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A COMPUTER METHOD FOR APPROXIMATING THE ZEROS OF CERTAIN ENTIRE FUNCTIONS

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1. Introduction. The computer revolution has greatly enhanced techni ques used in numerical calculations for a wide range of functions. This paper investigates some properties of a certain class of entire functions known as the Lindelöf functions. These functions are of the form:

$$f(z) = \prod_{N=1}^{\infty}\left(1 - \frac{z}{N^A}\right),\ A > 1,\ z \text{ complex.} \tag{1}$$

Since $f(z)$ is an infinite product, it is a generaliziation of a polynomial, and hence has a special appeal. In addition, when $A = 2$, we have the well-known equality

$$\frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}} = \prod_{N=1}^{\infty}\left(1 - \frac{z}{N^A}\right).$$

Thus these functions are also intimately related to the trigonometric functions.

In 1972 King and Shah (cf. [3]) exhibited selected properties of the Lindelöf Functions. As a partial proof of one theorem, it was necessary to approximate the bounds for the zeros of the derivatives of these functions. The techniques were slow and cumbersome due to the lack of computer facilities available to the authors at that time. We present here a method for locating the bounds, which is not only accurate to several decimal places, but greatly improves the previous values obtained for these bounds. A proof of a required theorem is given, along with sample computer calculations illustrating the method, and a discussion of the results.

2. Mathematical analysis. We shall examine the zeros of $f'(z)$, the derivative of the Lindelöf Function, $f(z) = \prod_{N=1}^{\infty} (1 - z/N^A)$, $A > 1$. These Lindelöf Functions are a special subset of Functions of Bounded Index [2]. Both f and all its derivatives are of Bounded Index [3]. From a theorem of Laguerre (cf. [1]) the zeros of $f'(z)$ are all real and are separated by the

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respective zeros of $f(z)$. Note that these occur when $z = N^A$. Thus the zeros of $f'(z)$ are in the range. $N^A < x < (N+1)^A$, where x is the real part of z.

Let $H(x) \equiv f'(x)/f(x)$. Taking logarithmic derivatives, it is observed that

$$H(x) = \sum_{J=1}^{\infty} \frac{1}{x - J^A}. \tag{2}$$

The zeros of $f'(x)$, here denoted by $B_N(A)$, are identical to those of $H(x)$, and will be indexed by the subscript N. $H(x)$ is a decreasing function of x, since $H'(x) < 0$. Thus

$$H(x) \equiv f'(x)/f(x) > 0 \rightarrow B_N(A) > x,$$

and

$$H(x) \equiv f'(x)/f(x) < 0 \rightarrow B_N(A) < x.$$

We now state and prove the essential theorem.

THEOREM. *Let* $H(x) - f'(x)/f(x)$, *as in* (2), *and*

(i)

$$H_1(x) = \sum_{J=1}^{K} \frac{1}{x - J^A} - \frac{(A + K - x)}{(A-1)(K+1-x)^A}, \tag{3}$$

$$H_2(x) = \sum_{J=1}^{K} \frac{1}{x - J^A} - \frac{1}{(A-1)(K+1)^{A-1}}. \tag{4}$$

If K is chosen such that $x < K + 1$, $A > 1$, then, for each N with $N^A < x < (N+1)^A$, we have

$$H_1 < H < H_2. \tag{5}$$

(ii) H_1 and H_2 are decreasing functions of x in each interval.

PROOF. Recall from calculus that if a function f is continuous, decreasing, and non-negative on $[1, \infty)$ and if $\int_1^\infty f(x)dx$ converges, then

$$\int_1^\infty f(x)\,dx \leqq \sum_{k=1}^{\infty} f(k) \leqq f(1) + \int_1^\infty f(x)\,dx. \tag{7}$$

