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$\mathfrak{F}$-PROJECTORS IN LOCALLY FINITE GROUPS

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ABSTRACT. The author discusses the class $\mathscr{X}$ of countable locally finite-solvable groups with min-p for all primes p. It is shown that if $\mathfrak{F}$ is a saturated formation which contains no non co-Hopfian groups then the $\mathfrak{F}$-projectors of a group $G \in \mathscr{X}$ are all conjugate if and only if the group G has only countably many such $\mathfrak{F}$-projectors.

1. Introduction. A group G is said to be locally finite-solvable if every finite subset of elements of G is contained in a finite solvable subgroup of G. In this paper we shall be concerned with the class $\mathfrak{X}$ of all countable locally finite-solvable groups with min-p for all primes p, which was first studied in [1]. Here a group G is said to have min-p if every p-subgroup of G has the minimal condition on subgroups. The structure of groups in the class $\mathfrak{X}$ has been well documented in [1], [4] and [6, chapter 3].

In [4] we obtained a theory of saturated formations in the class $\mathfrak{X}$. For the sake of completeness we now describe this theory. If G is in the class $\mathfrak{X}$ then G will be called an $\mathfrak{X}$-group. Suppose $\mathfrak{Z}$ is a QS-closed subclass of $\mathfrak{X}$; that is every $\mathfrak{Z}$-group in an $\mathfrak{X}$-group, and every section of a $\mathfrak{Z}$-group is a $\mathfrak{Z}$-group. Let π denote a non-empty set of primes and, for each $p \in \pi$, let $f(p)$ be a subclass of $\mathfrak{Z}$ satisfying

(i) $f(p)$ is Q-closed

(ii) If $G \in \mathfrak{Z}$ and

$$N = \bigcap \{C_G(H/K) \mid H/K \text{ is a } p\text{-chief factor of } G \text{ such that } G/C_G(H/K) \in f(p)\}$$

then $G/N \in f(p)$.

The saturated $\mathfrak{Z}$-formation defined locally by f is then the class of groups:

$$\mathfrak{F} = \mathfrak{F}(f) = \mathfrak{Z} \cap \mathfrak{S}_\pi \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'}, \mathfrak{S}_p f(p),$$

where $\mathfrak{S}_\pi$ denotes the class of locally finite-solvable π-groups. Moreover

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$\mathfrak{F}$ is called a co-Hopfian saturated $\mathfrak{Z}$-formation if no $\mathfrak{F}$-group contains a proper subgroup isomorphic to itself.

By an $\mathfrak{F}$-projector of a group $G \in \mathfrak{Z}$ we mean a subgroup E of G so that $E \in \mathfrak{F}$ and whenever $E \leqq H \leqq G$, $K \lhd H$ and $H/K \in \mathfrak{F}$ then $H = KE$. The main result of **[4]** then says that if $\mathfrak{F}$ is a co-Hopfian $\mathfrak{Z}$-formation and $G \in \mathfrak{Z}$ then G possesses $\mathfrak{F}$-projectors. Furthermore any two $\mathfrak{F}$-projectors have the property that their maximal σ-subgroups are conjugate for all finite sets of primes σ, a concept termed finite conjugacy in **[4]**.

It is therefore of interest to know under which conditions the $\mathfrak{F}$-projectors of a $\mathfrak{Z}$-group will actually be conjugate and it is this problem that is addressed in this note. Such conditions were actually obtained in **[2]** for the special case $\mathfrak{F} = L\mathfrak{N} \cap \mathfrak{Z}$, the class of locally nilpotent $\mathfrak{Z}$-groups. There we showed that if a group $G \in \mathfrak{Z}$ has only countably many $L\mathfrak{N}$-projectors then they are all conjugate and in this case G has a finite normal series, each factor of which is locally nilpotent.

In the present paper we show that if $\mathfrak{F}$ is a co-Hopfian saturated $\mathfrak{Z}$-formation and $G \in \mathfrak{Z}$ has only countably many $\mathfrak{F}$-projectors then these are all conjugate. The group G need no longer have a finite locally nilpotent series as is easily seen by using the examples in **[1**, Satz 5.3].

Our proof is somewhat different from that in **[2]** essentially because the formations occurring need not be subgroup closed. In fact the proof closely resembles the existence proof for $\mathfrak{F}$-projectors.

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2. Proofs of the Results. It will be convenient to prove several lemmata in much more generality than is needed here. If G is an $\mathfrak{X}$-group we shall let $\mathscr{S}$ be a set of finitely conjugate subgroups of G, with the property that whenever $F \in \mathscr{S}$ then every conjugate of F is also in $\mathscr{S}$. The first three results are easy to prove so we state them without proof.

LEMMA 2.1. *Suppose* $E \in \mathscr{S}$ *has the property.*

(*) *There exists a Sylow basis* $\{E_i\}$ *of* E *and an integer* n *such that if* $\langle E_1, \ldots, E_n \rangle \leqq F \in \mathscr{S}$ *then* $E = F$.

