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THE APPROXIMATE SOLUTION OF MONOTONE NONLINEAR OPERATOR EQUATIONS

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ABSTRACT. This paper is concerned with nonlinear equations involving monotone operators and compact perturbations of monotone operators. Projection methods determine approximate solutions. Such equations are put into the more general framework of regular operator approximation theory, which yields the convergence of approximate solutions under minimal hypothesis. Nonlinear integral equations of Urysohn type illustrate the theory.

1. Introduction and summary. There is a considerable literature on monotone nonlinear operator equations and compact perturbations of such equations. Principal applications are given by nonlinear integral equations of Urysohn and Hammerstein type. Some pertinent references are [5], [6], and [8].

Regular operator approximation theory [1], which is based on inverse-compactness concepts, provides a convenient general framework for the convergence of approximate solutions. The existence of solutions then follows in a natural way.

The gist of regular operator approximation theory is as follows. Let X and Y be Banach spaces. Let A, A_n: $X \to Y$, for $n = 1, 2, \ldots$. We shall compare equations

$$Ax = y, \qquad A_n x_n = y_n, \qquad x, x_n \in X, \qquad y, y_n \in Y.$$

Regular convergence $A_n \xrightarrow{r} A$ is a composite property. It includes continuous convergence,

$$A_n \xrightarrow{c} A: x_n \to x \Rightarrow A_n x_n \to Ax.$$

It also includes asymptotic regularity: if $\{x_n\}$ is bounded and $A_n x_n \to y$ on a subsequence, then $\{x_n\}$ has a convergent subsequence. Assume that $A_n \xrightarrow{r} A$, $y_n \to y$, and $\gamma > 0$. Define

$$S = \{x \in X: Ax = y, \|x\| \leqq \gamma\},$$
$$S_n = \{x_n \in X: A_n x_n = y_n, \|x_n\| \leqq \gamma\}.$$

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Then $S_n \to S$ in the sense that any ε-neighborhood of S contains S_n for all n sufficiently large. If $S_n \neq \emptyset$ for infinitely many n, then $S \neq \emptyset$. Further details are given in §2.

Suppose henceforth that X is a reflexive real Banach space with conjugate space X^*. Symbolize the relationship between X and X^* by

$$\langle y, x \rangle = y(x), \quad x \in X, \ y \in X^*.$$

For example, $X = L^p[0, 1]$ and X^* is identified with $L^q[0, 1]$, where $1/p + 1/q = 1$ and $1 < p, q < \infty$. Integral equation examples will be posed in a Hilbert space, namely $L^2[0, 1]$. The greater generality adopted here requires no more analytical effort.

For $n = 1, 2, \ldots$, let P_n be a bounded linear projection on X and P_n^* the adjoint projection on X^*. Under typical hypotheses, $P_n \to I$ and $P_n^* \to I^*$. For example, X is a Hilbert space and $P_n^* = P_n$ are orthogonal projections. Or P_n and P_n^* may project onto subspaces spanned by elements of countable bases of X and X^*. Projections onto spline subspaces of function spaces have great practical importance. Some pertinent references are [**2**, **4**, 7].

Let $A: X \to X^*$. We shall compare equations

$$Ax = y, \qquad P_n^* A x_n = P_n^* y, \qquad x, x_n \in X, \ y \in Y.$$

We shall verify that $P_n^* A \xrightarrow{r} A$ when A is bounded, continuous, and strongly monotone:

$$\langle Ax_1 - Ax_2, x_1 - x_2 \rangle \geqq \alpha \| x_1 - x_2 \|^2, \quad \alpha > 0.$$

Then regular operator approximation theory relates solutions of $Ax = y$ and $P_n^* A x_n = P_n^* y$.

Extensions of the theory pertain to equations

$$(A - K)x = y, \qquad P_n^*(A - K)x_n = P_n^* y,$$

where K is compact and continuous. In special cases, K is asymptotically linear or quasibounded.

Since our purpose is to present a convenient general framework for the convergence theory, we do not go into details of practical implementation in this paper.

