of $\hat{f}$ is finite, and let μ be the discrete measure on $\hat{G}$ with the property that $\mu(\{y\}) = \hat{f}(y)$ if $y \in B$, and $\mu(\{y\}) = 0$ otherwise. Then $\check{\mu}|I_0$, the restriction of the inverse transform of the measure μ to the set I_0, coincides with f. Moreover, $\check{\mu}(x + \alpha) = \check{\mu}(x)$ for all x in G and all α in A; hence, $||\check{\mu}||_{1,\infty} = ||f||_1$. As above, choose a function h supported by the set $\hat{I}_0$, with $|\check{h}| \geq 1$ on the set I_0, and with $\check{h} \in (L^\infty, \ell^1)(G)$. Let $g = \check{h} \cdot \check{\mu}$; then $g \in (L^1, \ell^1)(G)$ with

$$||g||_{1,1} \leq ||\check{h}||_{\infty,1} ||\check{\mu}||_{1,\infty} = ||\check{h}||_{\infty,1} ||f||_1.$$

On the other hand, $\hat{g}$ is the convolution of the function h with the measure μ. So $\hat{g}$ vanishes off the set $\hat{I}_0 + E$; that is, g is an F-function. As noted above, $||g||_{p,\infty} \leq c||g||_{1,1}$ Finally, $|g| \geq |f|$ on the set I_0, so that $||f||_p \leq ||g||_{p,\infty}$. Combining these inequalities yields that $||f||_p \leq c||\check{h}||_{\infty,1}||f||_1$ for all E-polynomials f; it follows that E is a $\Lambda(p)$ set as required.

Next, if G has a compact open subgroup, H say, we can let $I_0 = H$ and $\hat{I}_0 = H^{\perp}$. Then the group [B] is the dual group of H. Given any E-polynomial f on H, let $f^{\sim}$ be the function f extended to be 0 on $G\backslash H$. Then $f^{\sim}$ is an F-function, and it follows that

$$||f||_p = ||f^{\sim}||_{p,\infty} \leq c||f^{\sim}||_{1,1} = c||f||_1.$$

Hence E is a $\Delta(p)$ set.

Finally, let g be isomorphic to $R^N \times J$, where $N > 0$ and J is noncompact but has a compact open subgroup H. Arrange matters so that [B] is the dual group of $T^N \times H$, and let f be any E-polynomial on $T^N \times H$. Let $F^{\sim}$ be the fattened version of the set E in the dual group of $R^N \times H$. Argue as in the case where $G = R^N$ to get an $F^{\sim}$-function, say $g^{\sim}$, with $|g^{\sim}| \geq |f|$ on $T^N \times H$, and with $||g^{\sim}||_{1,1} \leq c||f||_1$. Let g be the function $g^{\sim}$ extended to be 0 on $G\backslash(R^N \times H)$. Then g is an F-function, and it follows as before that

$$||f||_p \leq ||g||_{p,\infty} \leq c||g||_{1,1} \leq c||f||_1.$$

This completes the proof of the lemma and the proof of the theorem.

The proof of our main implication (i) $\Rightarrow$ (iii) is much easier in the case where G has a compact open subgroup than it is in the other cases. Indeed if the compact open subgroup H is used as above in defining amalgams on G and $\hat{G}$, and if f is an F-function on G, then each of the functions f_α is also an f-function, and $||f_\alpha||_p \leq c||f_\alpha||_1$ for all α. We do not know whether this inequality also holds when $G = R^N$. All that our methods yield for the latter group are estimates for $||f_\alpha||_p$ in terms of the L^1-norm of the F-function f over the whole group G. Moreover, in this case, no function f_α can be an F-function unless it is trivial, because its transform $\hat{f}_\alpha$ is the restriction to R^N of an entire function of N variables.

Previous proofs of versions of statement (iii) for sets F of the form $E + \hat{I}_0$ have dealt only with special classes of $\Lambda(p)$ sets E, and only with the case of statement (iii) where $q = 2$. For instance, it is shown in [**8**] for $g = R$ and $E = \{4^n\}_{n=1}^{\infty}$ that

(2) every F-function in $L^2(R)$ belongs to $L^p(R)$ for all p in the interval $(2, \infty)$.