Then the members of $\mathscr{S}$ *are all conjugate.*

COROLLARY 2.2. *Suppose* $E \in \mathscr{S}$ *has property* (*), *relative to the Sylow basis* $\{E_i\}$. *Then*

(i) *All Sylow bases of* E *have property* (*) *and*

(ii) *For the same value of* n, *all elements of* $\mathscr{S}$ *have property* (*).

We shall use 2.1 and 2.2 in the case when $\mathscr{S}$ consists of the set of all $\mathfrak{F}$-projectors of an $\mathfrak{X}$-group. These results will be used implicitly most of the time.

LEMMA 2.3. *Suppose $G \in \mathfrak{X}$ and $N \lhd G$. Suppose G/N is a Černikov group and E is an $\mathfrak{F}$-projector of G, for some saturated formation $\mathfrak{F}$. Suppose the $\mathfrak{F}$-projectors of EN are all conjugate. Then the $\mathfrak{F}$-projectors of G are all conjugate.*

In the sequel it will be the contrapositive form of this result that will be required. To complete the list of preliminary results we recall from [3] that an $\mathfrak{X}$-group has a set $\mathcal{N}$ of normal subgroups which allows us to endow each $\mathfrak{X}$-group with a "co-Černikov topology." [The reader should note that for $\mathfrak{X}$-groups the topology is independent of $\mathcal{N}$].

LEMMA 2.4. *Suppose E, F are closed subgroups (relative to the co-Černikov topology defined by $\mathcal{N}$) of an $\mathfrak{X}$-group. If $EN = FN$ for all $N \in \mathcal{N}$ then $E = F$.*

PROOF. By proposition 2.3 of [3], we have

$$E = \bar{E} = \bigcap \{EN: N \in \mathcal{N}\} = \bigcap \{FN: N \in \mathcal{N}\} = \bar{F} = F$$

where the bars denote topological closures.

Since we wish to apply this result to $\mathfrak{F}$-projectors we also note:

LEMMA 2.5. *If $\mathfrak{F}$ is a saturated formation, the $\mathfrak{F}$-projectors of an $\mathfrak{X}$-group are closed, relative to the co-Černikov topology defined by $\mathcal{N}$.*

PROOF. If E is an $\mathfrak{F}$-projector we only need show that $\bar{E} \in \mathfrak{F}$ since this then forces us to have $E = \bar{E}$. However, by proposition 2.3 of [3] it is easily seen that for all $N \in \mathcal{N}$, $EN = \bar{E}N$. Hence

$$\frac{E}{E \cap N} \cong \frac{EN}{N} = \frac{\bar{E}N}{N} \cong \frac{\bar{E}}{\bar{E} \cap N}$$

so by the Q-closure of $\mathfrak{F}$, $\bar{E}/\bar{E} \cap N \in \mathfrak{F}$. Hence by the R-closure of $\mathfrak{F}$ and the fact that $\bigcap \{N: N \in \mathcal{N}\} = 1$ it follows that $\bar{E} \in \mathfrak{F}$.

Using the notation of [4], we shall let $\mathfrak{F} = \mathfrak{F}(f) = \mathfrak{S}_\pi \cap \mathfrak{Z} \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'} \mathfrak{S}_p f(p)$, for some set of primes $\pi = \{p_1, p_2, \ldots\}$ and $\mathfrak{Z}$-preformation function f. We shall let $\pi_i = \{p_1, \ldots, p_i\}$ and within $G \in \mathfrak{Z}$ we choose a set $\{N_i: i \geqq 1\}$ of normal subgroups so that for all integers $i \geqq 1$.

(a) $\bigcap \{N_i: i \geqq 1\} = 1$

(b) N_i is a π_i'-group and $N_i \geqq N_{i+1}$

(c) G/N_i is Černikov.

That such a system of normal subgroups can be chosen in G is evident from [4].

THEOREM 2.6. *Suppose $G \in \mathfrak{Z}$ has only countably many $\mathfrak{F}$-projectors. Then the $\mathfrak{F}$-projectors of G are all conjugate.*

PROOF. We shall assume for a contradiction that not all the $\mathfrak{F}$-projectors of G are conjugate. Then π must be infinite otherwise the $\mathfrak{F}$-projectors would be Černikov groups and hence would be conjugate. For a subgroup H of G we shall let $S_i(H)$ denote a Sylow p_i-subgroup of H. We shall inductively construct for each positive integer n a set, $\mathscr{S}_n = \{E_1, \ldots, E_{2^n}\}$, of 2^n distinct $\mathfrak{F}$-projectors of G and integers m_n satisfying the following properties:

(i) $n \leqq m_n < m_{n+1}$ and $\mathscr{S}_n \subseteq \mathscr{S}_{n+1}$ for all $n \geqq 1$.

(ii) $N_{m_n} = G_n$ is a π_n'-group and $E_iG_n \neq E_jG_n$ if $i \neq j$.

