ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 4, Fall 1986

PARA-UNIFORMITIES, PARA-PROXIMITIES, AND H-CLOSED EXTENSIONS

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ABSTRACT. A generalized uniformity, called a para-uniformity, and its induced generalized proximity, called a para-proximity, are introduced and applied to the investigation of H-closed spaces and H-closed extensions of Hausdorff spaces.

H-closed spaces are characterized in terms of these structures, and the H-closed extensions of a Hausdorff space are characterized in terms of extensions of these structures. Moreover, collections of para-uniformities called superstructures are used to obtain all strict H-closed extensions of a non-H-closed Hausdorff space. Thus, the S-equivalence classes of H-closed extensions are described by a method similar to that of Fedorčuk for describing the R-equivalence classes.

0. Introduction. Alexandroff [1] remarked in 1960 that no method of systematically determining the H-closed extensions of a Hausdorff space had been found. In classifying (the isomorphism classes of) such extensions, the introduction of two equivalence relations discussed in [**18**] is helpful. We declare two H-closed extensions of a given space to be R-equivalent if they are θ-isomorphic and to be S-equivalent if their corresponding strict (or simple) extensions are isomorphic. In attempts to answer Alexandroff's remark, various authors have sought methods for obtaining all isomorphism classes, all R-equivalence classes, or all S-equivalence classes. (See, for instance, [**2**, **4**, **7**, **10**, **11**, **17** or **21**].)

Fedorčuk [7] refers to the particular problem of constructing the H-closed extensions of a given Hausdorff space by means of uniformity or proximity-like structures as "Tychonoff's problem." He [7], Porter and Votaw [**18**] have shown that in general there are not enough such structures on a set to yield all isomorphism classes of either semiregular H-closed extensions or strict H-closed extensions of one of its Hausdorff topologies. According to results in [**18**] this implies that neither the R-equivalence classes nor the S-equivalence classes can be obtained in this manner, and

AMS *subject classification*: Primary 54D25, 54E05, 54E15; Secondary 54D30.
Key words and phrases: para-uniformity, para-proximity, H-closed extension.
Received by the editors on April 2, 1984 and in revised form on June 1, 1985.

thus Tychonoff's problem has no solution. However, Fedorčuk [7] uses "H-structures," which are collections of generalized proximities called "θ-proximities," to construct all semiregular H-closed extensions of a semiregular Hausdorff space. (In [16] θ-proximities on regular topological spaces are shown to coincide with particular f-proximities.) Thus, we may use H-structures to describe all R-equivalence classes of H-closed extensions of a given Hausdorff space. Although nearness structures have been used by J.W. Carlson [4] to construct the strict H-closed extensions (and, hence, the S-equivalence classes of H-closed extensions) of a given space, the following question has remained unanswered in the literature: can collections of generalized uniformities or proximities be used to obtain the S-equivalence classes of H-closed extensions of a given Hausdorff space? In this paper we provide an affirmative answer to this question.

In particular we shall introduce a generalized uniformity, called a para-uniformity, and its associated generalized proximity, called a para-proximity. These notions enable us to obtain new characterizations of H-closed spaces and of H-closed extensions of Hausdorff spaces. Canonical completions of these structures yield a rather large class of strict H-closed extensions (those with "relatively completely regular outgrowth"), and this class is shown to include the extensions with "relatively zero-dimensional outgrowth" studied by Flachsmeyer [8]. Moreover, collections of para-uniformities called superstructures will be used to obtain a representative from each isomorphism class of strict H-closed extensions. Thus, we obtain a new description of the S-equivalence classes of H-closed extensions of a given Hausdorff space by means of superstructures.

Fedorčuk [5] has previously introduced generalized uniformities called "θ-uniformities," which he later used to construct members of a class of H-closed extensions as canonical completions [6]. We shall develop a relation between these completions and those of para-uniformities. We are thankful to the referee for bringing to our attention [13], where Kulpa develops generalized covering uniformities which correspond to the (diagonal) para-uniformities introduced here. Hence, many results in this paper extend and illuminate results of [13].

The development of the theory of para-uniform and para-proximity spaces to a great extent parallels that of uniform and proximity spaces. Thus, many details of the proofs of the early basic results are left to the reader, who might find reference to [3], [14], [15], [22], or [24] helpful.

A few comments about notation and terminology are appropriate now. If (Y, σ) is a topological space, $X \subset Y$, and $y \in Y$, then $O_\sigma^{y,X}$ denotes $\{G \cap X : y \in G \in \sigma\}$. Thus, $O_\sigma^{y,Y}$ is the collection of open neighborhoods of y in Y. If (Y, σ) is an extension of (X, τ), then the associated strict (respectively, simple) extension is denoted by $(Y, \sigma^{\#})$ (respectively, (Y, σ^{+})).

Recall [18] that a basis for the topology $\sigma^{\#}$ on Y is $\{G^{\#}: G \in \tau\}$ where $G^{\#} = \{y \in Y: G \in O_{\sigma}^{y, X}\}$, while a basis for the topology σ^{+} on Y is $\{G \cup \{y\}: G \in O_{\sigma}^{y, X}, y \in Y\}$. Moreover, $\sigma^{\#} \subset \sigma \subset \sigma^{+}$ and $O_{\sigma}^{y, X} = O_{\sigma^{\#}}^{y, X} = O_{\sigma^{+}}^{y, X}$, for each $y \in Y$. An extension (Y, σ) of (X, τ) is called a strict (respectively, simple) extension if $\sigma = \sigma^{\#}$ (respectively, $\sigma = \sigma^{+}$). If $\mathscr{F}$ is a filter on a space (X, τ), then $\mathscr{F}$ will be called an open filter or τ-filter if $\mathscr{F}$ has a base of open sets.

The study of para-uniform spaces was initiated in the Ph. D. dissertation of the second author [23].

1. Para-uniform spaces. If X is a set and A is a subset of X, then we shall let $\Delta(A) = \{(x, x): x \in A\}$. If $U \subset X \times X$ we let dom $U = \{x: (x, y) \in U$ for some $y \in X\}$, $U^0 = \Delta(\text{dom } U)$, and $U^{-1} = \{(y, x): (x, y) \in U\}$. Also, if $U \subset X \times X$ and $A \subset X$, let $U[A] = \{y: (x, y) \in U$ for some $x \in A\}$. When $U, V \subset X \times X$ we let $U \circ V = \{(x, y)$: for some $z \in X$, $(x, z) \in V$ and $(z, y) \in U\}$.

DEFINITION 1.1. Let X be a set and let $\mathscr{U}$ be a collection of subsets of $X \times X$ which satisfies:

(U1) $X \times X \in \mathscr{U}$;

(U2) if $U \in \mathscr{U}$, then $U^0 \subset U$;

(U3) if $U \in \mathscr{U}$, then $U \cap U^{-1} \in \mathscr{U}$;

(U4) if $U, V \in \mathscr{U}$, then there is $W \in \mathscr{U}$ such that $W \circ W \subset U \cap V$ and $W^0 = (U \cap V)^0$;

(U5) if $U \in \mathscr{U}$ and $U \subset V \subset X \times X$ with $U^0 = V^0$, then $V \in \mathscr{U}$; and

(U6) if $U, V \in \mathscr{U}$ and $x \in X$ with $U[x] \neq \varnothing$, then $U[x] \cap \text{dom } V \neq \varnothing$.

Then $\mathscr{U}$ is called a para-uniformity on X, and $(X, \mathscr{U})$ is called a para-uniform space. The members of $\mathscr{U}$ are called entourages.

Note that if condition (U2) is strengthened to require that $\Delta(X) \subset U$ for every $U \in \mathscr{U}$, then $\mathscr{U}$ is a uniformity on X. Of course, in this case, some of the conditions (U1) – (U6) are redundant, but this shows that the conditions are consistent and that the collection of para-uniformities on a set is nontrivial in general.

If $\mathscr{U}$ is a para-uniformity on X and $x \in X$, let $\mathscr{U}(x) = \{U[x]: U \in \mathscr{U}\} - \{\varnothing\}$. It may be shown easily, using conditions (U1) – (U5), that $\{\mathscr{U}(x): x \in X\}$ is a neighborhood system on X. The resulting topology on X will be denoted by $\tau(\mathscr{U})$. Note that $G \in \tau(\mathscr{U})$ if and only if $x \in G$ implies there is some $U \in \mathscr{U}$ such that $x \in U[x] \subset G$. Condition (U6) simply says that, for each entourage $U \in \mathscr{U}$, dom U is $\tau(\mathscr{U})$-dense in X. Note that if $\mathscr{U}$ and $\mathscr{V}$ are para-uniformities on X and $\mathscr{U} \subset \mathscr{V}$, then $\tau(\mathscr{U}) \subset \tau(\mathscr{V})$.

DEFINITION 1.2. Let $(X, \mathscr{U})$ be a para-uniform space. (a) $\mathscr{U}$ is said to be compatible with a topology τ on X if $\tau = \tau(\mathscr{U})$. (b) If $(X, \tau(\mathscr{U}))$ is Hausdorff, then $\mathscr{U}$ is called a separated para-uniformity.

Throughout this paper many useful elementary results concerning para-uniform entourages and topologies will be needed. For example, if $\mathcal{U}$ is a para-uniformity on a set X, U is a symmetric entourage in $\mathcal{U}$, and $x \in \operatorname{dom} U$, then

(1) $U[x] \times U[x] \subset U \circ U$;

(2) $y \in U[x]$ implies $\overline{U[y]} \subset (U \circ U)[x]$; and

(3) $\overline{U[x]} \subset \operatorname{dom} U$ implies $\overline{U[x]} \subset (U \circ U)[x]$.

Similar results occasionally will be noted as needed.

As in the theory of uniform spaces, it is convenient to consider collections with certain properties which generate, in a specified manner, unique para-uniformities.

DEFINITION 1.3. Let X be a set. (a) Let $\mathcal{B}$ be a collection of subsets of $X \times X$ which satisfies (U2), (U4), and (U6) of Definition 1.1 and (B3): if $B \in \mathcal{B}$, then there is some $D \in \mathcal{B}$ such that $D \subset B \cap B^{-1}$ and $D^0 = B_0$. Then $\mathcal{B}$ is called a para-uniform basis on X. (b) Let $\mathcal{S}$ be a collection of subsets of $X \times X$ which satisfies (U2) and (U6) of Definition 1.1 and (S4): if $S \in \mathcal{S}$, then there is some $T \in \mathcal{S}$ such that $T \circ T \subset S \cap S^{-1}$ and $T^0 = S^0$. Then $\mathcal{S}$ is called a para-uniform subbasis on X.

If $\mathcal{B}$ is a para-uniform basis on X, then it may be shown easily that $\mathcal{U}(\mathcal{B}) = \{X \times X\} \cup \{U \subset X \times X\colon$ for some $B \in \mathcal{B}$, $B \subset U$ and $B^0 = U^0\}$ is the smallest para-uniformity on X which contains $\mathcal{B}$. If $\mathcal{S}$ is a para-uniform subbasis on X, then it may also easily be verified that $\mathcal{B}(\mathcal{S}) = \{\bigcap \mathcal{T} : \mathcal{T}$ is a finite subcollection of $\mathcal{S}\}$ (where $\bigcap \phi = X \times X$) is a para-uniform basis on X and that $\mathcal{U}(\mathcal{B}(\mathcal{S}))$ is the smallest para-uniformity on X containing $\mathcal{S}$. $\mathcal{U}(\mathcal{B}(\mathcal{S}))$ may also be denoted by $\mathcal{U}(\mathcal{S})$.

We will freely use the fact that the collection of symmetric entourages of a para-uniformity $\mathcal{U}$ which are open in the product topology $\tau(\mathcal{U}) \times \tau(\mathcal{U})$ is a basis for $\mathcal{U}$.

A para-uniformity on X may be described in terms of uniformities on subsets of X. This is the content of the next two propositions, whose straightforward proofs are omitted.

PROPOSITION 1.4. *Let $(X, \mathcal{U})$ be a para-uniform space and set $\mathcal{A}_{\mathcal{U}} = \{\operatorname{dom} U\colon U \in \mathcal{U}\}$. For each $A \in \mathcal{A}_{\mathcal{U}}$, $\mathcal{B}_A = \{V \in \mathcal{U}\colon V = V^{-1}$ and $\operatorname{dom} V = A\}$ is a basis for a uniformity $\mathcal{U}_A$ on A. Moreover, the following properties are satisfied*:

(i) *$X \in \mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\mathcal{U}}$ is closed under finite intersections*;

(ii) *if $A_1, A_2 \in \mathcal{A}_{\mathcal{U}}$ and $V_i \in \mathcal{U}_{A_i}$ $(i=1, 2)$, then $V_1 \cap V_2 \in \mathcal{U}_{A_1 \cap A_2}$*; and

(iii) *if $A_1, A_2 \in \mathcal{A}_{\mathcal{U}}$, $V \in \mathcal{U}_{A_1}$, and $x \in X$ with $V[x] \neq \emptyset$, then $V[x] \cap A_2 \neq \emptyset$.*

PROPOSITION 1.5. *Let X be a set, let $\mathcal{A}$ be a collection of subsets of X, and for each $A \in \mathcal{A}$ let $\mathcal{V}_A$ be a uniformity on $\mathcal{A}$. Moreover, assume that the following properties are satisfied*:

(i) *$X \in \mathscr{A}$ and $\mathscr{A}$ is closed under finite intersections;*
(ii) *if $A_1, A_2 \in \mathscr{A}$ and $V_i \in \mathscr{V}_{A_i}$ $(i = 1, 2)$, then $V_1 \cap V_2 \in \mathscr{V}_{A_1 \cap A_2}$; and*
(iii) *if $A_1, A_2 \in \mathscr{A}$, $V \in \mathscr{V}_{A_1}$, and $x \in X$ with $V[x] \neq \varnothing$, then $V[x] \cap A_2 \neq \varnothing$.*

Then $\mathscr{B} = \bigcup_{A \in \mathscr{A}} \mathscr{V}_A$ is a para-uniform basis for a para-uniformity $\mathscr{U}$ on X, and $\mathscr{V}_A = \mathscr{U}_A$, where $\mathscr{U}_A$ is the uniformity on A with basis $\{V \in \mathscr{U}: V = V^{-1}$ and dom *$V = A\}$.*

If $(X, \mathscr{U})$ is a para-uniform space, $A \in \mathscr{A}_{\mathscr{U}}$, and $\mathscr{U}_A$ is the uniformity on A induced by $\mathscr{U}$ as in Proposition 1.4, then it is clear that $\tau(\mathscr{U}_A) \subset \tau(\mathscr{U})$, where $\tau(\mathscr{U}_A)$ is the uniform topology on A induced by $\mathscr{U}_A$. (Observe first that A is $\tau(\mathscr{U})$-open in X since it is the domain of an entourage.)

The preceding characterization of a para-uniform space demonstrates that a para-uniformizable topology may be obtained from uniformizable topologies on dense subsets. It follows from the next result that every topology is of this type.

THEOREM 1.6. *Let (X, τ) be a topological space, and let β be a subbasis for τ. For each $G \in \beta$ let $S(G) = (G \times G) \cup [(X - \bar{G}) \times (X - \bar{G})]$. Then $\mathscr{S} = \{S(G): G \in \beta\}$ is a subbasis for a compatible para-uniformity on (X, τ).*

PROOF. For $G \in \beta$, $S(G) \circ S(G) = S(G)$, $S(G)^{-1} = S(G)$, and if $x \in X$ with $S(G)[x] \neq \varnothing$, then $S(G)[x] = G$ or $X - \bar{G}$. With these observations, it is straightforward to verify (U2), (U6), and (S4) for $\mathscr{S}$ and that $\tau(\mathscr{U}(\mathscr{S})) = \tau$.

Note that if σ is a subbase for the topology τ on X and σ consists of open dense subsets of X, then $\{G \times G: G \in \sigma\}$ also serves as a subbasis for a compatible para-uniformity on (X, τ).

Certain subsets of para-uniform spaces become para-uniform spaces in the natural manner.