Some of the ingredients in our proof that (ii′) $\Rightarrow$ (iii) were used in [**8**] to deduce for such sets F that every F-function in $(L^2, \ell^2)(R)$ belongs to all the amalgams $(L^p, \ell^2)(R)$ with $2 \le p < \infty$. A duality argument shows that a set F has property (2) above if and only if it has the property that, for each index p' in the interval $(1, 2)$ and every function f in $L^p(R)$, the restriction $\hat{f}|F$ of the Fourier transform of f to F belongs to $L^2(F)$. Suppose that E determines a Littlewood-Paley decomposition of R, as in [**9**]. Then it follows from Minkowski's inequality and the Hausdorff-Young theorem that $\hat{f}|F \in L^2(F)$ for all functions f that belong to $L^{p'}(R)$ for some index p' with $1 < p' \le 2$; the argument in [**8**] then shows that statement (iii) holds for the set F, in the special case where $q = 2$. Every set that determines a Littlewood-Paley decomposition of R is a $\Lambda(p)$ set for all $p < \infty$, but there are many other $\Lambda(p)$ sets. Theorem 1 applies to the full class of $\Lambda(p)$ sets, and also yields the cases of statement (iii) where $q \ne 2$.

These results can be summaraized by saying that whenever a function in a suitable amalgam $(L^p, \ell^q)(G)$ has a Fourier transform with a sufficiently thin support, then the index p controlling the *local* size of the function can be improved to a larger value. We now consider whether it is also possible in this situation to improve the index q controlling the *global* size of the function, that is to replace q by a smaller number. If, like the sets considered above, the set F has a nonempty interior, then no such global improvement is possible. To see this in the case where $G = R$, suppose without loss of generality that F includes an open interval, U say, containing the origin. Let f be the inverse transform of some nontrivial C^∞-function supported by the interval U. The fact that $\hat{f}$ is smooth and has compact support implies that $f \in (L^\infty, \ell^q)(G)$ for all $q > 0$, and that the same is true for every dilate of f. For each fixed number $t > 0$, let f_t be the dilate given by letting $f_t(x) = f(x/t)$ for all x. When $t > 1$, these dilates are F-functions too. Fix indices p, q, r, and s with $s < q$. If it were the case that every F-function in $(L^p, \ell^q)(G)$ belonged to $(L^r, \ell^s)(G)$, then the closed-graph theorem would yield a constant k so that $||g||_{r,s} \le k||g||_{p,q}$ for all such F-functions g. It is easy to verify, however, that $||f_t||_{r,s}/||f_t||_{p,q} \to \infty$ as $t \to \infty$. Much as in [**7**], a similar argument works for any nondiscrete abelian group G.

Let us briefly consider F-functions when the interior of F is empty but F has positive measure. Suppose first that $p \in [2, \infty)$; then there are no indices q and s with $s < q$ for which every F-function in $(L^p, \ell^p)(G)$ belongs to $(L^P, \ell^s)(G)$. Indeed, let K be a compact subset of F having positive measure, and let p' be the index conjugate to p; then $p' \leq 2$, since $p \geq 2$. It is shown in [**7**] that, for each index $r > p$, there is a function f in $L^{p'}(G)$ for which the restriction of $\hat{f}$ to K does not belong to $L^r(K)$. Regard $L^{p'}(K)$ as the subspace of functions in $L^{p'}(G)$ that vanish off K. It follows by duality that there is a function g in $L^{p'}(K)$ for which $\check{g} \notin L^{r'}(G)$. On the other hand, by [**8**] or the references cited in [**8**, p. 123], for each index u in the interval $[1,2]$ the inverse transform maps $L^u(K)$ into $(L^\infty, \ell)(G)$. Using complex interpolation as in [**8**, p. 129], we see that, for each index $t > 0$, there must in fact be a function g in $L^{r'}(K)$ with the property that $\check{g} \notin (L^t, \ell^p)(G)$. Applying the Baire category theorem as in [**7**] yields that there is a function g in $L^{p'}(K)$ for which $\check{g}$ belongs to none of the amalagams $(L^t, \ell^r)(G)$ with $r < p$. Clearly, $\check{g}$ is an F-function, and as noted above it belongs to $(L^\infty, \ell^p)(G)$.

Now let $p \in [1,2)$. It is shown in [**14**] and [**6**] that there are subsets S of $\hat{G}$, having positive measure, so that the only S-function in $L^p(G)$ is the trivial function. It is easy to verify that these sets S also have the property that the only S-function in $(L^1, \ell^p)(G)$ is 0. Of course, every such trivial function belongs to all amalgams. We do not know any example of a set F of positive measure for which, for some indices p and q, the set of F-functions in $(L^p, \ell^q)(G)$ is *nontrivial* and included in some amalgam $(L^r, \ell^s)(G)$ with $s < q$.

3. p-Sidon sets. Fix an index p in the interval $[1,2)$, and call a subset E of the discrete group B a *p-Sidon set* if every E-function in $L^\infty(I_0)$ has the property that $\hat{f} \in \ell^p(B)$. This condition is equivalent to the existence of a constant c so that $||\hat{f}||_p \leq c||f||_\infty$ for all E-polynomials f.

