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WHEN IS A FUZZY TOPOLOGY TOPOLOGICAL?

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ABSTRACT. In a 1981 paper R. Lowen defined topological fuzzy topologies (intuitively, fuzzy topologies that are not significantly fuzzy). He also proved that compact, Hausdorff fuzzy topologies must be topological. In this note Lowen's concept is described using semi-closure operators and the hypergraph functor. The two main results are:

1) The α-closure operators defined by a topological fuzzy topology are all identical and must be closure operators.

2) A fuzzy topology is topological if and only if the image under the hypergraph functor is a certain topological product.

Preliminaries. In this note the underlying lattice will be linearly ordered, complete, and completely distributive. The topology τ_r consists of L and all half-open intervals of the form $(\alpha, 1]$. A map into L is lower semi-continuous (lsc) if and only if it is continuous using τ_r on L. The Lowen definition [4] of fuzzy topology will be used: a family of fuzzy sets containing all constant maps and closed under arbitrary suprema and finite infima. If τ is a topology for X, $\omega(\tau)$ is the L-fuzzy topology consisting of all lsc maps from X into L. If T is an L-fuzzy topology for X, then $\iota(T)$ is the weakest topology that makes all elements of T lower semi-continuous. T is topological [5] provided $\omega(\iota(T)) = T$.

I. Relation to α-closure. For the definition of α-closure operators and their basic properties see [2] and [3]. The first lemma is easy to do directly and is implicit in [3].

LEMMA I.1. *Let (X, T) be an L-fts with* 0*-closure k_0. Then k_0 is a closure operator.*

PROPOSITION I.2. *Let (X, τ) be a topological space with closure c. For α in $L-\{1\}$, let k_α denote the α-closure operator induced by $\omega(\tau)$. Then $k_\alpha = c$.*

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PROOF. Let $A \subseteq X$ and let $x \in k_\alpha(A)$. For $U \in \tau$, χ_U is lsc and so is in $\omega(\tau)$. If $x \in U$, $\chi_U(x) > \alpha$ and so $\chi_A \wedge \chi_U \neq 0$, i.e., $A \cap U \neq \emptyset$. Thus, $x \in c(A)$.

Now let $x \in c(A)$ and let $G \in \omega(\tau)$ with $G(x) > \alpha$. Then $x \in G^{-1}(\alpha, 1]$, which is open in τ, and so $A \cap G^{-1}(\alpha, 1] \neq \emptyset$. Thus $\chi_A \wedge G \neq 0$ and $x \in k_\alpha(A)$.

REMARK. If L has an order-reversing involution, $\alpha \to \alpha'$, then it is easy to verify that $k_\alpha(A) = (\bar{\chi}_A)^{-1}[\alpha', 1]$, where $\bar{}$ denotes the fuzzy closure in $\omega(\tau)$. This observation yields another proof of I.2 since $\bar{\chi}_A = \chi_{c(A)}$.

COROLLARY I.3. *Let T be a topological L-fuzzy topology for X with α-closure operator k_α. Then for every $\alpha \in L-\{1\}$ $k_\alpha = k_0$.*

PROOF. Let c be the closure in $\iota(T)$. Since T is topological, $T = \omega(\iota(T))$ and by I.2 $k_\alpha = c = k_0$.

By the augmented L-fuzzy topology for I(L), the fuzzy unit interval **[1]**, I means the supremum of the Hutton topology and T_c, the set of constant maps into L. As noted in **[3]**, the augmented topology is a Lowen-type topology which induces the same α-closure operators as the Hutton topology. To discuss $I(L)$, one also must assume that L has an order-reversing involution.

COROLLARY I.4. *The augmented L-fuzzy topology for $I(L)$ is topological if and only if $L = \{0, 1\}$.*

PROOF. When $L \neq \{0, 1\}$, there exist α-closure operators which are not closure operators **[2]**. Thus, $k_\alpha \neq k_0$ for some α.

The last result of this section points out that the conclusion of I.3 is only necessary, not sufficient. If $\mathscr{A}$ is a family of semi-closure operators indexed by $L-\{1\}$, there are known necessary and sufficient conditions under which there is an L-fuzzy topology with $\mathscr{A}$ as its α-closure operators. (See **[3]**.) A family that satisfies those conditions is called an *L-FTP* collection. For an *L-FTP* collection $\mathscr{A}$, $\mathscr{F}(\mathscr{A})$ is the set of L-fuzzy topologies with $\mathscr{A}$ as α-closure operators.

PROPOSITION I.5 *Let c be a closure operator defined on a set X. Let $\mathscr{A} = \{k_\alpha | \alpha \in L-\{1\}\}$ where $k_\alpha = c$ for every α. Then*

i) *$\mathscr{A}$ is an L-FTP collection*

ii) *$\mathscr{F}(\mathscr{A})$ contains a unique topological element.*

PROOF. The first assertion is clear from the definition of an *L-FTP* collection in **[3]**. For ii) let τ be the topology with closure c. By I.2 $\omega(\tau)$, which is topological, is in $\mathscr{F}(\mathscr{A})$. For $T \in \mathscr{F}(\mathscr{A})$, with $\iota(T) \neq \tau$, by I.2

$\omega(\iota(T)) \notin \mathscr{F}(\mathscr{A})$ and so T is not topological. If $\tau = \iota(T)$, T is topological if and only if $T = \omega(\iota(T)) = \omega(\tau)$.