PROPOSITION 1.7. *Let $(Y, \mathscr{U})$ be a para-uniform space and let X be either $\tau(\mathscr{U})$-open or $\tau(\mathscr{U})$-dense in Y. Then $\mathscr{U}|_X = \{U \cap (X \times X): U \in \mathscr{U}\}$ is a para-uniformity on X. Moreover, if $\mathscr{B}$ is a basis (respectively, subbasis) for $\mathscr{U}$, then $\{B \cap (X \times X): B \in \mathscr{B}\}$ is a basis (respectively, subbasis) for $\mathscr{U}|_X$.*

PROOF. It is straightforward to verify (U1) – (U5) for $\mathscr{U}|_X$. Recall that these are the conditions of Definition 1.1 needed to insure that $\mathscr{U}|_X$ induces the topology $\tau(\mathscr{U}|_X)$ on X. Also, it is straightforward to then show that $\tau(\mathscr{U}|_X) = \tau(\mathscr{U})|_X$. So (U6) may be verified by a completely topological argument: the domain of any entourage in $\mathscr{U}|_X$ is $\tau(\mathscr{U}|_X)$-dense in X since it is the intersection of a $\tau(\mathscr{U})$-dense and $\tau(\mathscr{U})$-open subset of Y with a subset of Y which is either $\tau(\mathscr{U})$-dense or $\tau(\mathscr{U})$-open.

DEFINITION 1.8. Let $(Y, \mathscr{U})$ be a para-uniform space and let X be either a $\tau(\mathscr{U})$-dense or a $\tau(\mathscr{U})$-open subset of Y. Then $\mathscr{U}|_X$ is called the relative para-uniformity on X.

We shall conclude this section by discussing the generalization of uniformly continuous mappings to the para-uniform case. If $f: X \to Y$ is a function and $V \subset Y \times Y$, then we use $f^{-1}(V)$ to denote $\{(x, y) \in X \times X: (f(x), f(y)) \in V\}$.

DEFINITION 1.9. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be para-uniform spaces and let $f: X \to Y$ be a function.

(a) f is para-uniformly continuous if, for each $V \in \mathscr{V}$ with $f^{-1}(V) \neq \emptyset$, $f^{-1}(V) \in \mathscr{U}$.

(b) If f is a para-uniformly continuous bijection and f^{-1} is also para-uniformly continuous, then f is called a para-uniform isomorphism.

PROPOSITION 1.10. *If* $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ *is para-uniformly continuous, then* $f: (X, \tau(\mathscr{U})) \to (Y, \tau(\mathscr{V}))$ *is continuous.*

The proof is similar to the proof of the analogous result in the uniform case.

Note that if $(Y, \mathscr{U})$ is a para-uniform space and X is either $\tau(\mathscr{U})$-dense or $\tau(\mathscr{U})$-open in Y, then $\mathrm{id}_X: (X, \mathscr{U}|_X) \to (Y, \mathscr{U})$ is para-uniformly continuous. Also, it is clear that the composition of para-uniformly continuous functions is para-uniformly continuous.

2. Para-proximity spaces. A given para-uniformity on a set X can be realized in terms of uniformities on subsets of X, and each uniformity induces a proximity. So it is natural to seek a structure on X involving proximities on subsets of X whose relation to para-uniformities is analogous to the relation of proximities to uniformities.

DEFINITION 2.1. Let X be a set, let $\mathscr{A}$ be a collection of sbsets of X, and let $\mathscr{D} = \{\delta_A: A \in \mathscr{A}\}$ where δ_A is a proximity on A for each $A \in \mathscr{A}$.

(a) The triple $(X, \mathscr{A}, \mathscr{D})$ is called a para-proximity space if the following three conditions hold:

(P1) $X \in \mathscr{A}$ and $\mathscr{A}$ is closed under finite intersections;

(P2) if $A_1, A_2 \in \mathscr{A}$, then $\delta_{A_1 \cap A_2} \subset \delta_{A_1} \cap \delta_{A_2}$; and

(P3) if $A_1, A_2 \in \mathscr{A}$ and $x \in A_1$, then $x\ \delta_{A_1}(A_1 \cap A_2)$.

(b) If $(X, \mathscr{A}, \mathscr{D})$ is a para-proximity space, then the para-proximity on X associated with $(X, \mathscr{A}, \mathscr{D})$ is $\delta \subset \mathscr{P}(X) \times \mathscr{P}(X)$ defined by (for $B_1, B_2 \subset X$) $B_1\ \bar{\delta}\ B_2$ if and only if there is an $A \in \mathscr{A}$ with $B_1 \subset A$ and $B_1\ \bar{\delta}_A\ (B_2 \cap A)$.

If $(X, \mathscr{A}, \mathscr{D})$ is a para-proximity space with associated para-proximity δ, then we may define (for $B \subset X$) $B^{\delta} = \{x \in X: x\ \delta\ B\}$. It is straightforward to verify that $B \to B^\delta$ is a Kuratowski closure operator on $\mathscr{P}(X)$ and, hence, $\tau(\delta) = \{X - B^\delta: B \subset X\}$ is a topology on X.

Note that if δ_X is a proximity on X, then $(X, \{X\}, \{\delta_X\})$ is a para-proximity space whose associated para-proximity is δ_X. Thus, para-proximity spaces generalize proximity spaces.

The easy proof of the following proposition is omitted.

PROPOSITION 2.2. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ, and let $B, B_1, B_2 \subset X$.* (a) *$\emptyset\, \bar{\delta}\, B$ and $B\, \bar{\delta}\, \emptyset$.*

(b) *$x\, \delta\, x$ for all $x \in X$.*

(c) *$B\, \delta\, (B_1 \cup B_2)$ if and only if $B\, \delta\, B_1$ or $B\, \delta\, B_2$.*

(d) *If $B_1\, \bar{\delta}\, B_2$, then there is $C \subset X$ such that $B_1\, \bar{\delta}\, X - C$ and $C\, \bar{\delta}\, B_2$.*

(e) *If $B_1\, \bar{\delta}\, B_2$, then $B_1 \cap B_2 = \emptyset$.*

(f) *If $B_1\, \bar{\delta}\, B_2$ and $C_i \subset B_i$ $(i = 1, 2)$, then $C_1\, \bar{\delta}\, C_2$.*

It follows from this proposition that if the para-proximity δ associated with $(X, \mathscr{A}, \mathscr{D})$ is symmetric (that is, $B_1\, \delta\, B_2$ if and only if $B_2\, \delta\, B_1$), then δ is a proximity on X, and, hence, $\tau(\delta)$ is completely regular.

DEFINITION 2.3. Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ.

(a) δ is said to be compatible with a topology τ on X if $\tau = \tau(\delta)$.

(b) $(X, \mathscr{A}, \mathscr{D})$ is called separated if $\tau(\delta)$ is Hausdorff.

Note that $\tau(\delta)$ is T_0 if and only if $x, y \in X$ with $x\delta y$ and $y\delta x$ implies $x = y$, and $\tau(\delta)$ is T_1 if and only if $x,y \in X$ with $x\delta y$ implies $x = y$. Two other consequences concerning the para-proximity topology are recorded in the next proposition.

PROPOSITION 2.4. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ.*

(a) *For $x \in X$ and $B \subset X$, $x\, \bar{\delta}\, X - B$ if and only if $x \in \operatorname{int} B$.*

(b) *For $B_1, B_2 \subset X$, $B_1\, \bar{\delta}\, X - B_2$ implies $B_1 \subset \operatorname{int} B_2$.*

We begin to develop the relation between para-uniform spaces and para-proximity spaces in the next theorem, whose proof is quite similar to the proof of the analogous result in the uniform-proximity case.

THEOREM 2.5. *Let $(X, \mathscr{U})$ be a para-uniform space. Let $\mathscr{A}_{\mathscr{U}} = \{\operatorname{dom} U : U \in \mathscr{U}\}$, for each $A \in \mathscr{A}_{\mathscr{U}}$ let $\mathscr{U}_A$ be the uniformity on A induced by $\mathscr{U}$ (as in Proposition 1.4), and let δ_A be the proximity induced on A by $\mathscr{U}_A$. Set $\mathscr{D}_{\mathscr{U}} = \{\delta_A : A \in \mathscr{A}_{\mathscr{U}}\}$. Then $(X, \mathscr{A}_{\mathscr{U}}, \mathscr{D}_{\mathscr{U}})$ is a para-proximity space. Moreover, $\tau(\mathscr{U}) = \tau(\delta_{\mathscr{U}})$, where $\delta_{\mathscr{U}}$ is the para-proximity on X associated with $(X, \mathscr{A}_{\mathscr{U}}, \mathscr{D}_{\mathscr{U}})$ given by (for $B_1, B_2 \subset X$) $B_1\, \overline{\delta_{\mathscr{U}}}\, B_2$ if and only if there is $U \in \mathscr{U}$ with $B_1 \subset \operatorname{dom} U$ and $U[B_1] \cap U[B_2 \cap \operatorname{dom} U] = \emptyset$.*

The result, together with Theorem 1.6. tells us that any topology is para-proximizable.

Before we observe that every para-proximity space is induced by a para-uniform space in the manner prescribed by Theorem 2.5, it is appropriate to introduce the notion of a totally bounded para-uniformity.

DEFINITION 2.6. A para-uniform space $(X, \mathscr{U})$ is totally bounded if, for each $U \in \mathscr{U}$, there is a finite collection $\mathscr{C}$ of subsets of X such that $X = \bigcup \{\bar{C}:C \in \mathscr{C}\}$ and $\bigcup \{C \times C:C \in \mathscr{C}\} \subset U$.

Note that the definition is equivalent to the usual definition of totally bounded in case $\mathscr{U}$ is a uniformity, since then $\bar{C} \subset V[C]$ for every $V \in \mathscr{U}$. (Also, it is equivalent to assume that $\varnothing \notin \mathscr{C}$.) Moreover, we have the following straightforward characterizations of totally bounded para-uniformity.

PROPOSITION 2.7. *Let $(X, \mathscr{U})$ be a para-uniform space and let $\mathscr{A}_{\mathscr{U}} = \{$dom $U:U \in \mathscr{U}\}$. The following are equivalent:*

(a) *$(X, \mathscr{U})$ is totally bounded.*

(b) *For each $U \in \mathscr{U}$ there is a finite subset $F \subset X$ such that $X = \overline{U[F]}$.*

(c) *$\mathscr{U}_A$ is totally bounded, for each $A \in \mathscr{A}_{\mathscr{U}}$.*

THEOREM 2.8. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space. For each $A \in \mathscr{A}$, let $\mathscr{V}_A$ be the unique totally bounded uniformity on A which induces δ_A. Then $\mathscr{B} = \bigcup \{\mathscr{V}_A:A \in \mathscr{A}\}$ is a basis for the unique totally bounded para-uniformity $\mathscr{U}$ on X such that $(X, \mathscr{A}, \mathscr{D}) = (X, \mathscr{A}_{\mathscr{U}}, \mathscr{D}_{\mathscr{U}})$.*

PROOF. Everything follows easily once we have verified (ii) of Proposition 1.5. To this end note that a basis for $\mathscr{V}_A$ is $\mathscr{B}_A = \{\bigcup_{j=1}^{m} H_j \times H_j: K_j \overline{\delta}_A A - H_j, H_j \subset A\ (j = 1, 2, \ldots, m), A = \bigcup_{j=1}^{m} K_j\}$. Let $A_1, A_2 \in \mathscr{A}$. To verify (ii) of Proposition 1.5 it suffices to show that if $B_i \in \mathscr{B}_{A_i}$ $(i = 1, 2)$, then $B_1 \cap B_2 \in \mathscr{V}_{A_1 \cap A_2}$. So let $K_j^i \overline{\delta}_{A_i} A_i - H_j^i$, $H_j^i \subset A_i$ $(j = 1, 2, \ldots, m_i)$, $A_i = \bigcup_{j=1}^{m_i} K_j^i$ $(i = 1, 2)$ and $B_i = \bigcup_{j=1}^{m_i} H_j^i \times H_j^i$ $(i = 1, 2)$. Then $(A_1 \cap A_2) \times (A_1 \cap A_2) \supset B_1 \cap B_2 \supset \bigcup_{j=1}^{m_1} \bigcup_{k=1}^{m_2} ((H_j^1 \cap H_k^2) \times (H_j^1 \cap H_k^2)) \in \mathscr{B}_{A_1 \cap A_2}$. So $B_1 \cap B_2 \in \mathscr{V}_{A_1 \cap A_2}$.

If $(X, \mathscr{A}, \mathscr{D})$ is a para-proximity space, then it follows from Theorem 2.8 and the remarks following Proposition 1.5 that $\tau(\delta_A) \subset \tau(\delta)$, for each $A \in \mathscr{A}$. Also, in view of Theorem 2.8, the next results about relative para-proximities are not suprising.

PROPOSITION 2.9. *Let $(Y, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ, and let X be either $\tau(\delta)$-open or $\tau(\delta)$-dense in Y. Let $\mathscr{A}|_X = \{A \cap X:A \in \mathscr{A}\}$ and $\mathscr{D}|_X = \{\delta_A|_{A \cap X}: A \in \mathscr{A}\}$. Then $(X, \mathscr{A}|_X, \mathscr{D}|_X)$ is a para-proximity space with associated para-proximity $\delta|_X$ defined by (for $B_1, B_2 \subset X$) $B_1\ \delta|_X\ B_2$ if* and only if *$B_1\ \delta\ B_2$.* Also $\tau(\delta|_X) = \tau(\delta)|_X$.

DEFINITION 2.10. Let $(Y, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ and let X be either $\tau(\delta)$-open or $\tau(\delta)$-dense in Y. Then $\delta|_X$ is called the relative para-proximity on X.

PROPOSITION 2.11. *Let $(Y, \mathscr{U})$ be a para-uniform space, let $(Y, \mathscr{A}, \mathscr{D})$ be the para-proximity space induced by $(Y, \mathscr{U})$, and let δ be the associated para-proximity on Y. Let $\tau = \tau(\mathscr{U}) = \tau(\delta)$ and let X be either τ-open or τ-dense in Y. Then $(X, \mathscr{A}_{\mathscr{U}|X}, \mathscr{D}_{\mathscr{U}|X}) = (X, \mathscr{A}|_X, \mathscr{D}|_X)$ and $\delta_{\mathscr{U}|X} = \delta|_X$.*

The notion of a proximity mapping also generalizes easily to the para-proximity case.

DEFINITION 2.12. Let $(X_i, \mathscr{A}_i, \mathscr{D}_i)$ be a para-proximity space with associated para-proximity δ_i $(i = 1, 2)$ and let $f: X_1 \to X_2$ be a function. (a) f is a para-proximity mapping if, whenever $x, y \in X_1$ and $x \, \delta_1 \, y$, then $f(x) \, \delta_2 \, f(y)$. (b) If f is a para-proximity bijection and f^{-1} is also a para-proximity mapping, then f is called a para-proximity isomorphism.

PROPOSITION. 2.13. (a) *A para-proximity mapping is continuous with respect to the para-proximity topologies.* (b) *A para-uniformly continuous mapping is a para-proximity mapping with respect to the induced para-proximities.* (c) *The composition of para-proximity mappings is a para-proximity mapping.*

3. Extensions of para-uniform and para-proximity spaces. Throughout the remainder of this paper we shall use "para-uniform space" to mean separated para-uniform space, "para-proximity space" to mean separated para-proximity space, and "topological space" to mean Hausdorff topological space. It is worth noting now that when a para-uniform space $(X, \mathscr{U})$ is separated it is not necessarily true that the uniformity $\mathscr{U}_A$ induced on $A = \text{dom}\, U$ $(U \in \mathscr{U})$ is separated. (In fact, the compatible para-uniformity $\mathscr{U}$ induced on the real line with the usual topology, in the manner prescribed by Theorem 1.6 using the usual topology as its own subbase, has the property that, for all $A \in \mathscr{A}_{\mathscr{U}}$, $\mathscr{U}_A$ is not separated.) A similar word of caution holds for para-proximity spaces.

In order to develop the theory of para-uniform and para-proximal extensions, we need to introduce the notions of Cauchy filter and round filter.

DEFINITION 3.1. Let $(X, \mathscr{U})$ be a para-uniform space and let $\mathscr{F}$ be a filter on X. $\mathscr{F}$ is Cauchy (or $\mathscr{U}$-Cauchy) if for each $U \in \mathscr{U}$, there is an $x \in X$ such that $U[x] \in \mathscr{F}$.

The easy proof of the following characterizations of Cauchy filter is omitted.