THEOREM. *Let E be a subset of B, and let $F = E + \hat{I}_0$. Then the following statements are equivalent.*

(a) *The set E is a p-Sidon set.*

(b) *Every F-function in the amalgam $(L^\infty, \ell^1)(G)$ has the property that its transform belongs to $L^p(\hat{G})$.*

PROOF. Suppose first that statement (a) holds, and that $G = R^N$. Given an F-function f in the amalgam $(L^\infty, \ell^1)(G)$, from the functions $\tau_\alpha f_\alpha$ and their sum g_0 exactly as in the proof of Theorem 1. Regard g_0 as a function on the group I_0; then g_0 is an E-function with $||g_0||_\infty \leq ||f||_{\infty,1}$. Moreover, $\hat{g_0}(\beta) = \hat{f}(\beta)$ for all β in B. On the other hand, since E is a p-Sidon set, there is a constant c so that $||\hat{g_0}||_p \leq c||\hat{g_0}||_\infty$ for all E-functions g on the group I_0. Hence, $(\sum_{\beta\in B}|\hat{f}(\beta)|^p)^{1/p} \leq c||f||_{\infty,1}$. For each element γ of the set $\hat{I}_0$, form the functions $\tau_\alpha(\overline{\gamma}\cdot f_\alpha)$ and their sum g_γ. Then $\hat{g_\gamma}(\beta) = \hat{f}(\beta+\gamma)$ for all β in B. Therefore g_γ is also an E-function with $||g_\gamma||_{\infty,1} \leq ||f||_{\infty,1}$, and

$$\sum_{\beta\in B}|\hat{f}(\beta+\gamma)|^p \leq (c||f||_{\infty,1})^p \quad \text{for all } \gamma \text{ in } \hat{I}_0.$$

Recall that the set $\hat{I}_0$ was normalized to have total mass 1. Integrating the inequality above over this set, with respect to γ, yields that $(||\hat{f}||_p)^p \leq (c||f||_{\infty,1})^p$. So, (a) $\Rightarrow$ (b) in this case.

The same argument works when G has the form $R^N\times H$, where H is compact. Finally, suppose that $G = R^N\times J$, where J has a compact open subgroup H. Given an F-function f, let $f^\sim$ be its restriction to the subgroup $R^N\times H$, and let g be the function that coincides with $\tilde{f}$ on this subgroup and vanishes on the rest of G. We already know that (a) $\Rightarrow$ (b) for the group $R^N\times H$. So the function $(f^\sim)\hat{}$ on the dual group $R^N\times[\hat{J}/H^\perp]$ has the property that $||(f^\sim)\hat{}||^p \leq c||f^\sim||_{\infty,1}$. It follows that $||(g')\hat{}||_p \leq c||g'||_{\infty,1}$, too. Applying the same argument to translates of f yields that, if g' coincides with f on some coset of $R^N\times H$ and vanishes on the rest of G, then $||(g')\hat{}||_P \leq c||g'||_{\infty,1}$. Index the cosets of $R^N\times H$ in $R^N\times J$ as $\{C_\delta\}_{\delta\in\Delta}$, and for each index δ, let g_δ be the function that coincides with f on the coset C_δ and vanishes on the rest of G. For each finite subset D of Δ,

$$\sum_{\delta\in D}||g_\delta||_{\infty,1} \leq ||f||_{\infty,1}.$$

Therefore there are only countably-many indices δ for which $||g_\delta||_{\infty,1} > 0$. The series $\sum_{\delta\in\Delta}||g_\delta||_{\infty,1}$ converges to $||f||_{\infty,1}$ and the series $\sum_{\delta\in\Delta}g_\delta$ converges in the space $(L^\infty,\ell^1)(G)$ to f. By the Hausdorff-Young theorem for amalgams, the series $\sum_{\delta\in\Delta}\hat{g}_\delta$ converges in the space $(L^\infty,\ell^2)(\hat{G})$ to $\tilde{f}$. On the other hand, our estimate $||\hat{g}_\delta||_{\infty,1} \leq c||g_\delta||_{\infty,1}$ guarantees that the series $\sum_{\delta\in\Delta}\hat{g}_\delta$ converges in $L^p(\hat{G})$ to a function with L^p-norm at most $c\sum_{\delta\in\Delta}||g_\delta||_{\infty,1}$. Hence $||\hat{f}||_p \leq c||f||_{\infty,1}$ as required.