REMARKS. a) For any L-FTP collection $\mathscr{A}$, if $L \neq \{0, 1\}$ and T_0 is the largest element in $\mathscr{F}(\mathscr{A})$, $\iota(T_0)$ is discrete. Thus, in general, $\omega((\iota T))$ will not be in the same closure-class as T.

b) The results (except I.4) can easily be extended to a non-linear lattice by using an analogue of τ_r on L^a (see **[2]**) and defining lsc for a map into L^a as continuity using this topology.

II. Relation to the hypergraph functor. If (X, T) is an L-fts, the hypergraph functor maps it to the topological space $(X \times L - \{1\}, S(T))$ where $S(T) = \{S(G) | G \in T\}$ and $S(G) = \{(x, \alpha) | G(x) > \alpha\}$. (For more details on this functor see **[6]**.) In this section τ_{ℓ}, which consists of all half-open intervals of the form $[0, \alpha)$ for $\alpha \in L$, will be used on $L - \{1\}$. $\alpha\chi_A$ denotes $\alpha \wedge \chi_A$ where α is the constant map with value α.

PROPOSITION II.1 *Let (X, T) be an L-fts. Then $S(T) \subseteq \iota(T) \times \tau_{\ell}$.*

PROOF. Let $G \in T$ and let $(x, \alpha) \in S(G)$. If there is γ with $\alpha < \gamma < G(x)$, then $(x, \alpha) \in G^{-1}(\gamma, 1] \times [0, \gamma) \subseteq S(G)$. Otherwise let $\alpha_0 = G(x)$. Now $(x, \alpha) \in G^{-1}(\alpha, 1] \times [0, \alpha_0) \subseteq S(G)$.

LEMMA II. 2 *Let $A \subseteq X$ and let $\alpha \in L$. Then*
i) *$S(\alpha\chi_A) = A \times [0, \alpha)$ and*
ii) *if $S(F) = A \times [0, \alpha)$, $F = \alpha\chi_A$.*

PROOF. For i), $(y, \beta) \in S(\alpha\chi_A)$ if and only if $\alpha\chi_A(y) > \beta$, i.e., $y \in A$ and $\beta < \alpha$. For ii), suppose $F \neq \alpha\chi_A$ and pick x with $\beta = F(x) \neq \alpha\chi_A(x)$. Three cases can arise, all of which are routine. For example, if $x \in A$ and $\beta < \alpha$, $(x, \beta) \notin S(F)$ but $(x, \beta) \in A \times [0, \alpha)$, a contradiction.

PROPOSITION II.3 *Let (X, T) be an L-fts. Then T is topological if and only if $S(T) = \iota(T) \times \tau_{\ell}$.*

PROOF: If T is topological, then for $U \in \iota(T)$ and $\alpha \in L$ the lsc map $\alpha\chi_U$ is in T. Since $S(\alpha\chi_U) = U \times [0, \alpha)$, $\iota(T) \times \tau_{\ell} \subseteq S(T)$. For the converse, since $T \subseteq \omega(\iota(T))$ always holds, it is sufficient to show that every lsc map is in T. By II.2 ii), for every $U \in \iota(T)$ and $\alpha \in L$, $\alpha\chi_U \in T$. For any lsc F and $\alpha \in L$, let $\beta(\alpha) = \vee \{\gamma \in L | \gamma < \alpha\}$ (as usual $\vee \emptyset = 0$) and let $U(\alpha) = F^{-1}(\beta(\alpha), 1]$. Each $U(\alpha) \in \iota(T)$ and it is easy to check that $F = \vee \{\alpha\chi_{U(\alpha)} | \alpha \in L\}$. Thus $F \in T$.

Using Rodabaugh's α-property ([2] contains a definition and some related results), one can obtain an interesting variation of the last proposition.

PROPOSITION II.4 *Let (X, T) be an L-fts. Then T is topological if and* *only if T has the α-property for all $\alpha \in L$-$\{1\}$ and $S(T)$ is a product topology.*

PROOF: It is easy to check that a topological fuzzy topology has the α-property for all α. For the sufficiency of the condition, first note that if T has the α-property for all α the 0-closure k_0 is the closure operator in $\iota(T)$. Let $S(T) = \tau_1 \times \tau_2$. By II.1 $\tau_1 \subseteq \iota(T)$ and $\tau_2 \subseteq \tau_{\prime}$. For the constant map α, $S(\alpha) = X \times [0, \alpha)$. The second projection of $S(\alpha)$ is in τ_2 and so $\tau_2 = \tau_{\prime}$. Let $U \in \iota(T)$. For $A = X - U$, $k_0(A) = A$ and so by the 0-property there is $G \in T$ with $A = G^{-1}\{0\}$. The first projection of $S(G)$ is $G^{-1}(0, 1]$, which is U, and so $\iota(T) = \tau_1$. By II.3 T is topological.

For L-fuzzy topologies without the α-property for some α, it is an open question whether $S(T)$ can be a product. If $S(T)$ is a product, it can be shown that $S(T) = \tau_0 \times \tau_{\prime}$ where τ_0 is the topology with closure k_0 and that $k_\alpha(A)$ is closed relative to τ_0 for $A \subseteq X$ and every $\alpha \in L - \{1\}$.

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