To prove the left hand side of (5), we shall establish that

$$(J - x)^A \leqq J^A - x, \quad \text{for } J \geqq K + 1 > x.$$

Let $y \equiv J - x$. Then $y > 0$ and since $x > 1$ and $J = x + y$, we see that $(y + x)^A > y^A + x^A > y^A + x$. Therefore $J^A > (J - x)^A + x$ or

$$J^A - x > (J - x)^A. \tag{8}$$

Now $f(J) \equiv 1/(J + K - x)^A$ is a decreasing non-negative continuous function of J, so by (7) and (8)

$$
\begin{aligned}
\sum_{J=K+1}^{\infty} \frac{1}{(J-x)^A} &= \sum_{J=1}^{\infty} \frac{1}{(J+K-x)^A} \\
&\leqq \frac{1}{(1+K-x)^A} + \int_1^{\infty} \frac{dT}{(T+K-x)^A} \\
&= \frac{1}{(1+K-x)^A} + \frac{1}{(A-1)(K+1-x)^{A-1}} \\
&= \frac{(A+K-x)}{(A-1)(K+1-x)^A}.
\end{aligned}
\tag{9}
$$

By (8) and (9) we may conclude that

$$
\begin{aligned}
H(x) \equiv \sum_{J=1}^{\infty} \frac{1}{(x-J^A)} &\equiv \sum_{J=1}^{K} \frac{1}{(x-J^A)} - \sum_{J=K+1}^{\infty} \frac{1}{J^A - x} \\
&\geqq \sum_{J=1}^{K} \frac{1}{(x-J^A)} - \sum_{J=K+1}^{\infty} \frac{1}{(J-x)^A} \\
&\geqq \sum_{J=1}^{K} \frac{1}{(x-J^A)} - \frac{(A+K-x)}{(A-1)(K+1-x)^A} \equiv H_1.
\end{aligned}
$$

This proves the left hand side of (5). To complete this proof, we note

$$
\frac{1}{J^A - x} > \frac{1}{J^A}, \tag{10}
$$

since $J > K + 1 > x$, $x > 1$ and $A > 1$.

Therefore, $f(J) \equiv 1/(K + J)^A$ is a decreasing, non-negative, continuous function of J, so, by (7) and (10), we have

$$
\begin{aligned}
H \equiv \sum_{J=1}^{\infty} \frac{1}{x-J^A} &\equiv \sum_{J=1}^{K} \frac{1}{x-J^A} - \sum_{J=K+1}^{\infty} \frac{1}{J^A - x} \\
&\leqq \sum_{J=1}^{K} \frac{1}{x-J^A} - \sum_{J=K+1}^{\infty} \frac{1}{J^A} \equiv \sum_{J=1}^{K} \frac{1}{x-J^A} - \sum_{J=1}^{\infty} \frac{1}{K+J)^A} \\
&\leqq \sum_{J=1}^{K} \frac{1}{x-J^A} - \int_1^{\infty} \frac{dT}{(K+T)^A} \\
&= \sum_{J=1}^{K} \frac{1}{x-J^A} - \frac{1}{(A-1)(K+1)^{A-1}} \equiv H_2.
\end{aligned}
$$

Thus (5) has been verified.

For the proof of part (ii) of the Theorem, it is easily verified that H_2 is a decreasing function of x, since the second term does not involve x at all.

Now let

$$H_1 \equiv \sum_{J=1}^{K} \frac{1}{x - J^A} - \frac{(A + K - x)}{(A - 1)(K + 1 - x)^A}$$
$$\equiv S_1 + S_2,$$

i.e.,

$$S_2 = \frac{-(A + K - x)}{(A - 1)(K + 1 - x)^A}.$$

A direct calculation yields

$$dS_2/dx = \frac{(K + 1 - x)^A - (A + K - x)(A)(K + 1 - x)^{A-1}}{(A - 1)(K + 1 - x)^{2A}}.$$

Since $A > 1$,

$$K + 1 - x < K + A - x < A(K + A - x).$$

Multiplying each side of the inequality by the positive factor $(K + 1 - x)^A$, we see that

$$(K + 1 - x)(K + 1 - x)^{A-1} < A(K + A - x)(K + 1 - x)^{A-1}.$$

Thus the numberator of dS_2/dx is negative, so $dS_2/dx < 0$. Obviously $dS_1/dx < 0$, so we must have $dH_1/dx < 0$, i.e., H_1 is a decreasing function of x.

This completes the proof of the theorem.