PROPOSITION 3.2. *Let $(X, \mathscr{U})$ be a para-uniform space, let $\mathscr{A}_{\mathscr{U}} = \{$dom*

$U:U \in \mathscr{U}\}$, and let $\mathscr{U}_A$ be the uniformity induced on A by $\mathscr{U}$ for each $A \in \mathscr{A}_{\mathscr{U}}$. Then the following are equivalent for a filter $\mathscr{F}$ on X.

(a) *$\mathscr{F}$ is $\mathscr{U}$-Cauchy.*

(b) *For each $U \in \mathscr{U}$ there is $F \in \mathscr{F}$ with $F \times F \subset U$.*

(c) *$\mathscr{F}|_A$ is $\mathscr{U}_A$-Cauchy for each $A \in \mathscr{A}_{\mathscr{U}}$.*

Note that if $\mathscr{F}$ is $\mathscr{U}$-Cauchy, then dom $U \in \mathscr{F}$, for every $U \in \mathscr{U}$. Accordingly, there are examples of $\tau(\mathscr{U})$-convergent filters on some para-uniform spaces $(X, \mathscr{U})$ which are not $\mathscr{U}$-Cauchy. In fact, the neighborhood filter of a point $x \in X$ is $\mathscr{U}$-Cauchy if and only if $x \in$ dom U for every $U \in \mathscr{U}$. Thus, every neighborhood filter is $\mathscr{U}$-Cauchy if and only if $\mathscr{U}$ a uniformity on X.

As in the uniform case, a $\mathscr{U}$-Cauchy filter converges to each of its adherence points, as can be verified easily. Also every $\mathscr{U}$-Cauchy filter $\mathscr{F}$ on X contains a smallest $\mathscr{U}$-Cauchy filter $\mathscr{F}_m = \{U[F]:U \in \mathscr{U},\ F \in \mathscr{F}\}$ called the minimal $\mathscr{U}$-Cauchy filter contained in $\mathscr{F}$.

DEFINITION. 3.3. Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ.

(a) A filter $\mathscr{F}$ on X is round (or δ-round) if $\mathscr{A} \subset \mathscr{F}$ and $F_1 \in \mathscr{F}$ implies there is an $F_2 \in \mathscr{F}$ with $F_2 \bar{\delta} X - F_1$.

(b) Let $\mathscr{F}$ be a filter on X such that $\mathscr{A} \subset \mathscr{F}$. The δ-round hull of $\mathscr{F}$ is defined to be $\mathscr{F}_r = \{H \subset X:F \bar{\delta} X - H$ for some $F \in \mathscr{F}\}$.

The proof of the next proposition requires only a slight modification of the proof of the corresponding results in the proximity case (see, for instance **[20]**) and is omitted.

PROPOSITION 3.4. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space and let δ be the associated para-proximity on X.*

(a) *If $\mathscr{F}$ is a filter on X and $\mathscr{A} \subset \mathscr{F}$, then $\mathscr{F}_r$ is a δ-round filter and $\mathscr{F}_r \subset \mathscr{F}$.*

(b) *Each δ-round filter is contained in a maximal δ-round filter.*

(c) *If $\mathscr{F}$ is a maximal δ-round filter and B_1 and B_2 are subsets of X such that $B_1 \bar{\delta} X - B_2$ and B_1 meets $\mathscr{F}$, then $B_2 \in \mathscr{F}$.*

(d) *A δ-round filter $\mathscr{F}$ is a maximal δ-round filter if and only if $B_1, B_2 \subset X$ with $B_1 \bar{\delta} X - B_2$ implies $X - B_1 \in \mathscr{F}$ or $B_2 \in \mathscr{F}$.*

(e) *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two distinct maximal δ-round filters on X, then there are $F_1 \in \mathscr{F}_1$ and $F_2 \in \mathscr{F}_2$ with $F_1 \cap F_2 = \emptyset$.*

Note that if $(X, \mathscr{A}, \mathscr{D})$ is a para-proximity space, then $\mathscr{A}$ has f.i.p., and, hence, there are δ-round filters on X.

When we say that a filter on a para-uniform space is round, it is understood to be round with respect to the para-proximity induced by the para-uniformity.

PROPOSITION 3.5. (a) *A minimal Cauchy filter on a para-uniform space is a maximal round filter.* (b) *A maximal round filter on a totally bounded para-uniform space is a minimal Cauchy filter.*

PROOF. Let $(X, \mathscr{U})$ be a para-uniform space, $\mathscr{A} = \mathscr{A}_{\mathscr{U}} = \{\text{dom } U : U \in \mathscr{U}\}$, $\mathscr{D} = \mathscr{D}_{\mathscr{U}}$, and $\delta = \delta_{\mathscr{U}}$.

(a). Let $\mathscr{F}$ be a minimal $\mathscr{U}$-Cauchy filter. Then $\mathscr{F}_r$ is a δ-round filter contained in $\mathscr{F}$. Let $U \in \mathscr{U}$. If $V \in \mathscr{U}$ with $V^0 = U^0$, $V = V^{-1}$, and $V \circ V \circ V \subset U$, and if $F \in \mathscr{F}$ with $F \times F \subset V$, then $V[F] \in \mathscr{F}_r$ and $V[F] \times V[F] \subset U$. So $\mathscr{F}_r$ is $\mathscr{U}$-Cauchy, and $\mathscr{F} = \mathscr{F}_r$ is δ-round. To see that $\mathscr{F}$ is maximal δ-round, let $B_1, B_2 \subset X$ with $B_1 \bar{\delta}\, X - B_2$. Then there is $U \in \mathscr{U}$ with $B_1 \subset \text{dom } U = A$ and $B_1 \overline{\delta_A} (X - B_2) \cap A$; so $B_1 \overline{\delta_A} (A - B_2)$. Let $\mathscr{U}_A^T$ be the totally bounded uniformity on A induced by δ_A. Then $H = [(B_2 \cap A) \times (B_2 \cap A)] \cup [(A - B_1) \times (A - B_1)] \in \mathscr{U}_A^T \subset \mathscr{U}_A \subset \mathscr{U}$. Since $\mathscr{F}$ is $\mathscr{U}$-Cauchy, there is an $F \in \mathscr{F}$ with $F \times F \subset H$. So either $F \subset B_2 \cap A$ or $F \subset A - B_1$, whence either $B_2 \in \mathscr{F}$ or $X - B_1 \in \mathscr{F}$. By PROPOSITION 3.4(d), $\mathscr{F}$ is a maximal δ-round filter.

(b). Let $\mathscr{F}$ be a maximal δ-round filter. Since $\mathscr{U}$ is totally bounded, a basis for $\mathscr{U}$ is $\mathscr{B} = \bigcup \{\mathscr{B}_A : A \in \mathscr{A}\}$ where, for $A \in \mathscr{A}$, $\mathscr{B}_A = \{\bigcup_{j=1}^m H_j \times H_j : H_j \subset A,\ K_j \overline{\delta_A}\, A - H_j\ (j = 1, 2, \ldots, m),\ A = \bigcup_{j=1}^m K_j\}$. Let $B \in \mathscr{B}$. Then there is an $A \in \mathscr{A}$ such that $B = \bigcup_{j=1}^m H_j \times H_j$ where $H_j \subset A$, $K_j \overline{\delta_A}\, A - H_j\ (j = 1, 2, \ldots, m)$ and $A = \bigcup_{j=1}^m K_j$. Since $\mathscr{F}$ is a maximal δ-round filter, for each $j = 1, 2, \ldots, m$ either $X - K_j \in \mathscr{F}$ or $H_j \in \mathscr{F}$. But since $A \in \mathscr{F}$, there is some $j \in \{1, 2, \ldots, m\}$ for which $H_j \in \mathscr{F}$. Then $H_j \times H_j \subset B$. So $\mathscr{F}$ is $\mathscr{U}$-Cauchy. Now $\mathscr{F}_m$ is a maximal δ-round filter by (a), and $\mathscr{F}_m \subset \mathscr{F}$. Thus, $\mathscr{F} = \mathscr{F}_m$ is a minimal $\mathscr{U}$-Cauchy filter.

DEFINITION. 3.6. (a) A para-uniform space $(X, \mathscr{U})$ is complete if every $\mathscr{U}$-Cauchy filter on X is $\tau(\mathscr{U})$-convergent.

(b) A para-uniform space $(Y, \mathscr{V})$ is a para-uniform extension of a para-uniform space $(X, \mathscr{U})$ if X is a $\tau(\mathscr{V})$-dense subset of Y and $\mathscr{U} = \mathscr{V}|_X$.

(c) A para-uniform extension $(Y, \mathscr{V})$ of $(X, \mathscr{U})$ is said to have relatively uniform outgrowth (r.u.o.) if $Y - X \subset \text{dom V}$, for every $V \in \mathscr{V}$.

(d) A para-uniform completion of a para-uniform space is a complete para-uniform extension.

(e) A para-proximity space is full if each round filter has non-void adherence with respect to the para-proximity topology.

(f) A para-proximity space $(Y, \mathscr{A}', \mathscr{D}')$ is a para-proximal extension of a para-proximity space $(X, \mathscr{A}, \mathscr{D})$ if X is a $\tau(\delta')$-dense subset of Y, $\mathscr{A} = \mathscr{A}'|_X$, and $\mathscr{D} = \mathscr{D}'|_X$.

(g) A para-proximal extension $(Y, \mathscr{A}', \mathscr{D}')$ of $(X, \mathscr{A}, \mathscr{D})$ is said to have relatively proximal outgrowth (r.p.o.) if $Y - X \subset A$ for each $A \in \mathscr{A}'$.

Note that if $(Y, \mathscr{V})$ is a para-uniform extension of $(X, \mathscr{U})$, then $(Y,$

$\mathscr{A}_{\mathscr{V}}, \mathscr{D}_{\mathscr{V}})$ is a para-proximal extension of $(X, \mathscr{A}_{\mathscr{U}}, \mathscr{D}_{\mathscr{U}})$. Also note that if (Y, σ) is a topological extension of (X, τ) and $\mathscr{V}$ is compatible with σ, then $\mathscr{V}|_X$ is compatible with τ, and $(Y, \mathscr{V})$ is a para-uniform extension of $(X, \mathscr{V}|_X)$.

The next two propositions relate para-uniform extensions to simple and strict topological extensions. If $U \subset Y \times Y$ and $S \subset Y$, then we use $U(S)$ to denote $U \cap (S \times S)$.

PROPOSITION 3.7. *Let (Y, σ) be a topological extension of (X, τ) and let $\mathscr{V}$ be a compatible para-uniformity on Y. Then $\mathscr{B}^+ = \{U(S): X \subset S \subset Y, U \in \mathscr{V}\}$ is a basis for a para-uniformity $\mathscr{V}^+$ on Y, $\mathscr{V} \subset \mathscr{V}^+$, $\tau(\mathscr{V}^+) = \sigma^+$, and $(Y, \mathscr{V}^+)$ is a para-uniform extension of $(X, \mathscr{V}|_X)$.*

PROOF. It is straightforward to verify that $\mathscr{B}^+$ is a para-uniform basis on Y, and clearly $\mathscr{V} \subset \mathscr{V}^+$. To see that $\sigma^+ \subset \tau(\mathscr{V}^+)$, it is enough to observe that if $y \in Y$ and $G \in O_\sigma^{y,X}$, then there is $V \in \mathscr{V}$ for which $y \in V[X]$ and $\varnothing \neq X \cap V[y] \subset G$ and, hence, $V(\{y\} \cup X)[y] \subset G \cup \{y\}$. Now let $y \in Y$ and let $U \in \mathscr{V}$ be such that U is open in the product topology on $Y \times Y$ and $y \in U[y]$. Then $(U[y] \cap X) \cup \{y\} \in \sigma^+$ and $(U[y] \cap X) \cup \{y\} \subset U(S)[y]$, for all S with $X \cup \{y\} \subset S \subset Y$. Since the open entourages in $\mathscr{V}$ form a basis for $\mathscr{V}$, this shows that $\tau(\mathscr{V}^+) \subset \sigma^+$. It is then clear that X is $\tau(\mathscr{V}^+)$-dense in Y and that $\mathscr{V}^+|_X = \mathscr{V}|_X$.

Since $\mathscr{V} \subset \mathscr{V}^+$, $\mathrm{id}_Y : (Y, \mathscr{V}^+) \to (Y, \mathscr{V})$ is para-uniformly continuous.

PROPOSITION 3.8. *Let (Y, σ) be a topological extension of (X, τ) and let $\mathscr{V}$ be a compatible para-uniformity on Y such that $Y - X \subset \operatorname{dom} V$ for each $V \in \mathscr{V}$. If $V \in \mathscr{V}$, let $V^\sharp = V(X) \cup \{(x, y) \in Y \times Y: G \times G \subset V$, for some $G \in O_\sigma^{x,X} \cap O_\sigma^{y,X}\}$. Then $\mathscr{B}^\sharp = \{V^\sharp: V \in \mathscr{V}\}$ is a basis for $\mathscr{V}$. Moreover, $\sigma = \sigma^\sharp$. Thus, any para-uniform extension of a para-uniform space with r.u.o. yields a strict topological extension.*

PROOF. We must first show that $\mathscr{B}^\sharp$ is a para-uniform basis on Y. To verify (U2), let $U \in \mathscr{V}$ and let $V \in \mathscr{V}$ with $V^0 = U^0$, $V^{-1} = V$, $V \circ V \subset U$, and V open in the product topology on $Y \times Y$. Now if $y \in Y - X$, then $G = V[y] \cap X \in O_\sigma^{y,X}$ and $G \times G \subset U$. Thus, $\Delta(Y - X) \subset U^\sharp$. Also $(U^\sharp)^0(X) = U^0(X)$. Therefore, $(U^\sharp)^0 \subset U^\sharp$. To see that (B3) is satisfied, note that if $U, V \in \mathscr{V}$, then $U^\sharp \cap V^\sharp = (U \cap V)^\sharp$ and $U^\sharp \cap (U^\sharp)^{-1} = (U \cap U^{-1})^\sharp$. To verify (U4), let $U, V \in \mathscr{V}$ and let $W \in \mathscr{V}$ with $W^0 = (U \cap V)^0$, $W^{-1} = W$, W open in the product topology on $Y \times Y$, and $W \circ W \circ W \circ W \subset U \cap V$. One may easily verify that $(W^\sharp)^0 = (U^\sharp \cap V^\sharp)^0$ and $W^\sharp \subset U^\sharp \cap V^\sharp$. Finally, (U6) is satisfied for $\mathscr{B}^\sharp$ since (U6) is satisfied for $\mathscr{V}|_X$ and $Y - X \subset \operatorname{dom} V^\sharp$, for all $V \in \mathscr{V}$. So $\mathscr{B}^\sharp$ is a para-uniform basis on Y.

If $U \in \mathscr{V}$ and $V \in \mathscr{V}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \circ V \subset U$, then $U^0 = V^0 = (U^\#)^0 = (V^\#)^0$, $V \subset U^\#$, and $V^\# \subset U$. It follows that $\mathscr{V} = \mathscr{U}(\mathscr{B}^\#)$. So it remains to show that $\tau(\mathscr{U}(\mathscr{B}^\#)) = \sigma^\#$.

Let $y \in G \in \sigma^\#$ and assume, without loss of generality, that $G = \{x \in Y: G \cap X \in O_\sigma^{x,X}\}$. Since $\sigma^\# \subset \sigma$, there is $U \in \mathscr{V}$ such that $y \in U[y] \subset G$. Let $V \in \mathscr{V}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \subset U$. We claim that $V^\#[y] \subset G$. To see this, let $x \in V^\#[y]$. First observe that if $x \in V(X)[y]$, then $x \in G$. So assume $(y, x) \notin V(X)$. Then there is $H \in O_\sigma^{x,X} \cap O_\sigma^{y,X}$ such that $H \times H \subset V$. Now $H \cap V[y] \neq \emptyset$. Letting $h \in H \cap V[y]$, we have $H \subset V[h] \subset (V \circ V)[y] \subset U[y] \subset G$, whence $x \in G$. So indeed $V^\#[y] \subset G$. Thus, $\sigma^\# \subset \tau(\mathscr{U}(\mathscr{B}^\#))$.