2. Regular operator approximation theory. This material is adapted from [**1**], where proofs are available. We shall use the notation

$$x, x_n \in X, \qquad S, S_n \subset X, \qquad n \in N = \{1, 2, \ldots\}.$$

Denote infinite subsets of N by $N', N'', \ldots$. The sets of all cluster points (limits of subsequences) of $\{x_n\}$ and $\{S_n\}$ are

$$\{x_n\}^* = \{x \in X: x_n \to x,\ n \in N'\},$$
$$\{S_n\}^* = \{x \in X: x_n \to x,\ x_n \in S_n,\ n \in N'\}.$$

A sequence $\{x_n\}$ is d-compact (discretely compact) if every subsequence has a convergent subsequence. Similarly, $\{S_n\}$ is d-compact if every sequence $\{x_n \in S_n: n \in N'\}$ has a convergent subsequence. If $S_n = \emptyset$ for $n \in N$, then $\{S_n\}$ is trivially d-compact.

Set convergence is defined by $S_n \to S$ as $n \to \infty$ if any ε-neighborhood of S contains S_n for all n sufficiently large.

Such a set limit is not unique, i.e.,

$$S_n \to S \subset S' \Rightarrow S_n \to S'.$$

Void sets play a special role:

(2.1) $$S_n = \emptyset \text{ for } n \in N \Rightarrow S_n \to S \text{ for all } S \subset X,$$

(2.2) $$S_n \to \emptyset \Rightarrow S_n = \emptyset \text{ for } n \text{ sufficiently large},$$

(2.3) $$S_n \neq \emptyset \text{ for } n \in N', \qquad S_n \to S \Rightarrow S \neq \emptyset.$$

Set convergence and d-compactness are related by

(2.4) $$\{S_n\}\ d\text{-compact}, \qquad \{S_n\}^* \subset S \Rightarrow S_n \to S.$$

This is fundamental in the regular operator approximation theory.

Let $A, A_n: X \to Y$. Key properties are:

(i) A is regular if for $\{x_n\}$ bounded, $\{Ax_n\}$ d-compact $\Rightarrow$ $\{x_n\}$ d-com pact;

(ii) $\{A_n\}$ is asymptotically regular if, for $\{x_n\}$ bounded, $\{A_n x_n\}$ d-compact $\Rightarrow$ $\{x_n\}$ d-compact, and every subsequence of $\{A_n\}$ has this property.

Regular convergence is defined by $A_n \xrightarrow{r} A$ if $A_n \xrightarrow{c} A$ and $\{A_n\}$ is asymptotically regular.

By simple arguments,

(2.5) $$A_n \xrightarrow{c} A \Rightarrow A \text{ continuous},$$

(2.6) $$A_n \xrightarrow{r} A \Rightarrow A \text{ regular and continuous}.$$

Solutions of equations

$$Ax = y, \qquad A_n x_n = y_n, \qquad x, x_n \in X, \quad y, y_n \in Y,$$

are compared with the aid of (2.3) and (2.4):

THEOREM 2.1. *Let* $A_n \xrightarrow{r} A$, $y_n \to y$, $\gamma > 0$, *and*

$$S = \{x \in X: Ax = y,\ \|x\| \leqq \gamma\},$$
$$S_n = \{x_n \in X: A_n x_n = y_n,\ \|x_n\| \leqq \gamma\}.$$

Then $\{S_n\}$ *is d-compact,* $\{S_n\}^* \subset S$, *and* $S_n \to S$. *If* $S_n \neq \emptyset$ *for* $n \in N'$, *then* $S \neq \emptyset$.

The scope of Theorem 2.1 can be enlarged. Let $K, K_n: X \to Y$, for $n \in N$. Then:

(i) K is compact if $\{x_n\}$ bounded $\Rightarrow \{Kx_n\}$ d-compact;

(ii) $\{K_n\}$ is asymptotically compact if $\{x_n\}$ bounded $\Rightarrow \{K_n x_n\}$ d-compact.

Asymptotically compact convergence is defined by $K_n \xrightarrow{ac} K$ if $K_n \xrightarrow{c} K$ and $\{K_n\}$ is asymptotically compact. By simple arguments,

$$K_n \xrightarrow{ac} K \Rightarrow K \text{ compact, continuous,} \tag{2.7}$$

$$A \text{ regular, } K \text{ compact} \Rightarrow A - K \text{ regular,} \tag{2.8}$$

$$A_n \xrightarrow{r} A, \qquad K_n \xrightarrow{ac} K \Rightarrow A_n - K_n \xrightarrow{r} A - K. \tag{2.9}$$

Thus, equations $(A - K)x = y$ and $(A_n - K_n)x_n = y_n$ can be compared with the aid of Theorem 2.1.