The proof that (b)$\Rightarrow$(a) is much like the proof in the previous section that (ii$'$)$\Rightarrow$(i). If statement (b) holds, then by the closed-graph theorem there must be a constant k so that $||\hat{g}|| \leq K||g||_{\infty,1}$

for all F-functions g. Suppose that $G = R^N$. Given an E-polynomial f, form the discrete measure μ as in the proof that (ii$'$) $\Rightarrow$ (i), and observe that $||\check{\mu}||_{\infty,\infty} = ||f||_\infty$. Choose the function h as before, and again let $g = \check{h} \cdot \check{\mu}$; then g is an E-function in $(L^\infty, \ell^1)(G)$ with $||g||_{\infty,1} \leq ||\check{h}||_{\infty,1}||f||_\infty$, As noted above, $||\hat{g}||_p \leq k||g||_{\infty,1}$. On the other hand, $||\hat{g}||_p = ||h||_p||\hat{f}||_p$, so that $||\hat{f}||_p$ is majorized by the quantity $(\kappa||\check{h}||_{1,\infty}/||h||_p)||f||_\infty$. Therefore, E is a p-Sidon set in this case.

Again, the same argument works when $G = R^N \times H$, where H is compact. Finally, if $G = R^N \times J$, where J is noncompact but has a compact open subgroup, proceed as above to get a function g on the open subgroup $R^N \times H$, and simply extend g to be 0 on the rest of G. This completes the proof of the theorem.

As mentioned in the introduction, the Hausdorff-Young Theorem for amalgams suggests that the two statements in Theorem 2 ought to imply that $\hat{f} \in (L^\infty, \ell^p)(\hat{G})$. Moreover, there ought to be a version of the theorem with statement (b) replaced by the assertion that if f is an F-function in the amalgam $(L^\infty, \ell^p)(G)$, where q is an index in the interval $(1, 2]$, then $\hat{f} \in (L^p, \ell^p)(\hat{G})$. Perhaps the problem is our method of proof, which seems better adapted to *reverse amalgams.* as explained below, than to amalgams.

Suppose in the rest of this discussion that $G = R^N$. Given a measurable function f on G, and finite indices p and q, let

$$||f||^{\leftarrow}_{q,p} = \left(\int I_o\left(\left(\sum_{\alpha\in A}|f(x+\alpha)|^q\right)^{1/q}\right)^p dx\right)^{1/p}.$$

and make the appropriate changes for infinite indices p and q. Say that f belongs to the *reverse amalgam*, $(\ell^q, L^p)(G)$, if the norm $||f||^{\leftarrow}_{q,p}$ is finite. Our proof that (a)$\Rightarrow$(b) above actually shows that condition (a) implies that

(b$'$) $$||\hat{f}||^{\leftarrow}_{p,\infty} \leq c||f||^{\leftarrow}_{1,\infty} \text{ for all} F\text{-functions } f.$$

Statement (b) follows from (b$'$), because $||\hat{f}||_p \leq ||\hat{f}||^{\leftarrow}_{p,\infty}$ and $||f||^{\leftarrow}_{1,\infty} \leq ||f||_{\infty,1}$.

Statement (b$'$) above is an appropriate complement to the Hausdorff-Young theorem for reverse amalgams, which asserts that $||\hat{f}||^{\leftarrow}_{p,q} \leq$

$C\|f\|^{\leftarrow}_{p,q}$ for all functions f in $(\ell^q, L^p)(G)$, provided that $1 \leq q, p \leq 2$. This theorem does not seem to have been noticed before, although there is some precedent for it in the proof of [**1**; §12, Theorem 1] and in [**16**, §2]. To prove it, first consider four special cases. When $q = p = 1$ the theorem simply says that $\|\hat{f}\|_\infty \leq \|f\|_1$; when $q = p = 2$ it is the Parseval theorem. The case where $q = 1$ and $p = 2$ follows by Poisson summation, as in our proof above that (a)$\Rightarrow$(b); the case where $q = 2$ and $p = 1$ follows by duality from the validity of the previous case for the inverse transform. The other cases then follow by complex interpolation between these four endpoint cases. All this suggests that statement (a) should imply that if f is an F-function in the reverse amalgam $(\ell^q, L^\infty)(G)$, where the index q lies in the interval $(1, 2]$, then $\hat{f} \in (\ell^p, L^{q'})(\hat{G})$.

Translation and dilation, provided that the latter operation makes sense on the group G, do not affect the membership of a function in a given amalgam. It follows that if a set F has property (b), then so does every set obtained from F by translation and dilation; the same is true for properties (ii) and (iii). This has interesting consequences for the sets in certain discrete groups. We illustrate this point for the group Z. Given a subset E of Z, let "πE" be the subset of Z obtained by forming the set πE in R and replacing each element of πE by the element of Z closest to it. If E is a $\Lambda(p)$ set, then so is the set $F = E + [0, 1)$, and so is πF. But "πE"$+[0, 1)$ is a subset of $\pi F - 1/2$; hence "πE" $+[0, 1)$ is a $\Lambda(p)$ set. It then follows from Theorem 1 that "πE" is a $\Lambda(p)$ set. Similarly, Theorem 2 and the argument above show that if E is a p-Sidon set, then so is "πE".

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