Now let $U \in \mathscr{V}$ and let $y \in Y$ such that $y \in U^\#[y]$. Then we are able to find $G \in O_\sigma^{y,X}$ such that $G \times G \subset U$. (In case $y \in X$ take $G = V(X)[y]$, where $V \in \mathscr{V}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \subset U$.) Then $y \in G \cup \{x \in Y: G \in O_\sigma^{x,X}\} \subset U^\#[y]$. So $\tau(\mathscr{U}(\mathscr{B}^\#)) \subset \sigma^\#$.

In the remainder of this section we shall construct and investigate canonical para-uniform completions and canonical full para-proximal extensions. Recall that a filter on a topological space (X, τ) is τ-free (or simply free) when it has void adherence. We will call a filter on a para-uniform space or para-proximity space free if it is free with respect to the induced topology.

Definition 3.9. Let $(X, \mathscr{U})$ be a para-uniform space.

(a) Define $\mathscr{U}X$ to be $X \cup \{\mathscr{F}: \mathscr{F}$ is a free minimal $\mathscr{U}$-Cauchy filter on $X\}$.

(b) For $U \in \mathscr{U}$, define $U_* = U \cup \{(\mathscr{F}, x), (x, \mathscr{F}): \mathscr{F} \in \mathscr{U}X - X$, $x \in X$, and for some $F \in \mathscr{F} \cap \mathscr{U}(x)$, $F \times F \subset U\} \cup \{(\mathscr{F}, \mathscr{G}): \mathscr{F}, \mathscr{G} \in \mathscr{U}X - X$ and for some $F \in \mathscr{F} \cap \mathscr{G}$, $F \times F \subset U\}$.

Theorem 3.10. *Let $(X, \mathscr{U})$ be a para-uniform space. Then $\mathscr{B}_* = \{U_*: U \in \mathscr{U}\}$ is a basis for a para-uniformity $\mathscr{U}_*$ on $\mathscr{U}X$, and $(\mathscr{U}X, \mathscr{U}_*)$ is a para-uniform completion of $(X, \mathscr{U})$ with r. u. o. Thus, $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is a strict extension of $(X, \tau(\mathscr{U}))$.*

Proof. The proof that $\mathscr{B}_*$ is a para-uniform basis on X is similar to the proof for $\mathscr{B}^\#$ in Proposition 3.8 and is left to the reader. It is then easily verified that X is $\tau(\mathscr{U}_*)$-dense in $\mathscr{U}X$, $\mathscr{U}_*|_X = \mathscr{U}$, and $\mathscr{U}X - X \subset \operatorname{dom} V$ for all $V \in \mathscr{U}_*$. So it remains to show that $(\mathscr{U}X, \mathscr{U}_*)$ is complete.

Let $\mathscr{F}$ be a free minimal $\mathscr{U}_*$-Cauchy filter on $\mathscr{U}X$. Then $\mathscr{F}|_X$ is a $\tau(\mathscr{U})$-free minimal $\mathscr{U}$-Cauchy filter on X, so that there is $\mathscr{G} \in \mathscr{U}X - X$ for which $\mathscr{G} \subset \mathscr{F}|_X$. Then $F \cap V_*[\mathscr{G}] \neq \emptyset$, for every $F \in \mathscr{F}$ and every $V \in \mathscr{U}$, whence $\mathscr{G}$ is a $\tau(\mathscr{U}_*)$-adherence point of $\mathscr{F}$. This contradicts the assumption that $\mathscr{F}$ is free. So there are no free $\mathscr{U}_*$-Cauchy filters on $\mathscr{U}X$.

Note that if $\mathscr{U}$ is a uniformity on X, then $\mathscr{U}_*$ is a uniformity on $\mathscr{U}X$.

THEOREM 3.11. *Let $(X, \mathscr{U})$ be a para-uniform space. Then $\mathscr{U}^* = (\mathscr{U}_*)^+$ is a para-uniformity on $\mathscr{U}X$ and $(\mathscr{U}X, \mathscr{U}^*)$ is a para-uniform completion of $(X, \mathscr{U})$. Moreover, $\tau(\mathscr{U}_*)^+ = \tau(\mathscr{U}^*)$ and $\tau(\mathscr{U}^*)^\# = \tau(\mathscr{U}_*)$.*

PROOF. Everything is clear from Proposition 3.7 and Theorem 3.10 except for the completeness of $(\mathscr{U}X, \mathscr{U}^*)$. Suppose $\mathscr{G}$ is a free $\mathscr{U}^*$-Cauchy filter on $\mathscr{U}X$. Since $X \times X \in \mathscr{U}^*$, $\mathscr{G}$ contains a free minimal $\mathscr{U}$-Cauchy filter $\mathscr{F}$. But $\{F \cup \{\mathscr{F}\}: F \in \mathscr{F}\}$ is a $\tau(\mathscr{U}^*)$-neighborhood base at $\mathscr{F}$. Thus, $\mathscr{G}$ $\tau(\mathscr{U}^*)$-converges to $\mathscr{F} \in \mathscr{U}X$, which contradicts the assumption that $\mathscr{G}$ is free. So $(\mathscr{U}X, \mathscr{U}^*)$ is complete.

By Theorem, 3.10 it follows that every para-uniform space has a para-uniform completion which coincides with the unique uniform completion for a uniform space. Moreover, according to Theorem 3.11, any non-complete para-uniform space has more than one para-uniform completion. (Even a non-complete uniform space has more than one para-uniform completion!) Thus, it is natural to ask in what sense each of these canonical completions is unique. This question is answered by the next several results.

LEMMA 3.12. *Let $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a para-uniformly continuous mapping of para-uniform spaces with $f(X)$ $\tau(\mathscr{V})$-dense in Y and $(Y, \mathscr{V})$ complete. Then there is a unique para-uniformly continuous mapping g: $(\mathscr{U}X, \mathscr{U}^*) \to (Y, \mathscr{V})$ such that $f(x) = g(x)$, for all $x \in X$.*

PROOF. For each $x \in X$ define $g(x) = f(x)$. We must define $g(x)$ for $x \in \mathscr{U}X - X$. In this case, $x = \mathscr{F}$, a free minimal $\mathscr{U}$-Cauchy filter on X. Now $\{f(F): F \in \mathscr{F}\}$ is a filterbase on Y and is $\mathscr{V}$-Cauchy since $f(X)$ is dense in Y. Let $\mathscr{G}(x) = \{G \subset Y: f(F) \subset G$ for some $F \in \mathscr{F}\}$. Then $\mathscr{G}(x)$ is a $\mathscr{V}$-Cauchy filter on Y and converges to a unique point $g(x) \in Y$. Thus, g: $\mathscr{U}X \to Y$ is defined.

To show that g is para-uniformly continuous, let $V \in \mathscr{V}$ and let $W \in \mathscr{V}$ with $W^0 = V^0$, $W^{-1} = W$, and $W \circ W \circ W \subset V$. Then $f^{-1}(W) \in \mathscr{U}$ since $f(X)$ is dense in Y. Set $U = f^{-1}(W)$ and $H = X \cup \operatorname{dom} g^{-1}(W)$. Then $X \subset H \subset \mathscr{U}X$ and $U_* \in \mathscr{U}_*$. So $U_*(H) \in \mathscr{U}^*$. We claim that $U_*(H) \subset g^{-1}(V)$ and that $U_*(H)^0 = g^{-1}(V)^0$ (whence it follows that $g^{-1}(V) \in \mathscr{U}^*$). Let $(x, y) \in U_*(H)$. We shall verify that $(x, y) \in g^{-1}(V)$ in the case where $x = \mathscr{F}_1 \in \mathscr{U}X - X$ and $y = \mathscr{F}_2 \in \mathscr{U}X - X$. Let $G \in \mathscr{F}_1 \cap \mathscr{F}_2$ with $G \times G \subset U$. Now $\mathscr{G}(x)$ converges to $g(x)$, $\mathscr{G}(y)$ converges to $g(y)$, and $f(G) \in \mathscr{G}(x) \cap \mathscr{G}(y)$. So $g(x)$, $g(y) \in \mathrm{cl}_Y f(G)$. Also x, $y \in H - X$. So x, $y \in \operatorname{dom} g^{-1}(W)$, whence $g(x)$, $g(y) \in \operatorname{dom} W$. Thus, $W[g(x)]$ and $W[g(y)]$ are $\tau(\mathscr{V})$-neighborhoods of $g(x)$ and $g(y)$, respectively. Therefore, we may select $p \in W[g(x)] \cap f(G)$ and $q \in W[g(y)] \cap f(G)$. So $(p, q) \in f(G) \times f(G) \subset W$, $(g(x), p) \in W$, and $(q, g(y)) \in W$. It follows that $(g(x), g(y)) \in W \circ W \circ W \subset V$; i.e., $(x, y) \in g^{-1}(V)$. The proof that $U_*(H)^0$

$= g^{-1}(V)^0$ involves similar notions and is left to the reader.

That g is the unique para-uniformly continuous extension of f follows from the fact that g is continuous and X is dense in $\mathcal{U}X$.

THEOREM 3.13. *Let* $(X, \mathcal{U})$ *be a para-uniform space. Then* $(\mathcal{U}X, \mathcal{U}^*)$ *is the* (*up to para-uniform isomorphism*) *unique para-uniform completion of* $(X, \mathcal{U})$ *satisfying property* (C*): *If* $(Y, \mathcal{V})$ *is any para-uniform completion of* $(X, \mathcal{U})$, *then there is a unique para-uniformly continuous mapping* p^*: $(\mathcal{U}X, \mathcal{U}^*) \to (Y, \mathcal{V}^+)$ *such that* $p^*(x) = x$ *for all* $x \in X$; *i.e., the diagram*

$$\begin{array}{ccc} (\mathcal{U}X, \mathcal{U}^*) & \xrightarrow{p^*} & (Y, \mathcal{V}^+) \\ {\scriptstyle \mathrm{id}_X}\uparrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ (X, \mathcal{U}) & \xrightarrow{\mathrm{id}_X} & (Y, \mathcal{V}) \end{array}$$

commutes.

PROOF. To show that $(\mathcal{U}X, \mathcal{U}^*)$ satisfies (C^*) let $(Y, \mathcal{V})$ be an arbitrary para-uniform completion of $(X, \mathcal{U})$. Then $(Y, \mathcal{V}^+)$ is a para-uniform extension of $(X, \mathcal{U})$, and $(Y, \mathcal{V}^+)$ is complete since $\mathcal{V} \subset \mathcal{V}^+$. So id_X: $(X, \mathcal{U}) \to (Y, \mathcal{V}^+)$ satisfies the hypothesis of Lemma 3.12, whence there is a unique para-uniformly continuous mapping p^*: $(\mathcal{U}X, \mathcal{U}^*) \to (Y, \mathcal{V}^+)$ such that $p^*(x) = x$, for all $x \in X$.

Now suppose that $(Z, \mathcal{W})$ is a para-uniform completion of $(X, \mathcal{U})$ satisfying (C^*). Then we can find para-uniformly continuous mappings p^*: $(\mathcal{U}X, \mathcal{U}^*) \to (Z, \mathcal{W})$ and q^*: $(Z, \mathcal{W}) \to (\mathcal{U}X, \mathcal{U}^*)$ such that $p^*(x) = q^*(x) = x$, for all $x \in X$. So p^* is a para-uniform isomorphism.

LEMMA 3.14. *Let* f: $(Y, \mathcal{V}) \to (X, \mathcal{U})$ *be a para-uniformly continuous mapping of para-uniform spaces such that, for each* $U \in \mathcal{U}$, $f(Y) \cap \mathrm{dom}\, U \neq \emptyset$ *and such that if* $\mathcal{F}$ *is a free* $\mathcal{V}$-*Cauchy filter on* Y *then the filter induced by* $\mathcal{F}$ *under* f *is a free filter on* X. *If* $(Z, \mathcal{W})$ *is any para-uniform extension of* $(Y, \mathcal{V})$ *with* r.u.o., *then there is a unique para-uniformly continuous mapping* g: $(Z, \mathcal{W}) \to (\mathcal{U}X, \mathcal{U}_*)$ *such that* $g(y) = f(y)$, *for all* $y \in Y$.

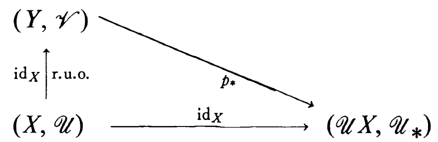
PROOF. For $y \in Y$ define $g(y) = f(y)$. Let $z \in Z - Y$. Then $O^{z,Z}_{\tau(\mathcal{W})}$ is a filterbase on Z, and since $(Z, \mathcal{W})$ is an r.u.o. para-uniform extension of $(Y, \mathcal{V})$, $O^{z,Z}_{\tau(\mathcal{W})}$ is a $\mathcal{W}$-Cauchy filterbase. So $O^{z,Y}_{\tau(\mathcal{W})}$ is a free $\mathcal{V}$-Cauchy filterbase on Y. Now the filter $\mathcal{F}(z)$ generated by $\{f(G)\colon G \in O^{z,Y}_{\tau(\mathcal{W})}\}$ is a $\tau(\mathcal{U})$-free filter on X. If $U \in \mathcal{U}$, then $f(Y) \cap \mathrm{dom}\, U \neq \emptyset$ so that $f^{-1}(U) \neq \emptyset$. Since f is para-uniformly continuous, $f^{-1}(U) \in \mathcal{V}$. Thus, there is $G \in O^{z,Y}_{\tau(\mathcal{W})}$ with $G \times G \subset f^{-1}(U)$. Then $f(G) \in \mathcal{F}(z)$ and $f(G) \times f(G) \subset U$. So $\mathcal{F}(z)$ is $\mathcal{U}$-Cauchy. Let $g(z) = \mathcal{G}(z)$, the unique minimal $\mathcal{U}$-Cauchy filter contained in $\mathcal{F}(z)$. $\mathcal{G}(z)$ is $\tau(\mathcal{U})$-free since $\mathcal{F}(z)$ is $\tau(\mathcal{U})$-free. So $g(z) \in \mathcal{U}X - X$. Thus, $g\colon Z \to \mathcal{U}X$ is defined.

To see that g is para-uniformly continuous, let $U \in \mathcal{U}$ and suppose that

$g^{-1}(U_*) \neq \phi$. We must show that $g^{-1}(U_*) \in \mathscr{W}$. To this end, let $V \in \mathscr{U}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \circ V \subset U$. Then, since $f(Y) \cap \text{dom } V \neq \varnothing$, $f^{-1}(V) \in \mathscr{V}$. So there is $W \in \mathscr{W}$ such that $f^{-1}(V) = W(Y)$. Now $W^\# = f^{-1}(V) \cup \{(x, y) \in Z \times Z$: for some $G \in O^{x,Y}_{\tau(\mathscr{W})} \cap O^{y,Y}_{\tau(\mathscr{W})}$, $G \times G \subset f^{-1}(V)\} \in \mathscr{W}$. Also, it is straightforward to verify that $W^\# \subset g^{-1}(U_*)$ and $(W^\#)^0 = g^{-1}(U_*)^0$. So indeed $g^{-1}(U_*) \in \mathscr{W}$.

That g is the unique extension of f follows from the continuity of g and the fact that Y is dense in Z.

THEOREM 3.15. *Let $(X, \mathscr{U})$ be a para-uniform space. Then $(\mathscr{U}X, \mathscr{U}_*)$ is the (up to para-uniform isomorphism) unique para-uniform completion of $(X, \mathscr{U})$ with r.u.o. satisfying property (C_*): If $(Y, \mathscr{V})$ is a para-uniform completion of $(X, \mathscr{U})$ with r.u.o., then there is a unique para-uniformly continuous mapping $p_*: (Y, \mathscr{V}) \to (\mathscr{U}X, \mathscr{U}_*)$ such that $p_*(x) = x$, for all $x \in X$; i.e., the diagram*



commutes.

PROOF. To see that $(\mathscr{U}X, \mathscr{U}_*)$ satisfies (C_*), let $(Y, \mathscr{V})$ be an arbitrary para-uniform completion of $(X, \mathscr{U})$ with r.u.o. Then $\text{id}_X: (X, \mathscr{U}) \to (X, \mathscr{U})$ and $(Y, \mathscr{V})$ satisfy the hypothesis of Lemma 3.14. So there is a unique para-uniformly continuous mapping $p_*: (Y, \mathscr{V}) \to (\mathscr{U}X, \mathscr{U}_*)$ such that $p_*(x) = x$, for all $x \in X$.