The identity operator I on X is regular. When $Y = X$, (2.8) and (2.9) yield

$$K \text{ compact} \Rightarrow I - K \text{ regular,} \tag{2.10}$$

$$K_n \xrightarrow{ac} K \Rightarrow I - K_n \xrightarrow{r} I - K. \tag{2.11}$$

These are the prototypes of the regular operator approximation theory.

3. Projections. Let $T, T_n \in L(X)$, the space of bounded linear operators on X. If $T_n \to T$, then $\{T_n\}$ is uniformly bounded, by the Banach-Steinhaus Theorem. It follows that $T_n \to T \Rightarrow T_n \xrightarrow{c} T$. This simplifies many proofs.

Let $P_n \in L(X), n \in N$, be projections with ranges and null spaces denoted by

$$E_n = \mathscr{R}(P_n), \qquad F_n = \mathscr{N}(P_n). \tag{3.1}$$

Assume that

$$\dim E_n < \infty, \qquad E_n \subset E_{n+1}, \qquad F_n \supset F_{n+1}, \tag{3.2}$$

$$P_n \to I \text{ as } n \to \infty. \tag{3.3}$$

Then $\{P_n\}$ is uniformly bounded, $P_n \xrightarrow{c} I$, and

$$\overline{\bigcup_n E_n} = X, \qquad \bigcap_n F_n = 0. \tag{3.4}$$

For example,

$$P_n x = \sum_{i=1}^{n} \langle f_i, x\rangle \phi_i, \quad \phi_i \in X, f_i \in X^*. \tag{3.5}$$

For $S \subset X$, define $S^\perp \subset X^*$ by

$$S^\perp = \{y \in X^*: \langle y, x\rangle = 0 \text{ for } x \in S\}, \tag{3.6}$$

and similarly for $S \subset X^*$ with $X^{**} = X$ since X is reflexive. Recall that $\bar{S} = X$ if and only if $S^\perp = 0^*$, and $\bar{S} = X^*$ if and only if $S^\perp = 0$.

The adjoint projections $P_n^* \in L(X^*)$ are determined by

$$\langle P_n^* y, x\rangle = \langle y, P_n x\rangle, \qquad x \in X,\ y \in X^*. \tag{3.7}$$

Since $\|P_n^*\| = \|P_n\|$, $\{P_n^*\}$ is uniformly bounded. Furthermore,

$$\begin{aligned} \mathscr{R}(P_n^*) &= \mathscr{N}(P_n)^\perp = F_n^\perp, \\ \mathscr{N}(P^*) &= \mathscr{R}(P_n)^\perp = E_n^\perp, \end{aligned} \tag{3.8}$$

$$\begin{gathered} \dim \mathscr{R}(P_n^*) < \infty, \qquad \mathscr{R}(P_n^*) \subset \mathscr{R}(P_{n+1}^*), \\ \mathscr{N}(P^*) \supset \mathscr{N}(P_{n+1}^*), \end{gathered} \tag{3.9}$$

$$\overline{\bigcup_n \mathscr{R}(P_n^*)} = X^*, \qquad \bigcap_n \mathscr{N}(P_n^*) = 0^*. \tag{3.10}$$

Then $P_n^* y = P_m^* y = y$, for $y \in \mathscr{R}(P_m^*)$ and $n \geqq m$. Hence $P_n^* y \to y$, for $y \in \mathscr{R}(P_m^*)$, $m \in N$. It follows that

$$P_n^* \to I^* \text{ as } n \to \infty. \tag{3.11}$$

For example, corresponding to (3.5),

$$P_n^* y = \sum_{i=1}^n \langle y, \phi_i\rangle f_i, \quad \phi_i \in X,\ f_i \in X^*. \tag{3.12}$$

There is a reciprocal relationship between P_n and P_n^*. Either could be introduced first. As mentioned above, if X is a Hilbert space we may have orthogonal projections $P_n^* = P_n$. See **[4]** for examples of projections onto spline subspaces.