(iii) There are 2^{n-1} distinct subgroups of the form E_kG_{n-1} (that is, given k, there exists precisely one integer $\ell \neq k$ so that $E_\ell G_{n-1} = E_kG_{n-1}$).

(iv) E_k has a Sylow basis $\{S_i(E_k)\}$ so that for the integer ℓ above

$$S_i(E_k) = S_i(E_\ell) \text{ for } i = 1, \ldots, n.$$

To begin the construction, let E_1 be an $\mathfrak{F}$-projector of G and suppose E_1 has a Sylow generating basis $\{S_i(E_1)\}$. Then by 2.1 and 2.2, it follows that G has an $\mathfrak{F}$-projector $E_2 \neq E_1$ such that

$$S_1(E_1) \leqq E_1 \cap E_2.$$

Since $S_1(E_1)$ is a Černikov group and **[4]** (2.6) guarantees that Sylow bases of Černikov groups can be extended, we can choose a Sylow basis $\{S_i(E_2)\}$ of E_2 satisfying $S_1(E_1) = S_1(E_2)$, so (iv) is satisfied with $n = 1$. Setting $G_0 = G$, (iii) is satisfied. Since $E_1 \neq E_2$, there exists, by 2.4, an integer $m_1 \geqq 1$ so that

$$E_1N_{m_1} \neq E_2N_{m_1}.$$

Set $G_1 = N_{m_1}$. Clearly (i) and (ii) are then satisfied. To complete the construction suppose the set $\mathscr{S}_n$ and the integer m_n have been constructed satisfying the required properties. Suppose $E_k \in \mathscr{S}_n$. From 2.3, the $\mathfrak{F}$-projectors of E_kG_n are not conjugate so there is an $\mathfrak{F}$-projector F_k of E_kG_n different from E_k so that

$$\{S_1(E_k), \ldots, S_{n+1}(E_k)\} \subseteq E_k \cap F_k.$$

By **[5]** (Lemma 5.3), F_k is an $\mathfrak{F}$-projector of G so is finitely conjugate to E_k and since $\{S_i(E_k)\}$ is a Sylow generating basis of E_k it follows that F_k has a Sylow generating basis $\{S_i(E_k)\}$ satisfying

$$S_i(E_k) = S_i(F_k) \text{ for } i = 1, \ldots, n+1.$$

Let $\mathscr{S}_{n+1} = \{E_k, F_k\colon 1 \leqq k \leqq 2^n\}$ and pick $m_{n+1} > m_n$ so that $G_{n+1} = N_{m_{n+1}}$ is a π_{n+1}'-group, with the property that

$$E_kG_{n+1} \neq F_kG_{n+1} \text{ for all } k = 1, \ldots, 2^n.$$

This is possible by 2.4. Since F_k is an $\mathfrak{F}$-projector of E_kG_n it follows that

$$F_kG_n = E_kG_n \text{ for all } k,$$

since $\mathfrak{F}$-projectors cover $\mathfrak{F}$-factor groups. If now $E_i = F_j$ for some choice of i and j then $E_iG_n = F_jG_n = E_jG_n$ which is impossible (by condition (ii) applied to $\mathscr{S}_n$) unless $i = j$. Similarly $F_i \neq F_j$ if $i \neq j$ so $\mathscr{S}_{n+1}$ has 2^{n+1} distinct elements. Conditions (ii) and (iii) are then easily verified and the construction proceeds. We remark also that if $E_k \in \mathscr{S}_n$ then $S_i(E_k) \in \mathrm{Syl}_{p_i}(E_kG_n)$ for $i = 1, \ldots, n$. This follows because G_n is a π_n'-group.

It is then possible to construct $2^{\aleph_0}$ descending chains of subgroups of the form

$$L_1 \geqq L_2 \geqq \cdots \geqq L_n \geqq \cdots$$

with the property that $L_n = E_iG_n$ for some $E_i \in \mathscr{S}_n$, with i dependent on n. We set $L = \bigcap_{n\geqq 1} L_n$. Just as in the proof of [4] (3.4) one can show, using property (iv), that $L_n = LG_n$ and hence that $L \in \mathfrak{F}$. Furthermore the facts that $\mathfrak{F}$ is co-Hopfian and G is countable imply that L is an $\mathfrak{F}$-projector of G.

Finally if $L = \bigcap_{n\geqq 1} L_n$ and $M = \bigcap_{n\geqq 1} M_n$ are the subgroups formed from 2 distinct chains then $L \neq M$. For if $L = M$ then $L_n = LG_n = MG_n = M_n$ for all integers $n \geqq 1$, a contradiction. Thus we have constructed $2^{\aleph_0}$ $\mathfrak{F}$-projectors and this final contradiction completes the proof.

By modifying the proof slightly it is easily seen that the proof of 2.6 can be made independent of the existence of $\mathfrak{F}$-projectors. Furthermore it is clear that all $\mathfrak{F}$-projectors occur as an intersection of some chain of subgroups, as in the construction.

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