The proof of uniqueness is essentially identical to the proof of uniqueness in Theorem 3.13.

We now turn our attention to finding full para-proximal extensions of para-proximity spaces.

DEFINITION 3.16. Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ.

(a) Let δX denote $X \cup \{\mathscr{F}: \mathscr{F}$ is a free maximal δ-round filter on $X\}$.

(b) For $B \subset X$, define $O(B) = B \cup \{\mathscr{F} \in \delta X - X: B \in \mathscr{F}\}$.

Note that $\delta X = O(X)$ and that if $A \in \mathscr{A}$, then $O(A) = A \cup (\delta X - X)$.

THEOREM 3.17. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space with associated para-proximity δ. Let $\mathscr{A}_* = \{O(A): A \in \mathscr{A}\}$ and, for $A \in \mathscr{A}$, define $\delta_{O(A)} \subset \mathscr{P}(O(A)) \times \mathscr{P}(O(A))$ by (for $T_1, T_2 \subset O(A)$) $T_1 \overline{\delta_{O(A)}} T_2$ if and only if there are $B_1, B_2 \subset A$ with $B_1 \overline{\delta_A} B_2$ and $T_i \subset O(B_i)$ $(i = 1, 2)$. Then $\delta_{O(A)}$ is a proximity on $O(A)$. Moreover, if we set $\mathscr{D}_* = \{\delta_{O(A)}: A \in \mathscr{A}\}$, then $(\delta X, \mathscr{A}_*, \mathscr{D}_*)$ is a full para-proximal extension of $(X, \mathscr{A}, \mathscr{D})$ with r.p.o. We shall let δ_* denote the associatrd para-proximity on δX.*

PROOF. We shall verify the strong axiom of proximities for $\partial_{O(A)}$. Suppose that $T_1, T_2 \subset O(A)$ and $T_1 \overline{\delta_{O(A)}} T_2$. Then there are $B_1, B_2 \subset A$ such that $B_1 \overline{\delta_A} B_2$ and $T_i \subset O(B_i)$ $(i = 1, 2)$. Since δ_A is a proximity on A, there is $C \subset A$ such that $B_1 \overline{\delta_A} A - C$ and $C \overline{\delta_A} B_2$. Set $T = O(C)$. Then it is clear that $T \overline{\delta_{O(A)}} T_2$. Since $B_1 \overline{\delta_A} A - C$, there is $D \subset A$ such that $B_1 \overline{\delta_A} A - D$ and $D \overline{\delta_A} A - C$. Then $O(A) - O(A - D) \subset O(C)$ and hence $O(A) - O(C) \subset O(A) - [O(A) - O(A - D)] = O(A - D)$. So $O(A) - T \subset O(A - D)$, $T_1 \subset O(B_1)$, and $B_1 \overline{\delta_A} A - D$. Thus, $T_1 \overline{\delta_{O(A)}} O(A) - T$.

Verification of the other proximity axioms for $\delta_{O(A)}$ is routine, as is the proof that $(\delta X, \mathscr{A}_*, \mathscr{D}_*)$ is a para-proximal extension of $(X, \mathscr{A}, \mathscr{D})$. Noting that $\{O(B): B \in \tau(\delta)\}$ is an open basis for the topology $\tau(\delta_*)$, it is clear that X is a $\tau(\delta_*)$-dense subset of δX.

It remains to show that $(\delta X, \mathscr{A}_*, \mathscr{D}_*)$ is full. Suppose that $\mathscr{F}$ is a free δ_*-round filter on δX. Then $\mathscr{F}|_X$ is a δ-round filter on X, and so is contained in a maximal δ-round filter $\mathscr{G} \in \delta X - X$. Let $F \in \mathscr{F}$. Then $F \cap O(A) \in \mathscr{F}$, for each $A \in \mathscr{A}$ and $F \cap A \in \mathscr{F}|_X \subset \mathscr{G}$. So $\mathscr{G} \in O(F \cap A)$. Thus, $\mathscr{G}\, \delta_{O(A)} F \cap A$, for all $A \in \mathscr{A}$, and so $\mathscr{G}\, \delta_* F$. Therefore, $\mathscr{G}$ is a $\tau(\delta_*)$ -adherence point of $\mathscr{F}$.

Note that for the para-proximity space $(X, \{X\}, \{\delta_X\})$, where δ_X is a separated proximity on X, $(\delta X, \tau(\delta_*))$ is the Smirnov compactification of (X, δ_X) **[20]**.

THEOREM 3.18. *Let $(X, \mathscr{U})$ be a totally bounded para-uniform space. Then* $(\mathscr{U}X, \mathscr{A}_{\mathscr{U}_*}, \mathscr{D}_{\mathscr{U}_*}) = (\delta_{\mathscr{U}}X, (\mathscr{A}_{\mathscr{U}})_*, (\mathscr{D}_{\mathscr{U}})_*)$.

PROOF. Clearly $\mathscr{U}X = \delta_{\mathscr{U}}X$ (see Proposition 3.5) and $\mathscr{A}_{\mathscr{U}_*} = (\mathscr{A}_{\mathscr{U}})_*$. $\mathscr{D}_{\mathscr{U}_*} = (\mathscr{D}_{\mathscr{U}})_*$ will follow once we have shown that, for $T_1, T_2 \subset O(A) \in \mathscr{A}_{\mathscr{U}_*} = (\mathscr{A}_{\mathscr{U}})_*$, $T_1 \overline{\delta_{O(A)}} T_2$ if and only if T_1 and T_2 are distant in the proximity induced on $O(A)$ by $(\mathscr{U}_*)_{O(A)}$. Suppose $T_1 \overline{\delta_{O(A)}} T_2$. Then there are $B_1, B_2 \subset A$ such that $B_1 \overline{\delta_A} B_2$ and $T_i \subset O(B_i)$ $(i = 1, 2)$. So there is $V \in \mathscr{U}$ with dom $V = A$, $V^{-1} = V$, and $V[B_1] \cap V[B_2] = \emptyset$. Then $V_* \in \mathscr{U}_*$, dom $V_* = O(A)$, and it is a straightforward exercise to verify that $V_*[T_1] \cap T_2 = \emptyset$. So T_1 and T_2 are distant in the proximity induced on $O(A)$ by $(\mathscr{U}_*)_{O(A)}$.

Conversely, suppose that T_1 and T_2 are distant in the proximity induced on $O(A)$ by $(\mathscr{U}_*)_{O(A)}$. Then there is $U_* \in \mathscr{U}_*$ with dom $U_* = O(A)$, $U_*^{-1} = U_*$, and $U_*[T_1] \cap U_*[T_2] = \emptyset$. Let $V_* \in \mathscr{U}_*$ such that dom $V_* = O(A)$, $V_*^{-1} = V_*$ and $V_* \circ V_* \circ V_* \subset U_*$. Set $B_i = V[V_*[T_i] \cap A]$ $(i = 1, 2)$. Then $B_1 \overline{\delta_A} B_2$ and $T_i \subset O(B_i)$ $(i = 1, 2)$. Therefore, $T_1 \overline{\delta_{o(A)}} T_2$.

COROLLARY 3.19. *Let $(X, \mathscr{A}, \mathscr{D})$ be a para-proximity space. Then $(\delta X, \mathscr{A}_*, \mathscr{D}_*)$ is the (up to para-proximity isomorphism) unique full para-proximal extension of $(X, \mathscr{A}, \mathscr{D})$ with r.p.o. satisfying property* (F$_*$): *If $(Y, \mathscr{A}', \mathscr{D}')$ is a full para-proximal extension of $(X, \mathscr{A}, \mathscr{D})$ with r.p.o., then there* *is a unique para-proximity mapping* $q_*: (Y, \mathscr{A}', \mathscr{D}') \to (\delta X, \mathscr{A}_*, \mathscr{D}_*)$ *such that* $q_*(x) = x$, *for all* $x \in X$.

PROOF. To see that $(\delta X, \mathscr{A}_*, \mathscr{D}_*)$ satisfies (F_*), suppose that $(Y, \mathscr{A}', \mathscr{D}')$ is an arbitrary full para-proximal extension of $(X, \mathscr{A}, \mathscr{D})$ with r.p.o. Let $\mathscr{V}$ be the unique totally bounded para-uniformity on Y inducing $(Y, \mathscr{A}', \mathscr{D}')$, and let $\mathscr{U}$ be the unique totally bounded para-uniformity on X inducing $(X, \mathscr{A}, \mathscr{D})$. Then $\mathscr{U} = \mathscr{V}|_X$. So $(Y, \mathscr{V})$ is a para-uniform completion of $(X, \mathscr{U})$ with r.u.o. and, thus, by Theorem 3.15 there is a para-uniformly continuous mapping $q_*: (Y, \mathscr{V}) \to (\mathscr{U}X, \mathscr{U}_*)$ such that $q_*(x) = x$, for all $x \in X$. By Proposition 2.13 $q_*: (Y, \mathscr{A}', \mathscr{D}') \to (\delta X, \mathscr{A}_*, \mathscr{D}_*)$ is a para-proximity mapping. The uniqueness, again, follows easily.

Another canonical full para-proximal extension can be constructed which corresponds to the para-uniform completion $\mathscr{U}^*$. The next few results are analogous to Theorem 3.17, Theorem 3.18, and Corollary 3.19, and their proofs are left to the reader.

THEOREM 3.20. *Let* $(X, \mathscr{A}, \mathscr{D})$ *be a para-proximity space with associated para-proximity* δ. *Let* $\mathscr{A}^* = \{O(A) \cap H: A \in \mathscr{A}, X \subset H \subset \delta X\}$ *and let* $\mathscr{D}^* = \{\delta_{O(A)}|_{O(A) \cap H}: A \in \mathscr{A}, X \subset H \subset \delta X\}$. *Then* $(\delta X, \mathscr{A}^*, \mathscr{D}^*)$ *is a full para-proximal extension of* $(X, \mathscr{A}, \mathscr{D})$. *We shall let* δ^* *denote the associated para-proximity on* δX.

THEOREM 3.21. *Let* $(X, \mathscr{U})$ *be a totally bounded para-uniform space. Then* $(\mathscr{U}X, \mathscr{A}_{\mathscr{U}^*}, \mathscr{D}_{\mathscr{U}^*}) = (\delta_{\mathscr{U}}X, (\mathscr{A}_{\mathscr{U}})^*, (\mathscr{D}_{\mathscr{U}})^*)$.

COROLLARY 3.22. *Let* $(X, \mathscr{A}, \mathscr{D})$ *be a para-proximity space. Then* $(\delta X, \mathscr{A}^*, \mathscr{D}^*)$ *is the (up to para-proximity isomorphism) unique full para-proximal extension of* $(X, \mathscr{A}, \mathscr{D})$ *satisfying property* (F*): *If* $(Y, \mathscr{A}', \mathscr{D}')$ *is a full para-proximal extension of* $(X, \mathscr{A}, \mathscr{D})$, *then there is a unique para-proximity mapping* $q^*: (\delta X, \mathscr{A}^*, \mathscr{D}^*) \to (Y, \mathscr{A}', \mathscr{D}')$ *such that* $q^*(x) = x$, *for all* $x \in X$.

4. H-closed extensions. In this section we shall investigate the relationship between H-closed extensions of a topological space and its compatible para-uniformities and para-proximities. We begin with another characterization of totally bounded para-uniformity which leads to a characterization of H-closed spaces.

PROPOSITION 4.1. *Let* $(X, \mathscr{U})$ *be a para-uniform space. Then* $\mathscr{U}$ *is totally bounded if and only if every* $\tau(\mathscr{U})$*-open ultrafilter on* X *is* $\mathscr{U}$*-Cauchy.*

PROOF. First suppose that $\mathscr{U}$ is totally bounded, and let $\mathscr{F}$ be an open ultrafilter on $(X, \tau(\mathscr{U}))$. Let $U \in \mathscr{U}$, and let $V \in \mathscr{U}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \subset U$. There is a finite set $F \subset X$ for which $\overline{V[F]} = X$. Now, for each $a \in F$, $V[a] \subset \operatorname{int} U[a]$, and so $\overline{\bigcup \{\operatorname{int} U[a]: a \in F\}} = X$. Since $\mathscr{F}$ is an open ultrafilter, $\mathscr{F}$ must contain int $U[a]$ for some $a \in F$, and for that a we have $U[a] \in \mathscr{F}$. Thus, $\mathscr{F}$ is $\mathscr{U}$-Cauchy.

Conversely, suppose that every open ultrafilter on $(X, \tau(\mathscr{U}))$ is $\mathscr{U}$-Cauchy, and let $U \in \mathscr{U}$. Let $V \in \mathscr{U}$ with $V^0 = U^0$, $V^{-1} = V$, and $V \circ V \subset U$. If, for every finite subset $F \subset X$, $\overline{V[F]} \neq X$, then $\{X - \overline{V[F]}: F \text{ is finite}\}$ forms a base for an open filter on $(X, \tau(\mathscr{U}))$ which is contained in an open ultrafilter $\mathscr{F}$. Now $\mathscr{F}$ is $\mathscr{U}$-Cauchy; so there is $a \in X$ for which $V[a] \in \mathscr{F}$, a contradiction since $X - \overline{V[a]} \in \mathscr{F}$. Thus, there is a finite subset $F \subset X$ such that $\overline{V[F]} = X$, and $\overline{U[F]} = X$ too. Therefore, $\mathscr{U}$ is totally bounded.

THEOREM 4.2. *The following are equivalent for a topological space X.*

(a) *X is H-closed.*
(b) *Every compatible para-uniformity on X is complete.*
(c) *Every compatible totally bounded para-uniformity on X is complete.*
(d) *There is a complete, compatible, totally bounded para-uniformity on X.*
(e) *Every compatible para-proximity on X is full.*
(f) *There is a full compatible para-proximity on X.*

PROOF. (a) $\Rightarrow$ (b). Since X is H-closed, every open ultrafilter on X is convergent. If $\mathscr{U}$ is a compatible para-uniformity on X and $\mathscr{F}$ is a $\mathscr{U}$-Cauchy filter on X, then $\mathscr{F}$ is an open filter and is contained in an open ultrafilter $\mathscr{G}$. Since $\mathscr{G}$ converges and $\mathscr{F}$ is Cauchy, $\mathscr{F}$ converges too. So $(X, \mathscr{U})$ is complete.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (d). The compatible para-uniformity on X provided by Theorem 1.6 is totally bounded. If (c) holds, then it is also complete.

(d) $\Rightarrow$ (a). Let $\mathscr{F}$ be an open ultrafilter on X and let $\mathscr{U}$ be any complete, compatible, totally bounded para-uniformity on X. By Proposition 4.1, $\mathscr{F}$ is $\mathscr{U}$-Cauchy and, hence, converges. So X is H-closed.

The equivalences (c) $\Leftrightarrow$ (e) and (d) $\Leftrightarrow$ (f) follow from results of §3.

The H-closed extensions of a given topological space may also be characterized in terms of para-uniformities and para-proximities.

THEOREM 4.3. *Let (Y, σ) be a topological extension of (X, τ). The following are equivalent.*

(a) *(Y, σ) is H-closed.*

(b) *(Y, σ) is the underlying topological space of a para-uniform completion of a compatible totally bounded para-uniformity on (X, τ).*

(c) *(Y, σ) is the underlying topological space of a full para-proximal extension of a compatible para-proximity on (X, τ).*

PROOF. (a) $\Rightarrow$ (b). By Theorem 4.2 there is a complete, compatible, totally bounded para-uniformity $\mathscr{V}$ on (Y, σ). So $\mathscr{V}|_X$ is a compatible totally bounded para-uniformity on (X, τ) and $(Y, \mathscr{V})$ is a completion of $(X, \mathscr{V}|_X)$.

(b) $\Rightarrow$ (a). Suppose there is a complete compatible para-uniformity $\mathscr{V}$ on (Y, σ) such that $\mathscr{V}|_X$ is totally bounded. Then $\mathscr{V}$ must be totally bounded, too. So, by Theorem 4.2, (Y, σ) is H-closed.

(b) $\Leftrightarrow$ (c). This follows from results of §3.