Weak convergence will be used occasionally:

$$x_n \rightharpoonup x \text{ if } \langle y, x_n\rangle \to \langle y, x\rangle, \quad \text{for } y \in X^*. \tag{3.13}$$

It is easy to verify that

$$x_n \rightharpoonup x, \qquad y_n \to y \Rightarrow \langle y_n, x_n\rangle \to \langle y, x\rangle. \tag{3.14}$$

This will simplify some arguments. Recall that any bounded sequence in X is weakly sequentially compact:

$$\begin{gathered} \{x_n\} \text{ bounded} \Rightarrow x_n \rightharpoonup x,\ n \in N', \\ \text{for some } x \in X \text{ and } N' \subset N. \end{gathered} \tag{3.15}$$

4. Strongly monotone operators. Let $A: X \to X^*$. We shall be concerned with equations

(4.1)
$$Ax = y, \qquad P_n^* A x_n = P_n^* y,$$
$$x \in X,\ x_n \in E_n,\ y \in X^*.$$

By (3.6) and (3.8),

(4.2)
$$P_n^* A x_n = P_n^* y \Leftrightarrow \langle Ax_n - y, z\rangle = 0, \text{ for } z \in E_n,$$

(4.3)
$$P_n^* A x_n = P_n^* y, \qquad x_n \in E_n \Rightarrow \langle Ax_n - y, x_n\rangle = 0.$$

In order to apply Theorem 2.1, we shall need $P_n^* A \xrightarrow{r} A$ and, in particular, $P_n^* A \xrightarrow{c} A$. Since $P_n^* \xrightarrow{c} I^*$,

(4.4)
$$P_n^* A \xrightarrow{c} A \Leftrightarrow A \text{ continuous}.$$

Thus, we shall assume that A is continuous. We shall also need to assume that A is bounded: A maps bounded sets into bounded sets.

The operator $A: X \to X^*$ is strongly monotone with constant α if

(4.5)
$$\langle Ax_1 - Ax_2, x_1 - x_2\rangle \geqq \alpha\|x_1 - x_2\|^2, \quad \alpha > 0.$$

In this case, any solution of $Ax = y$ is unique.

LEMMA 4.1 *If A is strongly monotone, then A is regular.*

PROOF. Assume $\{x_n\}$ bounded and $\{Ax_n\}$ d-compact. By (3.15),

$$x_n \rightharpoonup x, \quad n \in N', \text{ for some } x \in X,\ N' \subset N,$$
$$Ax_n \rightharpoonup y, \quad n \in N'', \text{ for some } y \in X^*,\ N'' \subset N'.$$

Then (3.14) yields

$$\langle Ax_n - Ax, x_n - x\rangle \to 0, \quad n \in N''.$$

By (4.5), $x_n \to x$, $n \in N''$. Thus $\{x_n\}$ is d-compact and A is regular.

LEMMA 4.2. *If A is strongly monotone and bounded, then $\{P_n^* A\}$ is asympotically regular.*

PROOF. Assume $\{x_n\}$ bounded and $\{P_n^* A x_n\}$ d-compact. Then $\{Ax_n\}$is bounded. By (3.15),

$$x_n \to x, \quad n \in N', \text{ for some } x \in X,\ N' \subset N,$$
$$P_n^* A x_n \to y, \quad n \in N'', \text{ for some } y \in X^*,\ N'' \subset N'.$$

Now $P_n \to I$, $P_n x_n = x_n$, and (3.14) yield, for $n \in N''$,

$$\langle Ax, x_n - x\rangle \to 0,$$
$$\langle Ax_n, P_n x - x\rangle \to 0,$$
$$\langle Ax_n, x_n - P_n x\rangle = \langle P^* A x_n, x_n - x\rangle \to 0,$$
$$\langle Ax_n - Ax, x_n - x\rangle \to 0.$$

By (4.5), $x_n \to x$, $n \in N''$. Thus, $\{x_n\}$ is d-compact and $\{P_n^* A\}$ is asymptotically regular.

LEMMA 4.3. *If A is strongly monotone, bounded, and continuous, then* $P_n^* A \xrightarrow{r} A$.