3. Method. The calculations were performed on an Apple II + with 48K. Those wishing to trace the logic of the program may contact either author.

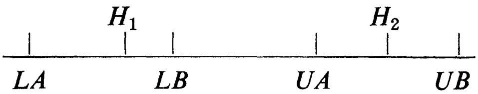
As in the theorem just proved, $f(z)$ indicates the Lindelöf Functions. By Laguerre's theorem, the zeros of $f'(z)$ are real and are separated by the zeros of $f(z)$. Hence, for each N,

$$N^A < x < (N + 1)^A,$$

where x represents the zeros of $f'(z)$. The method is by bisection of each interval. The user enters the number of bisections (M), as well as the values of N and A desired. With the notation used previously, the zeros of $f'(z)$ are the same as those of $H(x)$ and

$$H_1 < H < H_2$$

After each bisection, the "half" which contains the zero is located, and upper and lower bounds for H_1 and H_2 are given. At the end of the requested number of bisections, the lower and upper bounds are printed. The diagram illustrates the location of the quantities studied.



4. Results. Tables I and II are included, which exhibit values obtained. Table I contains those from the computer program and Table II gives results using the following formulas: (cf. [2])

$$\frac{3.6 + 2^A}{3} < B(1) < \frac{1 + 2^{A+1}}{4},$$

$$1 + 2^{A+1} < B(2) < \frac{2^A + 3^{A+1}}{4},$$

$$N^A < \frac{N^A(N+1)+(N+1)^A}{(N+2)} < B(N) < \frac{N^A+(N+1)^{A+1}}{(N+2)} < (N+1)^A;$$

$$N \geqq 3; \text{ and } A \geqq 3$$

Here $B(N)$ denotes the N^{th} zero of the derivative. When comparisons are made between the two tables, it is to be noted that Table I yields much more accurate values. This is even more evident as the values of A and N are increased; however, the computer time required is considerably more. In addition, when $1 < A < 2$, another more complicated formula was required (cf. [3]) and the computations become very messy. On the other hand, the computer program yields these values quickly and accurately.

Table I illustrates some results obtained.

		Lower Bound	Upper Bound
$M = 10 \quad A = 1.01$	$N = 2$	2.024	2.025
	$N = 4$	4.066	4.067
	$N = 6$	6.119	6.120
	$N = 10$	10.243	10.244
$M = 10 \quad A = 2$	$N = 2$	5.528	6.070
	$N = 3$	11.133	12.062
	$N = 4$	18.707	20.061
	$N = 5$	28.266	30.060
	$N = 10$	105.947	110.069
$M = 5 \quad A = 3$	$N = 2$	17.5	18.093
	$N = 4$	98.312	100.219
	$N = 5$	179.031	181.875
$M = 5 \quad A = 4$	$N = 3$	190.375	195.844

TABLE I.

		Lower Bound	Upper Bound
$A = 2$	$N = 3$	10.40	14.60
	$N = 4$	17.50	23.50
	$N = 5$	26.57	34.43
	$N = 10$	101.25	119.25
$A = 3$	$N = 2$	17	22.25
	$N = 4$	74.17	114.83
	$N = 5$	138	203
$A = 4$	$N = 3$	116	221

TABLE II

5. Conclusion. A computer program was designed to estimate the value of the zeros of $f'(z)$, where $f(z) = \prod_{N=1}^{\infty}(1 - z/N^A)$, $A > 1$, z complex. In these calculations the lower and upper bounds of these zeros are given respectively by

$$H_1(x) = \sum_{J=1}^{K} \frac{1}{x - J^A} - \frac{(A + K - x)}{(A - 1)(K + 1 - x)^A}$$

and

$$H_2(z) = \sum_{J=1}^{K} \frac{1}{x - J^A} - \frac{1}{(A - 1)(K + 1)^{A-1}},$$

where in both H_1 and H_2 K is chosen such that $1 < x < K + 1$, $A > 1$, and, for each N,

$$N^A < x < (N + 1)^A.$$

By the methods illustrated, we were able to approximate the zeros of $f'(z)$, quite rapidly and accurately, as long as N and A are not too large. Even in this case, the results were obtained more quickly and precisely than by the calculator computations.

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