It is clear from [7] or [18] that no one-to-one correspondence between the H-closed extensions of (X, τ) and compatible totally bounded para-uniformities (or compatible para-proximities) on (X, τ) exists, in spite of Theorem 4.3. In fact, Theorem 4.3 can be established only since a given para-uniform space may have many completions. Also note that a para-uniform space $(X, \mathscr{U})$ may indeed be H-closed even when $\mathscr{U}$ is not totally bounded (see Example 4.6 below).

Definition 4.4. A para-uniformity $\mathscr{U}$ on a set X is called pre-H-closed if every $\tau(\mathscr{U})$-free, $\tau(\mathscr{U})$-open ultrafilter on X is $\mathscr{U}$-Cauchy.

Note that, according to Proposition 4.1, a totally bounded para-uniformity is pre-H-closed.

Theorem 4.5. *Let $(X, \mathscr{U})$ be a para-uniform space. The following are equivalent.*

(a) *$\mathscr{U}$ is pre-H-closed.*

(b) *$(\mathscr{U}X, \tau(\mathscr{U}_*))$ is H-closed.*

(c) *$(\mathscr{U}X, \tau(\mathscr{U}^*))$ is H-closed.*

(d) *$(Y, \tau(\mathscr{V}))$ is H-closed for every para-uniform completion $(Y, \mathscr{V})$ of $(X, \mathscr{U})$.*

(e) *$(Y, \tau(\mathscr{V}))$ is H-closed for some para-uniform completion $(Y, \mathscr{V})$ of $(X, \mathscr{U})$ with r.u.o.*

Proof. (a) $\Rightarrow$ (b). Let $\mathscr{F}$ be a $\tau(\mathscr{U}_*)$-open ultrafilter on $\mathscr{U}X$. Either $\mathscr{F}$ converges or $\mathscr{F}$ is free. If $\mathscr{F}$ is free, then $\mathscr{F}|_X$ is a $\tau(\mathscr{U})$-free, $\tau(\mathscr{U})$-open ultrafilter on X and so is $\mathscr{U}$-Cauchy. Let $\mathscr{G}$ be the minimal $\mathscr{U}$-Cauchy filter contained in $\mathscr{F}|_X$. Then $\mathscr{G}$ is a $\tau(\mathscr{U}_*)$-adherence point of $\mathscr{F}$.

(b) $\Rightarrow$ (a). If $\mathscr{F}$ is a free $\tau(\mathscr{U})$-open ultrafilter on X, then $\{G \in \tau(\mathscr{U}_*): G \cap X \in \mathscr{F}\}$ generates a $\tau(\mathscr{U}_*)$-open filter $\mathscr{G}$ on $\mathscr{U}X$. Since $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is H-closed, $\mathscr{G}$ has an adherence point $\mathscr{H} \in \mathscr{U}X - X$. Now $O^{\mathscr{H}, X}_{\tau(\mathscr{U}_*)} = \mathscr{H} \cap \tau(\mathscr{U})$, and so every member of $\mathscr{H}$ must meet every member of $\mathscr{F}$. So $\mathscr{H} \subset \mathscr{F}$, since $\mathscr{F}$ is an open ultrafilter. Thus, $\mathscr{F}$ is $\mathscr{U}$-Cauchy, since $\mathscr{H}$ is $\mathscr{U}$-Cauchy.

(b) $\Rightarrow$ (c). Since $\tau(\mathscr{U}^*) = \tau(\mathscr{U}_*)^+$, $(\mathscr{U}X, \tau(\mathscr{U}^*))$ is H-closed if $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is H-closed.

(c) $\Rightarrow$ (d). Any para-uniform completion of $(X, \mathscr{U})$ is a para-uniformly continuous image of $(\mathscr{U}X, \mathscr{U}^*)$ by Theorem 3.13. So $(Y, \tau(\mathscr{V}))$ is H-closed as the continuous image of $(\mathscr{U}X, \tau(\mathscr{U}^*))$.

(d) $\Rightarrow$ (e). $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is H-closed, by (d).

(e) $\Rightarrow$ (b). By Theorem 3.15, $(\mathscr{U}X, \mathscr{U}_*)$ is a para-uniformly continuous image of $(Y, \mathscr{V})$. So $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is H-closed as the continuous image of $(Y, \tau(\mathscr{V}))$.

The example which follows may be helpful to the reader in distinguishing some of the para-uniform concepts being discussed.

EXAMPLE 4.6. Let $Y = [0, 1]$, and let $\mathscr{V}$ be the para-uniformity on Y, with subbasis consisting of all entourages in the usual (metric) uniformity on $[0, 1]$ along with $B = \bigcup_{n=1}^{\infty} [(1/(n+1), 1/n) \times (1/(n+1), 1/n)]$. Then $\tau(\mathscr{V})$ is the usual topology on $[0, 1]$. Let $X = (0, 1]$, and let $\mathscr{U} = \mathscr{V}|_X$. Then:

(a) $\mathscr{V}$ is pre-H-closed but not totally bounded. (No open ultrafilter converging to 0 can be $\mathscr{V}$-Cauchy.)

(b) $(Y, \tau(\mathscr{V}))$ is an H-closed extension of $(X, \tau(\mathscr{U}))$, but $\mathscr{U}$ is not pre-H-closed. (Note that $(Y, \mathscr{V})$ does not have r.u.o. as a para-uniform extension of $(X, \mathscr{U})$.) Thus, a non-pre-H-closed para-uniform space may have some H-closed para-uniform completions.

For a given totally bounded para-uniform space (or a given para-proximity space) the canonical para-uniform completions (or the canonical full para-proximal extensions), which we constructed in §3, yield the strict and simple H-closed extensions belonging to a particular S-equivalence class. It is clear from **[18]** that the set of S-equivalence classes so obtained cannot include all S-equivalence classes of H-closed extensions. Thus, it is of interest to characterize these classes. Such a characterization (in terms of the strict representative) is provided next.

DEFINITION 4.7. **[19]** A topological extension (Y, σ) of (X, τ) is said to have relatively completely regular outgrowth. (r.c.r.o.) if, whenever $y \in G \in \sigma$, there is $H \in \sigma$ with $\{y\} \cup (Y - X) \subset H$ and a continuous function $f\colon (H, \sigma|_H) \to [0, 1]$ such that $f(y) = 0$ and $f(H - G) \subset \{1\}$.

THEOREM 4.8. *Let (Y, σ) be a topological extension of (X, τ). The following are equivalent.*

(a) *(Y, σ) is an H-closed extension of (X, τ) with r.c.r.o.*

(b) *(Y, σ) is isomorphic to $(\mathscr{U}X, \tau(\mathscr{U}_*))$ for some compatible totally bounded para-uniformity $\mathscr{U}$ on (X, τ).*

(c) *(Y, σ) is isomorphic to $(\delta X, \tau(\delta_*))$ for some compatible para-proximity space $(X, \mathscr{A}, \mathscr{D})$ on (X, τ).*

PROOF. (b) $\Rightarrow$ (a). We show (more generally) that if $(Y, \mathscr{V})$ is a para-uniform completion of $(X, \mathscr{U})$ with r.u.o. and $\mathscr{U}$ is totally bounded, then $(Y, \tau(\mathscr{V}))$ is an H-closed extension of $(X, \tau(\mathscr{U}))$ with r.c.r.o. That $(Y, \tau(\mathscr{V}))$ is H-closed follows from Theorem 4.5. Let $y \in G \in \tau(\mathscr{V})$. Then there is

$V \in \mathscr{V}$ with $V^{-1} = V$ and $y \in V[y] \subset G$. Set $H = \operatorname{dom} V$ and let $\mathscr{V}_H$ be the uniformity induced on H by $\mathscr{V}$. (Recall that $\mathscr{V}_H$ need not be separated.) Then $\{y\} \cup (Y - X) \subset H$, since $(Y, \mathscr{V})$ is an r.u.o. para-uniform completion of $(X, \mathscr{U})$, and $H \in \tau(\mathscr{V})$. Since $(H, \tau(\mathscr{V}_H))$ is completely regular (not necessarily Tychonoff), there is a continuous function f: $(H, \tau(\mathscr{V}_H)) \to [0, 1]$ such that $f(y) = 0$ and $f(H - V[y]) \subset \{1\}$. But $\tau(\mathscr{V}_H) \subset \tau(\mathscr{V})$; so f: $(H, \tau(\mathscr{V})|_H) \to [0, 1]$ is continuous, and $f(H - G) \subset \{1\}$ since $V[y] \subset G$.

(a) $\Rightarrow$ (b). Let (Y, σ) be an H-closed extension of (X, τ) with r.c.r.o. For each $y \in Y$ and $G \in \sigma$ with $y \in G$, let $H(G, y) \in \sigma$ such that there is a continuous function $f(G, y)$: $H(G, y) \to [0, 1]$ with $\{y\} \cup (Y - X) \subset H(G, y)$, $f(G, y)(y) = 0$, and $f(G, y)(H(G, y) - G) \subset \{1\}$. Let $F = \{f(G, y)$: $y \in G \in \sigma\}$, and, for each $f = f(G, y) \in F$, let $H(f) = H(G, y)$. For $f \in F$ and $\varepsilon > 0$, let $V(f, \varepsilon) = (Y - \overline{H(f)}) \times (Y - \overline{H(f)}) \cup \{(x, y) \in H(f) \times H(f): |f(x) - f(y)| < \varepsilon\}$. It is straightforward to show that $\{V(f, \varepsilon)$: $f \in F$, $\varepsilon > 0\}$ is a subbasis for a compatible totally bounded para-uniformity $\mathscr{V}$ on (Y, σ). Let $\mathscr{U} = \mathscr{V}|_X$. Then it is clear that $(Y, \mathscr{V})$ is a para-uniform completion of $(X, \mathscr{U})$ with r.u.o. We claim that $(Y, \mathscr{V})$ and $(\mathscr{U}X, \mathscr{U}_*)$ are para-uniformly isomorphic completions of $(X, \mathscr{U})$. By Theorem 3.15, p_*: $(Y, \mathscr{V}) \to (\mathscr{U}X, \mathscr{U}_*)$ is para-uniformly continuous, and $p_*(x) = x$, for all $x \in X$. Define j: $\mathscr{U}X \to Y$ as follows. Set $j(x) = x$, for all $x \in X$, and, for $\mathscr{F} \in \mathscr{U}X - X$, let $j(\mathscr{F})$ be the unique point of Y to which the $\mathscr{V}$-Cauchy filter $\{G \subset Y: F \subset G$ for some $F \in \mathscr{F}\}$ converges. To see that j is para-uniformly continuous, let $V = V(f, \varepsilon) \in \mathscr{V}$ and let $W = V(f, \varepsilon/3)$. Set $U = W \cap (X \times X)$. It is straightforward to show that $U_* \subset j^{-1}(V)$ and $(U_*)^0 = j^{-1}(V)^0$, so that $j^{-1}(V) \in \mathscr{U}_*$. Thus, j: $(\mathscr{U}X, \mathscr{U}_*) \to (Y, \mathscr{V})$ is para-uniformly continuous and $j(x) = x$, for all $x \in X$. It follows that $(Y, \mathscr{V})$ and $(\mathscr{U}X, \mathscr{U}_*)$ are para-uniformly isomorphic para-uniform extensions of $(X, \mathscr{U})$. Thus, $(\mathscr{U}X, \tau(\mathscr{U}_*))$ and $(Y, \tau(\mathscr{V}))$ are is omorphic topological extensions of $(X, \tau(\mathscr{U}))$.

(b) $\Leftrightarrow$ (c) follows from previous results.

It follows that an extension of a topological space with r.c.r.o. is a strict extension. (This was also pointed out in [**19**].) It is clear that there must be strict extensions of some topological spaces without r.c.r.o. An example of such an extension is given now.

EXAMPLE 4.9. Let $X = \{(n, m): n \in \mathbf{N}, m \in \mathbf{Z} - \{0\}\}$ and τ be the discrete topology on X. Let $p = (0, 1)$, $q = (0, -1)$, and $Y = X \cup \{p, q\} \cup \{(n, 0): n \in \mathbf{N}\}$. For $n, k \in \mathbf{N}$, let $G(n, k) = \{(n, 0)\} \cup \{(n, m) \in X: |m| > k\}$, let $G(p, k) = \{p\} \cup \{(j, m) \in X: j > k, m > 0\}$, and $G(q, k) = \{q\} \cup \{(j, m) \in X: j > k, m < 0\}$. Let σ be the topology on Y generated by the basis $\{\{x\}: x \in X\} \cup \{G(n, k): n, k \in \mathbf{N}\} \cup \{G(y, k): y \in \{p, q\}, k \in \mathbf{N}\}$.

Then (Y, σ) is a strict H-closed extension of (X, τ), but p and q cannot be separated by any continuous real-valued function on any neighborhood of $Y - X$. So (Y, σ) does not have r.c.r.o. as an extension of (X, τ).

A special class of totally bounded para-uniformities may be used to obtain H-closed extensions studied by Flachsmeyer **[8]**.

DEFINITION 4.10. (a) **[8]** A topological extension (Y, σ) of (X, τ) is said to have relatively zero-dimensional outgrowth (r.z.d.o.) if σ has a base β such that $\mathrm{cl}_Y B - B \subset X$, for every $B \in \beta$.

(b) A collection $\mathscr{B}$ of subsets of $X \times X$ is called transitive if $B \circ B \subset B$, for every $B \in \mathscr{B}$.

THEOREM 4.11. *Let (Y, σ) be a topological extension of (X, τ). The following are equivalent.*

(a) *(Y, σ) is an H-closed extension of (X, τ) with* r.z.d.o.

(b) *(Y, σ) is isomorphic to $(\mathscr{U}X, \tau(\mathscr{U}_*))$ for some compatible totally bounded para-uniformity $\mathscr{U}$ on (X, τ) with a transitive basis.*

PROOF. (b) $\Rightarrow$ (a). If $U \in \mathscr{U}$ with $U^{-1} = U$ and $U \circ U \subset U$, then $U \circ U \circ U \circ U \subset U$, whence $U_* \circ U_* \subset U_*$. Let $\beta = \{U_*[p]\colon U \in \mathscr{U},\ U^{-1} = U,\ U \circ U \subset U, \text{ and } p \in U_*[p]\}$. Then β is a base for σ, and it is straightforward to show that if $B \in \beta$, then $\mathrm{cl}_{\mathscr{U}X}\, B - B \subset X$. So (Y, σ) has r.z.d.o. as an extension of (X, τ).

(a) $\Rightarrow$ (b). Let $\beta = \{G \in \sigma\colon \mathrm{cl}_Y\, G - G \subset X\}$. Since (Y, σ) has r.z.d.o. as an extension of (X, τ), β is a base for σ. Let $\mathscr{V}$ be the para-uniformity on Y generated by the subbasis $\{S(G)\colon G \in \beta\}$, where $S(G) = (G \times G) \cup [(Y - \mathrm{cl}_Y G) \times (Y - \mathrm{cl}_Y G)]$ (as in Theorem 1.6). Then $\mathscr{V}$ is a compatible para-uniformity on (Y, σ), and it is easy to verify that $\mathscr{V}$ has a transitive basis. Let $\mathscr{U} = \mathscr{V}|_X$. Then $\mathscr{U}$ has a transitive basis too. Further, if $y \in G \in \beta$, then define $f(G, y)\colon G \cup (Y - \mathrm{cl}_Y G) \to [0, 1]$ by $f(G, y)\,(G) \subset \{0\}$ and $f(G, y)\,(Y - \mathrm{cl}_Y G) \subset \{1\}$. As in the proof of Theorem 4.8, $\{V(f(G, y), \varepsilon)\colon y \in G \in \beta,\ 0 < \varepsilon < 1/2\}$ generates a compatible para-uniformity $\mathscr{V}'$ on (Y, σ) such that, when we set $\mathscr{U}' = \mathscr{V}'|_X$, $(Y, \mathscr{V}')$ and $(\mathscr{U}'X, \mathscr{U}'_*)$ are para-uniformly isomorphic completions of $(X, \mathscr{U}')$. But when $y \in G \in \beta$ and $0 < \varepsilon < 1/2$, $V(f(G, y), \varepsilon) = S(G)$. So $\mathscr{V} = \mathscr{V}'$, $\mathscr{U} = \mathscr{U}'$, and so $(Y, \mathscr{V})$ and $(\mathscr{U}X, \mathscr{U}_*)$ are para-uniformly isomorphic completions of $(X, \mathscr{U})$. Therefore, (Y, σ) and $(\mathscr{U}X, \tau(\mathscr{U}_*))$ are isomorphic topological extensions of (X, τ).