PROOF. Use (4.4) and Lemma 4.2.

This enables us to apply Theorem 2.1 to the equations $Ax = y$ and $P_n^* A x_n = P_n^* y$. First, however, we shall establish the existence of solutions x_n of $P_n^* A x_n = P_n^* y$. Since rather standard ideas are involved, we give a concise outline. The first result is a consequence of the Brouwer fixed point theorem. See [**8**].

LEMMA 4.4. *Let R^n be real Euclidean space with the inner product $\langle\ ,\ \rangle$. Assume $D \subset R^n$ is bounded, open, convex, and $0 \in D$. Assume $F\colon \bar{D} \to R^n$ is continuous and $\langle Fx, x\rangle > 0$ for $x \in \partial D$. Then $Fx = 0$ for some $x \in D$.*

LEMMA 4.5. *Assume A is strongly monotone with constant α. Let $y \in X^*$. Then*

$$\langle Ax - y, x\rangle > 0, \textit{for } \|x\| \geqq \gamma > \|AO - y\|/\alpha. \tag{4.6}$$

PROOF. $\langle Ax - y,\ x\rangle = \langle Ax - AO,\ x - O\rangle + \langle AO - y,\ x\rangle$.

$$\langle Ax - y, x\rangle \geqq \alpha\|x\|^2 - \|AO - y\|\ \|x\|, \tag{4.7}$$

$$\langle Ax - y, x\rangle > \alpha\|x\|^2 - \alpha\,\gamma\,\|x\| \geqq 0. \tag{4.8}$$

LEMMA 4.6. *Assume A is continuous, $y \in X^*$, $\gamma > 0$, and*

$$\langle Ax - y, x\rangle > 0, \quad \textit{for } \|x\| = \gamma. \tag{4.9}$$

Then $P_n^* A x_n = P_n^* y$, for some $x_n \in E_n$ with $\|x_n\| < \gamma$, $n \in N$.

PROOF. Without loss of generality, dim $E_n = n$. Let $\{\phi_1, \ldots, \phi_n\}$ be any basis for E_n. Then

$$x_n = \sum_{i=1}^{n} x_i^n \phi_i, \quad x_n \in E_n.$$

The correspondence $x_n \leftrightarrow (x_1^n, \ldots, x_n^n)$ defines an isomorphism $E_n \leftrightarrow R^n$. Let

$$F\colon R^n \to R^n, \qquad Fx^n = (\langle Ax_n - y, \phi_1\rangle, \ldots, \langle Ax_n - y, \phi_n\rangle).$$

Then F is continuous. By (4.2),

$$Fx^n = 0 \Leftrightarrow P_n^* A x_n = P_n^* y.$$

Let $D_n = \{x_n \in E_n\colon \|x_n\| < \gamma\}$ and $D_n \leftrightarrow D^n \subset R^n$. Then D^n is bounded, open, convex, and $0 \in D^n$. Also, $\partial D_n \leftrightarrow \partial D^n$. By (4.9) and a brief calculation,

$$\langle Fx^n, x^n \rangle = \langle Ax_n - y, x_n \rangle > 0, \text{ for } x^n \in \partial D^n.$$

By Lemma 4.4, $Fx^n = 0$, for some $x^n \in D$. Hence, there exists $x_n \leftrightarrow x^n$ such that $x_n \in D_n$ and $P_n^* A x_n = P_n^* y$.

With this preparation, we apply Theorem 2.1 to the equations $Ax = y$ and $P_n^* A x_n = P_n^* y$.

THEOREM 4.7. *Assume A is bounded, continuous, and strongly monotone with constant α. Then $P_n^* A \xrightarrow{r} A$. Let $y \in E^*$ and $\gamma > \|AO - y\|/\alpha$. Then*

$$S_n = \{x_n \in E_n \colon P^* A x_n = P^* y, \|x_n\| \leqq \gamma\} \neq \emptyset,$$

for $n \in N$, $Ax = y$ has a unique solution x with $\|x\| \leqq \gamma$, and $S_n \to \{x\}$.

PROOF. By Lemma 4.3, $P^* A \xrightarrow{r} A$. By Lemmas 4.5 and 4.6, $S_n \neq \emptyset$ for $n \in N$. As mentioned before, A strongly monotone implies any solution of $Ax = y$ is unique. Therefore, Theorem 2.1 applies, with $S_n \neq \emptyset$ and $S = \{x\} \neq \emptyset$.

COROLLARY 4.8. *In Theorem 4.7, let $x_n \in S_n$ for $n \in N$. Then there exists c such that*

$$\|x_n - x\| \leqq c \|P_n x - x\|^{1/2} \to 0. \tag{4.10}$$

PROOF. From (4.3) and (4.5),

$$\alpha \|x_n - x\|^2 \leqq \langle Ax_n - Ax, x_n - x \rangle = \langle Ax_n - y, -x \rangle$$
$$= \langle Ax_n - y, P_n x - x \rangle \leqq \|Ax_n - y\| \, \|P_n x - x\| \leqq \beta \|P_n x - x\|,$$

for some $\beta < \infty$. Let $c = \sqrt{\beta/\alpha}$. Since $P_n \to I$, (4.10) follows.

5. Compact perturbations of strongly monotone operators. Let $K \colon X \to X^*$. By (2.7), (4.4) and $P_n^* \xrightarrow{c} I^*$,

$$P_n^* K \xrightarrow{ac} K \Leftrightarrow K \text{ compact, continuous.} \tag{5.1}$$

Now (2.9) and Lemma 4.3 yield

LEMMA 5.1. *If A is strongly monotone, bounded, and continuous, and if K is compact and continuous, then*

$$P_n^*(A - K) \xrightarrow{r} A - K.$$

Therefore, Theorem 2.1 relates solutions of equations

$$(A - K)x = y, \qquad P_n^*(A - K)x_n = P_n^* y.$$

The existence of solutions can be guaranteed in various ways. For example, If A is strongly monotone with constant α and if

$$\langle Kx_1 - Kx_2, x_1 - x_2 \rangle \leqq \beta \|x_1 - x_2\|^2, \tag{5.2}$$

with $\beta < \alpha$, then $A - K$ is strongly monotone with constant $\alpha - \beta$. Then Theorem 4.7 applies to $(A - K)x = y$ and $P_n^*(A - K)x_n = P_n^* y$.

An operator $K: X \to X^*$ is asymptotically linear if

$$\frac{\|Kx - K_\infty x\|}{\|x\|} \to 0 \text{ as } \|x\| \to \infty, \quad K_\infty \in L(X, X^*). \tag{5.3}$$

Equivalently, for any $\varepsilon > 0$, there is an $R < \infty$ such that

$$\|Kx - K_\infty x\| \leqq \varepsilon \|x\|, \text{ for } \|x\| \geqq R. \tag{5.4}$$

Then

$$\|Kx\| \leqq (\|K_\infty\| + \varepsilon)\|x\|, \text{ for } \|x\| \geqq R. \tag{5.5}$$

If $K \in L(X, X^*)$, then K is asymptotically linear with $K_\infty = K$.

An operator $K: X \to X^*$ is quasibounded [3] if there exist $\beta < \infty$ and $R < \infty$ such that

$$\|Kx\| \leqq \beta \|x\|, \text{ for } \|x\| \geqq R. \tag{5.6}$$

Then

$$\langle Kx, x \rangle \leqq \beta \|x\|^2, \text{ for } \|x\| \geqq R. \tag{5.7}$$

If K is asymptotically linear, then K is quasibounded with any $\beta > \|K_\infty\|$ in (5.6) and (5.7).

With this preparation, we give another comparison of equations $(A - K)x = y$ and $P_n^*(A - K)x_n = P_n^* y$.

THEOREM 5.2. *Assume A is bounded, continuous, and strongly monotone with constant α. Assume K is compact, continuous, and satisfies* (5.7) *with $\beta < \alpha$ and $R < \infty$. (For example, K is asymptotically linear or quasibounded.) Let $y \in X^*$, $\gamma \geqq R$, and $\gamma > \|AO - y\|/(\alpha - \beta)$. Define*

$$S = \{x \in X: (A - K)x = y, \|x\| \leqq \gamma\},$$
$$S_n = \{x_n \in E_n: P_n^*(A - K)x_n = P_n^* y, \|x_n\| \leqq \gamma\}.$$

Then $S_n \to S$, $S_n \neq \emptyset$, for $n \in N$ and $S \neq \emptyset$.