EXAMPLE 4.12. Let (X, τ) be a topological space, and let $\mathscr{U}$ be the compatible para-uniformity on X with transitive basis $\{S(G)\colon G \in \tau\}$ (as in Theorem 1.6). Then $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is an H-closed extension of (X, τ) with r.z.d.o. In fact, $(\mathscr{U}X, \tau(\mathscr{U}_*))$ and $(\mathscr{U}X, \tau(\mathscr{U}^*))$ are, respectively, the strict and simple filter extensions of (X, τ) based on the collection of free τ-open ultrafilters. Thus, $(\mathscr{U}X, \tau(\mathscr{U}_*))$ is the Fomin extension of (X, τ) **[9]**, and $(\mathscr{U}X, \tau(\mathscr{U}^*))$ is the Katětov extension of (X, τ) **[12]**.

Flachsmeyer **[8]** studied H-closed extensions with r.z.d.o. and noted that, up to isomorphism, they could be obtained as filter extensions based on the set of maximal filters from a collection of open sets called a π-basis. (A π-basis on (X, τ) is a base β for τ such that $G \in \beta$ implies $X - \bar{G} \in \beta$.) Using the idea of a full π-basis, he showed that there is a one-to-one correspondence between the full π-bases on (X, τ) and the isomorphism classes of H-closed extensions of (X, τ) with r.z.d.o. (A full π-basis may be defined as a π-basis, β, with the property that $G \in \beta$ if every open ultrafilter containing G contains a subset of G which is an element of β.) This yields the result that there is a one-to-one correspondence between the isomorphism classes of H-closed extensions of (X, τ) with r.z.d.o. and the para-uniformities $\mathscr{U}$ on X generated by $\{S(G)\colon G \in \beta\}$ when β ranges through the full π-bases for τ.

Also note that it follows immediately from Theorems 4.8 and 4.11 that an H-closed extension with r.z.d.o. has r.c.r.o. However, there are some topological spaces which have H-closed extensions with r.c.r.o. and without r.z.d.o., as the existence of Hausdorff compactifications of a non-rim compact Tychonoff space shows. Thus, the method presented in Theorem 4.8 for obtaining H-closed extensions yields a larger class of H-closed extensions than does the method of Flachsmeyer.

We shall conclude this section by developing a relationship between the H-closed extensions obtained as canonical para-uniform completions and those obtained as canonical θ-uniform completions. The notion of θ-uniformity was introduced by Fedorčuk in **[5]**.

DEFINITION 4.13. **[5, 6]** Let (X, τ) be a topological space.

(a) A family α of subsets of X is a θ-cover of locally finite type if the members of α are regular open and if, for any point $x \in X$, there exist finitely many members $V_1, \ldots, V_n$ of α with $x \in \operatorname{int} \bigcup_{i=1}^{n} \operatorname{cl} V_i$.

(b) A collection μ of θ-covers of locally finite type is a θ-uniformity on (X, τ) (and (X, μ) is a θ-uniform space on (X, τ)) if the following conditions are satisfied:

(F1) if $\alpha \in \mu$ and β is a θ-cover of locally finite type such that α refines β, then $\beta \in \mu$;

(F2) if $\alpha, \beta \in \mu$, then α and β have a common star-refinement $\gamma \in \mu$;

(F3) if x and y are distinct points in X, then there are neighborhoods G of x and H of y and $\alpha \in \mu$ such that $G \cap \operatorname{st}(H, \alpha) \neq \varnothing$; and

(F4) if $x \in X$ and G is a regular open neighborhood of x, then there is a neighborhood N of x and $\alpha \in \mu$ such that $\operatorname{st}(N, \alpha) \subset G$.

A θ-uniformity does not determine the underlying topology, although a certain amount of "compatibility" is required. Also a θ-uniformity μ on (X, τ) is a θ-uniformity on (X, σ) if σ and τ are θ-homeomorphic.

If μ is a θ-uniformity on a topological space (X, τ), then a filter $\mathscr{F}$ on X is called μ-Cauchy if, for every $\alpha \in \mu$, $\mathscr{F} \cap \alpha \neq \varnothing$. Every μ-Cauchy filter $\mathscr{F}$ contains a unique minimal μ-Cauchy filter $\mathscr{F}_0$ and the regular open members of $\mathscr{F}_0$ form a filterbase which generates $\mathscr{F}_0$.

If μ is a θ-uniformity on a topological space (Y, σ) and X is a dense subset of Y, then $\mu_X = \{\alpha_X: \alpha \in \mu\}$, where $\alpha_X = \{V \cap X: V \in \alpha\}$, forms a θ-uniformity on $(X, \sigma|_X)$.

DEFINITION 4.14. [**5**, **6**] (a) A θ-uniformity μ on a topological space (Y, σ) is complete if every minimal μ-Cauchy filter converges

(b) Let (Y, σ) be a topological extension of (X, τ), let ν be a θ-uniformity on (Y, σ), and let μ be a θ-uniformity on (X, τ). (Y, ν) is a θ-uniform extension of (X, μ) if $\mu = \nu|_X$.

(c) A θ-uniform completion is a complete θ-uniform extension.

(d) A θ-uniformity μ on (X, τ) is pre-compact if μ has a (covering-type) basis consisting of finite θ-covers of locally finite type.

It is clear from Proposition 10 in [**6**] that a θ-uniform space may have a number of distinct completions. Let μ be a θ-uniformity on (X, τ). A canonical θ-uniform completion of (X, μ) is constructed in [**6**] as follows. Let $\hat{X}$ be the set whose members are elements of X or free minimal μ-Cauchy filters on X. Define a topology $\hat{\tau}$ on $\hat{X}$ by taking as a neighborhood basis at each point of X, all its neighborhoods in X, and at $\mathscr{F} \in \hat{X} - X$, all sets of the form $\{\mathscr{F}\} \cup G$ where $G \in \tau$ and int cl $G \in \mathscr{F}$. Then $(\hat{X}, \hat{\tau})$ is a topological (Hausdorff) extension of (X, τ). Define a θ-uniformity $\hat{\mu}$ on $(\hat{X}, \hat{\tau})$ as follows. For $G \in \tau$, let $\hat{G}$ denote the largest open subset of X such that $G = X \cap \hat{G}$. (Note that if G is regular open, then $\hat{G} = \operatorname{int}_{\hat{X}} \operatorname{cl}_{\hat{X}} G$). For $\alpha \in \mu$ let $\hat{\alpha} = \{\hat{G}: G \in \alpha\}$ and set $\hat{\mu} = \{\hat{\alpha}: \alpha \in \mu\}$. Then $\hat{\mu}$ is indeed a complete θ-uniformity on $(\hat{X}, \hat{\tau})$ and $(\hat{X}, \hat{\mu})$ is a θ-uniform completion of (X, μ). Moreover, if μ is pre-compact, then $(\hat{X}, \hat{\tau})$ is H-closed.

The theorem which follows shows that any totally bounded para-uniform space induces a pre-compact θ-uniformity on its underlying topological space in a natural way.

THEOREM 4.15. *Let $(X, \mathscr{U})$ be a totally bounded para-uniform space, and let $\tau = \tau(\mathscr{U})$.*

(a) *For $U \in \mathscr{U}$, U symmetric, $\alpha(U) = \{\text{int cl } U[x]: x \in X\}$ is a θ-cover of locally finite type on (X, τ).*

(b) *$\mu(\mathscr{U}) = \{\beta$: β is a θ-cover of locally finite type refined by $\alpha(U)$ for some symmetric $U \in \mathscr{U}\}$ is a pre-compact θ-uniformity on (X, τ).*

PROOF. (a) Let $U \in \mathscr{U}$ be symmetric. Clearly, $\alpha(U)$ is a family of regular open subsets of (X, τ). Now let $V \in \mathscr{U}$ be symmetric and open in the product topology on $X \times X$ with $V^0 = U^0$ and $V \subset U$. Since $\mathscr{U}$ is totally bounded, there is a finite set $F = \{x_1, \ldots, x_n\} \subset X$ such that $X = \text{cl } V[F] = \text{cl } \bigcup_{i=1}^n V[x_i] = \bigcup_{i=1}^n \text{cl } V[x_i]$. Noting that each $V[x_i]$ is open, we have $X = \text{int } X = \text{int } \bigcup_{i=1}^n \text{cl } V[x_i] = \text{int } \bigcup_{i=1}^n \text{cl int cl } V[x_i] \subset \text{int } \bigcup_{i=1}^n \text{cl int cl } U[x_i]$.

(b) We shall verify (F2) and the pre-compactness of $\mu(\mathscr{U})$, leaving the verification of (F3) and (F4) to the reader and noting that (F1) is obvious. Let $\alpha_1, \alpha_2 \in \mu(\mathscr{U})$. Then there are symmetric entourages $U_1, U_2 \in \mathscr{U}$ with $\alpha(U_i)$ refining α_i $(i = 1, 2)$. Let $V \in \mathscr{U}$ be symmetric and open in the product topology on $X \times X$ with $V^0 = (U_1 \cap U_2)^0$ and $V \circ V \circ V \subset U_1 \cap U_2$. Then it is straightforward to verify that $\alpha(V)$ is a common star refinement of $\alpha(U_1)$ and $\alpha(U_2)$, hence of α_1 and α_2. Thus (F2) holds.

In order to verify that $\mu(\mathscr{U})$ is pre-compact, let $\alpha \in \mu(\mathscr{U})$. We must find a finite family $\beta \in \mu(\mathscr{U})$ such that β refines α. Let $U \in \mathscr{U}$ be symmetric with $\alpha(U)$ refining α. Let $V \in \mathscr{U}$ be symmetric and open in the product topology on $X \times X$ with $V^0 = U^0$ and $V \circ V \circ V \subset U$. Since $\mathscr{U}$ is totally bounded, there is a finite set $F = \{x_1, \ldots, x_n\} \subset X$ such that $\text{cl } V[F] = X$. Now, as in the proof of (a), $\beta = \{\text{int cl } U[x_i]: i = 1, \ldots, n\}$ is a finite θ-cover of locally finite type. Also it may be verified easily that $\alpha(V)$ refines β. So $\beta \in \mu(\mathscr{U})$ and clearly β refines $\alpha(U)$ (and hence α).

Now, for a totally bounded para-uniform space $(X, \mathscr{U})$, the canonical θ-uniform completion of $(X, \mu(\mathscr{U}))$ is $(\hat{X}, \hat{\mu}(\mathscr{U}))$ whose underlying topological space $(\hat{X}, \hat{\tau}(\mathscr{U}))$ is an H-closed extension of $(X, \tau(\mathscr{U}))$. Of course, the extensions $(\mathscr{U}X, \tau(\mathscr{U}_*))$ and $(\mathscr{U}X, \tau(\mathscr{U}^*))$ are also H-closed extensions of $(X, \tau(\mathscr{U}))$ which represent a single R-equivalence class of H-closed extensions. The next theorem asserts that $(\hat{X}, \hat{\tau}(\mathscr{U}))$ also represents this R-equivalence class.

THEOREM 4.16. *Let $(X, \mathscr{U})$ be a totally bounded para-uniform space. Then $(\mathscr{U}X, \tau(\mathscr{U}^*))$ is θ-isomorphic to $(\hat{X}, \hat{\tau}(\mathscr{U}))$ as topological extensions of $(X, \tau(\mathscr{U}))$.*

PROOF. First note that if $\mathscr{F}$ is a $\mathscr{U}$-Cauchy filter on $(X, \mathscr{U})$, then $\mathscr{F}$ is a $\mu(\mathscr{U})$-Cauchy filter on the θ-uniform space $(X, \mu(\mathscr{U}))$. Moreover, if $\mathscr{F}$ is a minimal $\mathscr{U}$-Cauchy filter on $(X, \mathscr{U})$, then the unique minimal $\mu(\mathscr{U})$-Cauchy filter $\mathscr{F}_0$ on $(X, \mu(\mathscr{U}))$ contained in $\mathscr{F}$ has $\{\text{int cl } F: F \in \mathscr{F}\}$ as a filterbase.

Now define $j: \mathscr{U}X \to \hat{X}$ by $j(x) = x$ $(x \in X)$ and $j(\mathscr{F}) = \mathscr{F}_0$ $(\mathscr{F} \in \mathscr{U}X - X)$, where $\mathscr{F}_0$ is the unique minimal $\mu(\mathscr{U})$-Cauchy filter contained in $\mathscr{F}$. (We have $j(\mathscr{F}) = \mathscr{F}_0 \in \hat{X} - X$, since $\mathscr{F}_0$ being free follows from $\mathscr{F}$ being free.) If $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{U}X - X$ and $\mathscr{F}_1 \neq \mathscr{F}_2$, then there are open members $F_1 \in \mathscr{F}_1$ and $F_2 \in \mathscr{F}_2$ with $F_1 \cap F_2 = \phi$. So $\text{int cl } F_1 \cap \text{int cl } F_2 = \varnothing$. Since $\text{int cl } F_i \in (\mathscr{F}_i)_0$ $(i = 1, 2)$, it follows that $(\mathscr{F}_1)_0 \neq (\mathscr{F}_2)_0$. Thus, j is one-to-one.

If $\mathcal{F} \in \mathcal{U}X - X$, then a neighborhood basis at $\mathcal{F}$ in $\tau(\mathcal{U}^*)$ consists of sets of the form $\{\mathcal{F}\} \cup G$, where G is an open member of $\mathcal{F}$, and a neighborhood basis at $\mathcal{F}_0 = j(\mathcal{F})$ in $\hat{\tau}(\mathcal{U})|_{j(\mathcal{U}X)}$ consists of sets of the form $\{\mathcal{F}_0\} \cup G$ where G is open and int cl $G \in \mathcal{F}_0$. It follows in a straightforward manner that j: $\mathcal{U}X \to j(\mathcal{U}X)$ is a θ-continuous, open surjection. Now, $j(\mathcal{U}X)$, being the θ-continuous image of an H-closed space, is H-closed itself. Since $j(\mathcal{U}X)$ contains X, $j(\mathcal{U}X)$ is dense in $\hat{X}$. Thus, $j(\mathcal{U}X) = \hat{X}$, whence j is onto.

Therefore, j is a θ-isomorphism, as desired.

If we identify the points of $\hat{X}$ with the points of $\mathcal{U}X$ via the θ-isomorphism of the preceding theorem, then $\hat{\mu}(\mathcal{U})$ becomes a complete θ-uniformity on $(\mathcal{U}X, \tau(\mathcal{U}^*))$ (and also on $(\mathcal{U}X, \tau(\mathcal{U}_*))$, according to Proposition 10 in [6]). (Also $\mu(\mathcal{U}^*)$ and $\mu(\mathcal{U}_*)$ are θ-uniformities on $\mathcal{U}X$ with any topology σ for which $(\mathcal{U}X, \sigma)$ is θ-isomorphic to $(\mathcal{U}X, \tau(\mathcal{U}_*))$. In fact it can be shown that, as θ-uniformities on $(\mathcal{U}X, \sigma)$, $\hat{\mu}(\mathcal{U})$, $\mu(\mathcal{U}^*)$, and $\mu(\mathcal{U}_*)$ are identical.) Thus, we have the following corollary.

COROLLARY 4.17. *Let (Y, σ) be an H-closed extension of (X, τ) with r.c.r.o. Then there is a pre-compact θ-uniformity μ on (X, τ) such that the underlying topological space $(\hat{X}, \hat{\tau})$ of the canonical θ-uniform completion $(\hat{X}, \hat{\mu})$ belongs to the R-equivalence class of (Y, σ).*

The R-equivalence classes of H-closed extensions of a given topological space which are represented by canonical θ-uniform completions of pre-compact θ-uniformities on the space have not been characterized. However, it is clear that any such R-equivalence class contains an H-closed extension whose outgrowth is completely regular. The example which follows shows not only that an H-closed extension with completely regular outgrowth need not have r.c.r.o., but in fact need not be θ-isomorphic to an extension with r.c.r.o.

EXAMPLE 4.18. Let (X, τ) and (Y, σ) be the topological spaces introduced in Example 4.9. Recall that (Y, σ) is a strict H-closed extension of (X, τ) but does not have r.c.r.o. since the points p and q in $Y - X$ cannot be separated by any real-valued continuous function on any neighborhood of $Y - X$.

Now $Y - X$ is completely regular since it is discrete in the relative topology inherited from Y. Moreover, (Y, σ) cannot be θ-isomorphic to any extension of (X, τ) with r.c.r.o. For, suppose that (Z, η) is an extension of (X, τ) with r.c.r.o. and h: $Y \to Z$ is a θ-isomorphism. Since $h(p)$ and $h(q)$ are two distinct points of $Z - X$, there is a neighborhood $H \in \eta$ of $Z - X$ and a continuous function f: $H \to [0, 1]$ with $f(h(p)) \neq f(h(q))$. Set $K = h^{-1}(H)$ and define g: $K \to [0, 1]$ by $g = f \circ h$. Then $K \in \sigma$, $Y - X \subset K$, and g is continuous since $[0, 1]$ is regular. Thus g separates p and q, *a* contradiction.

5. Superstructures and H-closed extensions. In [7] Fedorčuk uses collections of θ-proximities called H-structures to construct all semiregular H-closed extensions of a given semiregular topological space. Thus, the R-equivalence classes of H-closed extensions of a given space may be described in terms of these H-structures. In this section we develop properties of certain collections of para-uniformities, which we shall call superstructures, and we shall be able to describe the S-equivalence classes of H-closed extensions of a given space in terms of these collections.

DEFINITION 5.1. Let $\mathscr{C}$ be a nonempty collection of pair-wise compatible para-uniformities on a set X. (I.e., if $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{C}$, then $\tau(\mathscr{U}_1) = \tau(\mathscr{U}_2)$.)

(a) A filter $\mathscr{F}$ on X is called $\mathscr{C}$-Cauchy if $\mathscr{F}$ is $\mathscr{U}$-Cauchy for some $\mathscr{U} \in \mathscr{C}$.

(b) $M(\mathscr{C})$ denotes the collection of all free $\mathscr{C}$-Cauchy filters on X (where adherence is computed with respect to the topology induced commonly by $\mathscr{U} \in \mathscr{C}$).

(c) Two filters $\mathscr{F}$ and $\mathscr{G}$ in $M(\mathscr{C})$ are said to be contiguous if there is a finite set $\{\mathscr{F}_1, \ldots, \mathscr{F}_n\} \subset M(\mathscr{C})$ such that $\mathscr{F}_1 = \mathscr{F}$, $\mathscr{F}_n = \mathscr{G}$, and $\mathscr{F}_i$ meets $\mathscr{F}_{i+1}$ for $i = 1, \ldots, n - 1$.

The relation of "being contiguous" is an equivalence relation on $M(\mathscr{C})$, as may be verified easily.

DEFINITION 5.2. Let $\mathscr{C}$ be as in Definition 5.1. For $\mathscr{F} \in M(\mathscr{C})$, let $m(\mathscr{F})$ denote the equivalence class under "being contiguous" of $\mathscr{F}$, and let $\mathscr{F}^\#$ denote the filter on X which is the intersection of all filters in $m(\mathscr{F})$. A set M of filters on a topological space (X, τ) is free if each filter in M is free, and M is separated if any two distinct filters $\mathscr{F}$ and $\mathscr{G}$ in M contain disjoint members $F \in \mathscr{F}$ and $G \in \mathscr{G}$.

DEFINITION 5.3. Let $\mathscr{C}$ be a nonempty collection of pair-wise compatible para-uniformities on a set X.

(a) $\mathscr{C}$ is called a superstructure on X if $\{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ is free and separated.

(b) $\mathscr{C}$ is said to be compatible with a topology τ on X if $\tau(\mathscr{U}) = \tau$ for every $\mathscr{U} \in \mathscr{C}$, in which case we write $\tau(\mathscr{C}) = \tau$.

DEFINITION 5.4. Let $\mathscr{C}$ be a compatible superstructure on a topological space (X, τ).

(a) A topological extension (Y, σ) of (X, τ) is a $\mathscr{C}$-completion of (X, τ) if every $\mathscr{C}$-Cauchy filter on X has a σ-adherence point in Y.

(b) Let $\mathscr{C}X = X \cup \{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ and let $\mathscr{C}\tau$ be the topology on $\mathscr{C}X$ such that $(\mathscr{C}X, \mathscr{C}\tau)$ is the strict filter extension of (X, τ) based on $\{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ [2].

PROPOSITION 5.5. *Let $\mathscr{C}$ be a compatible superstructure on (X, τ).*

(a) *$(\mathscr{C}X, \mathscr{C}\tau)$ is a $\mathscr{C}$-completion of (X, τ).*

(b) *$(\mathscr{C}X, \mathscr{C}\tau)$ is H-closed if and only if, for each free τ-ultrafilter $\mathscr{F}$ on X, there is $\mathscr{G} \in M(\mathscr{C})$ such that $\mathscr{G}^\# \subset \mathscr{F}$.*

(c) *If each free τ-ultrafilter is $\mathscr{C}$-Cauchy, then $(\mathscr{C}X, \mathscr{C}\tau)$ is H-closed.*

(d) *If some $\mathscr{U} \in \mathscr{C}$ is pre-H-closed, then $(\mathscr{C}X, \mathscr{C}\tau)$, is H-closed.*

PROOF. (a). That $(\mathscr{C}X, \mathscr{C}\tau)$ is Hausdorff follows from the fact that $\{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ is free and separated. If $\mathscr{F} \in M(\mathscr{C})$, then $\mathscr{F}^\#$ is a $\mathscr{C}\tau$-adherence point of $\mathscr{F}$ in $\mathscr{C}X$. So $(\mathscr{C}X, \mathscr{C}\tau)$ is a $\mathscr{C}$-completion of (X, τ).

(b) is easily verified, and (c) follows from (b) since $\mathscr{F}^\# \subset \mathscr{F}$ for every $\mathscr{F} \in M(\mathscr{C})$. Moreover, (d) follows from (c) since each free τ-ultrafilter will be $\mathscr{U}$-Cauchy when $\mathscr{U}$ is a pre-H-closed member of $\mathscr{C}$.

We are now able to describe all isomorphism classes of strict H-closed extensions of a given, non-H-closed, topological space (X, τ) as canonical $\mathscr{C}$-completions $(\mathscr{C}X, \mathscr{C}\tau)$ for certain compatible superstructures $\mathscr{C}$ on (X, τ). But first we need a lemma. Recall that the Katětov extension $(\kappa X, \kappa)$ of a topological space (X, τ) is projectively larger than any other H-closed extension of (X, τ). We can take $\kappa X = X \bigcup \{\mathscr{F}: \mathscr{F}$ is a free τ-open ultrafilter on $X\}$ so that $(\kappa X, \kappa)$ is the simple filter extension based on the set of free open ultrafilters on (X, τ). If (Y, σ) is any H-closed extension of (X, τ) and $f: \kappa X \to Y$ is the unique continuous surjection fixing the points of X, then, for any free τ-open ultrafilter $\mathscr{F}$ on X, we have $f(\mathscr{F}) = y \in Y - X$ if and only if $O_\sigma^{y,X} \subset \mathscr{F}$.

LEMMA 5.6. *Let (Y, σ) be an H-closed extension of a non-H-closed space (X, τ), and let $y \in Y - X$ be fixed. Then there is a compatible para-uniformity $\mathscr{U}(y)$ on (X, τ) such that:*

(a) *$\mathscr{U}(y)$ is totally bounded and has a transitive basis, and*

(b) *the filter on X generated by $O_\sigma^{y,X}$ is a free minimal $\mathscr{U}(y)$-Cauchy filter, and the other free minimal $\mathscr{U}(y)$-Cauchy filters are the members of $\kappa X - X$ which do not contain $O_\sigma^{y,X}$.*

PROOF. Let $f: \kappa X \to Y$ be the unique continuous surjection which fixes the points of X. Since $\tau - \bigcup f^{-1}(y)$ is a base for τ, $\beta = O_\sigma^{y,X} \bigcup (\tau - \bigcup f^{-1}(y))$ is a base for τ. Let $\mathscr{U}(y)$ denote the para-uniformity on X generated by $\{S(G): G \in \beta\}$ as in Theorem 1.6. It is clear that $\mathscr{U}(y)$ is compatible, totally bounded, and has a transitive basis. So (a) follows.

(b). Let $\mathscr{F} \in \kappa X - X$ such that $f(\mathscr{F}) = y$. Then $O_\sigma^{y,X} \subset \mathscr{F}$ and $\mathscr{F}$ is $\mathscr{U}(y)$-Cauchy since $\mathscr{U}(y)$ is totally bounded. Let $\mathscr{G}$ be the minimal $\mathscr{U}(y)$-Cauchy filter contained in $\mathscr{F}$. Recall that $\mathscr{G} = \{U[F]: U \in \mathscr{U}, F \in \mathscr{F}\}$. Now if $G \in O_\sigma^{y,X}$, then $S(G)[G] = G$. Thus, $O_\sigma^{y,X} \subset \mathscr{G}$. Let $B \in \tau - \bigcup f^{-1}(y)$. Then $y \notin \mathrm{cl}_Y B$ and so $X - \mathrm{cl}_X B = X \bigcap (Y - \mathrm{cl}_Y B) \in O_\sigma^{y,X}$. So $S(B)[x] \in O_\sigma^{y,X}$, for any $x \in X - \mathrm{cl}_X B$. Therefore, the filter generated on X by

$O_\sigma^{y,X}$ is $\mathscr{U}(y)$-Cauchy (and hence equals $\mathscr{G}$) and is a free, minimal $\mathscr{U}(y)$-Cauchy filter.

Now suppose that $\mathscr{F} \in \kappa X - X$ and $O_\sigma^{y,X} \not\subset \mathscr{F}$. Then $\mathscr{F}$ is $\mathscr{U}(y)$-Cauchy since $\mathscr{U}(y)$ is totally bounded. There are open sets $B \in \mathscr{F}$ and $G \in O_\sigma^{y,X}$ such that $B \cap G = \emptyset$. If $F \in \mathscr{F}$, then $B \cap F \in \mathscr{F}$ and $B \cap F \in \tau - \bigcup f^{-1}(y)$. So $S(B \cap F)\ [B \cap F] = B \cap F \subset F$. Thus, $\mathscr{F}$ is a free, minimal $\mathscr{U}(y)$-Cauchy filter. On the other hand, it is straightforward to verify that if $\mathscr{F} \in \mathscr{U}(y)X - X$ and $O_\sigma^{y,X} \not\subset \mathscr{F}$, then $\mathscr{F} \in \kappa X - X$.

THEOREM 5.7. *Let (Y, σ) be a topological extension of a non-H-closed space (X, τ). The following are equivalent.*

(a) *(Y, σ) is a strict H-closed extension of (X, τ).*

(b) *(Y, σ) is isomorphic to $(\mathscr{C}X, \mathscr{C}\tau)$ for some compatible superstructure $\mathscr{C}$ on (X, τ) whose members are pre-H-closed.*

PROOF. (b) $\Rightarrow$ (a) follows from previous results.

(a) $\Rightarrow$ (b). For each $y \in Y - X$, let $\mathscr{U}(y)$ be the para-uniformity on X guaranteed by Lemma 5.6. and let $\mathscr{C} = \{\mathscr{U}(y): y \in Y - X\}$. Then $\mathscr{C}$ is a nonempty collection of compatible, pre-H-closed para-uniformities on (X, τ). We claim that $\{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ is precisely the collection of filters generated on X by $O_\sigma^{y,X}$ for some $y \in Y - X$. To see this, note that (as in the proof of Lemma 5.6) $O_\sigma^{y,X}$ generates a free $\mathscr{C}$-Cauchy filter for each $y \in Y - X$, and also, for each $\mathscr{F} \in M(\mathscr{C})$, there is some $y \in Y - X$ such that $O_\sigma^{y,X} \subset \mathscr{F}$. Since (Y, σ) is Hausdorff, it follows that, for each $\mathscr{F} \in M(\mathscr{C})$ there is some $y \in Y - X$ such that $\mathscr{F}^\#$ equals the filter generated by $O_\sigma^{y,X}$ and, hence, $\mathscr{F}^\# \in M(\mathscr{C})$ for each $\mathscr{F} \in M(\mathscr{C})$. The claim follows immediately. Therefore, $\{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ is a free and separated set of filters, whence $\mathscr{C}$ is a superstructure. Moreover, $\mathscr{C}X - X = \{\mathscr{F}^\#: \mathscr{F} \in M(\mathscr{C})\}$ consists precisely of the filters generated on X by $O_\sigma^{y,X}$ for some $y \in Y - X$. The mapping h: $(\mathscr{C}X, \mathscr{C}\tau) \to (Y, \sigma)$ defined by $h(x) = x$ (if $x \in X$) and $h(\mathscr{F}) = y$ (if $\mathscr{F} \in \mathscr{C}X - X$ and $O_\sigma^{y,X} \subset \mathscr{F}$) is an isomorphism since $(\mathscr{C}X, \mathscr{C}\tau)$ and (Y, σ) are strict extensions of (X, τ) with identical filter traces.

REFERENCES

1. P. Alexandroff, *Some results in the theory of topological spaces obtained within the last twenty-five years*, Russian Math. Surveys **15** (1960), 23–83.

2. B. Banaschewski, *Extensions of topological spaces*, Canad. Math. Bull. **7** (1964), 1–22.

3. N. Bourbaki, *General Topology, Part I*, Addison-Wesley Publishing Co., Reading, Mass., 1966.

4. John W. Carlson, *H-closed and countably compact extensions*, Pacific J. Math **81** (2) (1979), 317–326.

5. V.V. Fedorčuk, *Uniform spaces and perfect irreducible mappings of topological spaces*, DAN SSSR **192** (1970), 1228–1230.

6. ———, *Completeness and products of θ-uniform spaces*, Colloquia Mathematica Societatis Janos Bolyai, 1972.

7. ———, *A problem of A. N. Tychonoff on the classification of H-closed extensions*, Fund. Math. **LXXXVI** (1974).

8. J. Flachsmeyer, *Zur Theorie der H-abgeschlossen Erweiterungen*, Math. Z. **94** (1966), 349–381.

9. S.V. Fomin, *The theory of H-closed extensions of topological spaces*, (Russian) Mat. Sb. **8** (1940), 285–294.

10. A.A. Gryzlov, *H-closed extensions*, Izvestiya VUZ Mathematika, Vol. **18**, No. 7 (1974), 21–26.

11. A.A. Ivanov, *Regular extensions of topological spaces*, Izv. Akad. Nauk USSR Ser. Fiz. Mat. **1** (1966), 28–35.

12. M. Katětov, *Über H-abgeschlossene und bikompakte Raume*, Čas. Mat. Fys. **69** (1940), 36–49.

13. W. Kulpa, *On a generalization of uniformities*, Coll. Math. **25** (1972), 227–240.

14. M.G. Murdeshwar and S.A. Naimpally, *Quasi-uniform topological spaces*, Noordhoff, 1966.

15. S.A. Naimpally and B.D. Warrack, *Proximity Spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, No. **59**, Cambridge University Press, 1960.

16. R.G. Ori and W.J. Thron, *On θ-proximities and related f-proximities*, Quaes. Math. **5** (1982), 317–329.

17. S.G. Ovsepjan, *Construction of all Hausdorff and all H-closed extensions of topological spaces*, DAN SSSR, **224** (1975), 764–767.

18. J.R. Porter and C. Votaw, *H-closed extensions. I*, General Topology and its Applications **3** (1973), 211–224.

19. ——— and ———, *H-closed extensions. II*, Trans. A.M.S. **202** (1975), 193–209.

20. Ju. M. Smirnov, *On Proximity Spaces*, A.M.S. Transl. **38** (1964), 5–38.

21. M.H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. A.M.S. **41** (1937), 375–401.

22. Wolfgang J. Thron, *Topological Structures*, Rinehart and Winston, New York, 1966.

23. C. Votaw, *H-closed extensions as para-uniform completions*, Ph.D. Dissertation, Univ. of Kansas, Lawrence, Kansas, 1971.

24. Stephen Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass., 1970.

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