PROOF. By Lemma 5.1, $P_n^*(A - K) \xrightarrow{r} A - K$. From (4.5), (4.7), and (5.7),

$$\langle (A - K)x - y, x \rangle \geqq (\alpha - \beta)\|x\|^2 - \|AO - y\| \|x\| > 0,$$
$$\text{for } \|x\| \geqq \gamma.$$

Lemma 4.6 yields $S_n \neq \emptyset$, for $n \in N$. Theorem 2.1 implies that $S_n \to S$ and $S \neq \emptyset$.

Suppose that X is a real Hilbert space. As usual, identify X^* with X. Then A, K: $X \to X$. For example, let $A = I$, the identity operator on X. Then I is strongly monotone with $\alpha = 1$. Theorem 5.2 pertains to equations

$$(5.8) \qquad (I - K)x = y, \qquad P_n^*(I - K)x_n = P_n^* y,$$

where K is compact, continuous, and satisfies (5.7) with $\beta < 1$ and $R < \infty$, and the constant γ in the theorem satisfies $\gamma \geqq R$ and $\gamma > \|y\|/(1 - \beta)$.

6. Integral equations. We illustrate the theory with two well known integral equations in the real Hilbert space $L^2 = L^2[0, 1]$. For further details, see **[6]**.

EXAMPLE 6.1. Consider

$$(6.1) \qquad x(s) - \int_0^1 k(s, t) f(x(t), t) dt = y(s),$$

with typical hypotheses

$$(6.2) \qquad \|k\| = \left(\int_0^1 \int_0^1 k(s, t)^2 \, ds dt\right)^{1/2} < 1,$$

$$(6.3) \qquad f(s, t) \text{ continuous on } R \times [0, 1],$$

$$(6.4) \qquad |f(s, t) - s| \text{ bounded on } R \times [0, 1].$$

Express (6.1) in operator form:

$$(6.5) \qquad (I - KF)x = y;$$

$$(6.6) \qquad Kx(s) = \int_0^1 k(s, t) x(t) dt;$$

$$(6.7) \qquad Fx(t) = f(x(t), t).$$

Then K is a compact linear operator on L^2 with $\|K\| \leqq \|k\| < 1$. Also, F: $L^2 \to L^2$, F is continuous and asymptotically linear with $F_\infty = I$. It follows that KF is compact, continuous, asymptotically linear with $(KF)_\infty = K$, and KF satisfies (5.7) with $\|K\| < \beta < 1$. Therefore, Theorem 5.2 pertains to the equations

$$(6.8) \qquad (I - KF)x = y, \qquad P_n^*(I - KF)x_n = P_n^* y.$$

See (5.8) and the accompanying remarks. In particular, $(I - KF)x = y$ has a solution with $\|x\| \leqq \|y\|/(1 - \beta)$.

EXAMPLE 6.2. Consider a generalization of (6.1),

$$a(x(s), s) - \int_0^1 k(x, t)f(x(t), t)dt = y(s), \tag{6.9}$$

where $\|k\| < \infty$, f satisfies (6.3) and (6.4), and

$$a(x, s) \text{ is continuous on } R \times [0, 1], \tag{6.10}$$

$$|a(x, s)| \leqq c + d|x|, \quad c, d \text{ constants}, \tag{6.11}$$

$$\frac{\partial a}{\partial x} \geqq \alpha > 0 \quad \text{on } R \times [0, 1]. \tag{6.12}$$

Express (6.9) in operator form:

$$(A - KF)x = y; \tag{6.13}$$

$$Ax(s) = a(x(s), s). \tag{6.14}$$

Then A: $L^2 \rightarrow L^2$, A is continuous, bounded, and monotone with constant α. See [**6**, Chapter 1]. Assume that $\|K\| < \beta < \alpha$. Then Theorem 5.2 pertains to

$$(A - KF)x = y, \qquad P_n^*(A - KF)x_n = P_n^*y. \tag{6.15}$$

Thus, $(A - KF)x = y$ has a solution in a sufficiently large ball.

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