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A REMARK ON THE ENERGY OF HARMONIC MAPS BETWEEN SPHERES

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1. Introduction. A harmonic map between two Riemannian manifolds is a critical point of the energy integral. The conformal invariance of this integral in two dimensions makes this variational problem especially tractable. A fact that is special to two dimensions is that a harmonic map $\phi: S^2 \to S^2$ is energy minimizing among all C^2 maps homotopic to ϕ. Furthermore, it is well known that the energy of a harmonic map $\phi: S^2 \to S^2$ is given by

$$E(\phi) = \frac{1}{2}\int_{S^2} |d\phi|^2 dv = |\deg \phi| \mathrm{vol}(S^2). \tag{1}$$

In contrast to the two dimensional situation, Eells and Sampson **[2]** showed that any differentiable map $\phi: S^n \to S^n$ ($n \geqq 3$) of nonzero degree does not minimize energy within its homotopy class. It is then natural to ask if there exist stable, harmonic maps $\phi: S^n \to S^n$ when $n \geqq 3$. This question was answered in the negative by Y.L. Xin **[5]** who proved the following more general theorem.

THEOREM. (XIN). *If $n \geqq 3$, there exists no nonconstant, stable harmonic map from S^n to any Riemannian manifold.*

Xin proved this result by computing the second variation of the energy along the conformal vector fields of S^n. A conformal vector field on S^n is of the form $\nu = \mathrm{grad}(\lambda|_{S^n})$, where λ is a linear functional on $\mathbf{R}^{n+1}$. Let $\phi_t: S^n \to M$ be a one parameter variation of a harmonic map $\phi = \phi_0$ such that

$$\left.\frac{d\phi_t}{dt}\right|_{t=0} = \phi_*(\nu), \tag{2}$$

where ν is a conformal vector field on S^n. Xin proves

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$$\left.\frac{d^2E(\phi_t)}{dt^2}\right|_{t=0} = (2-n)\int_{S^n} \|\phi_*(\nu)\|^2 d\,\mathrm{vol}. \tag{3}$$

Xin's result follows easily from this.

In this note we study the energy of a harmonic map between higher dimensional spheres. We obtain information about the behavior of the energy functional along an orbit of a harmonic map under the action of the conformal group. A weak form of formula (1) is derived from this.

MAIN THEOREM. *Let $\phi: S^n \to S^p$ ($n \geqq 3$) be a non-nullhomotopic, harmonic map. Then*

$$E(\phi) = \max_{g\in G} \frac{1}{2}\int_{S^n} \|d(\phi\circ g)\|^2 d\,\mathrm{vol}, \tag{a}$$

where G is the group of orientation preserving, conformal diffeomorphisms of S^n.

(b) *Moreover, we have the estimate*

$$E(\phi) \geqq \frac{n}{2}\mathrm{vol}(S^n).$$

It would be interesting to improve this estimate in the case $n = p$ so that it takes into account the degree of ϕ.

A basic reference to the subject of harmonic maps is the survey of Eells and Lemaire [1]. The terminology used in this paper is explained there.

2. Preliminaries about the conformal group. It will be useful to recall the basic facts about the conformal group and hyperbolic geometry.

Denote by $\mathbf{L}^{n+2} \cong \mathbf{R} \times \mathbf{R}^{n+1}$ the Lorentz space with the nondegenerate bilinear form $\langle\, ,\rangle: \mathbf{L}^{n+2} \times \mathbf{L}^{n+2} \to \mathbf{R}$ defined by

$$\langle v, w\rangle = -v^0w^0 + \sum_{i=1}^{n+1} V^iw^i,$$

for $v, w \in \mathbf{L}^{n+2}$. The positive light cone of $\mathbf{L}^{n+2}$ is given by

$$C_+ = \{v \in \mathbf{L}^{n+2}: v^0 > 0 \text{ and } \langle v, v\rangle = 0\}.$$

Denote by $O(1, n+1)$ the group of linear transformations of $\mathbf{L}^{n+2}$ that preserve the bilinear form $\langle\, ,\rangle$. We will be interested in the subgroup $G \subseteq O(1, n+1)$ that preserves C_+ and the orientation of $\mathbf{L}^{n+2}$. It is well known that G is the identity component of $O(1, n+1)$.

The subgroup G can be identified with the group of conformal diffeomorphisms of S^n. We review this identification. The vectors in $\mathbf{L}^{n+2}$ will be written as column vectors. Define maps $q: S^n \to C_+$ and $p: C_+ \to S^n$ by

$$\xi \overset{q}{\mapsto} \begin{bmatrix} 1 \\ \xi \end{bmatrix}$$

and

$$\begin{bmatrix} v^0 \\ \vdots \\ v^{n+1} \end{bmatrix} \overset{p}{\mapsto} \begin{bmatrix} v^1/v^0 \\ \vdots \\ v^{n+1}/v^0 \end{bmatrix} \in S^n \subseteq \mathbf{R}^{n+1}.$$

The conformal diffeomorphism $\hat{g}\colon S^n \to S^n$ corresponding to a $g \in G$ is given by

$$\hat{g}(x) = p(g \cdot (q(x))).$$

We write the matrix of g in terms of the standard basis of $\mathbf{L}^{n+2} \cong \mathbf{R}^{n+2}$ as

$$\begin{bmatrix} \lambda & c_1 & \cdots & c_{n+1} \\ b^1 & a_1^1 & \cdots & a_{n+1}^1 \\ \vdots & \vdots & & \vdots \\ b^{n+1} & a_1^{n+1} & \cdots & a_{n+1}^{n+1} \end{bmatrix} = \begin{bmatrix} \lambda & c \\ b & a \end{bmatrix}.$$

The action $v \mapsto g \cdot v$, $v \in \mathbf{L}^{n+2}$ can be written in matrix form as follows.

$$\begin{bmatrix} v^0 \\ \vdots \\ v^{n+1} \end{bmatrix} \overset{g}{\mapsto} \begin{bmatrix} \lambda & c_1 & \cdots & c_{n+1} \\ b_1 & a_1^1 & \cdots & a_{n+1}^1 \\ \vdots & \vdots & & \vdots \\ b^{n+1} & a_1^{n+1} & \cdots & a_{n+1}^{n+1} \end{bmatrix} \begin{bmatrix} v^0 \\ \vdots \\ v^{n+1} \end{bmatrix}$$

Then

$$\hat{g}(x) = \begin{bmatrix} \left(\sum_{i=1}^{n+1} a_i^1 x^i + b^1\right) \Big/ \left(\sum_{i=1}^{n+1} c_i x^i + \lambda\right) \\ \vdots \\ \left(\sum_{i=1}^{n+1} a_i^{n+1} x^i + b^{n+1}\right) \Big/ \left(\sum_{i=1}^{n+1} c_i x^i + \lambda\right) \end{bmatrix}, \tag{4}$$

where the column vector $x \in S^n \subseteq \mathbf{R}^{n+1}$. Using these formulas one easily computes that

$$\hat{g}^*(ds^2_{S^n}) = \left(\frac{1}{\left(\sum_{i=1}^{n+1} c_i x^i\right) + \lambda} \right)^2 ds^2_{S^n} \tag{5}$$

and that the Jacobian is given by

$$J\hat{g}(x) = \left(\frac{1}{c \cdot x + \lambda} \right)^n. \tag{6}$$

We regard $SO(n+1)$ as a subgroup of G, the inclusion being given by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Also set $\mathbf{H}^{n+1} = \{(\lambda, c): \lambda^2 + c^t c = -1 \text{ and } \lambda > 0\}$. $\mathbf{H}^{n+1}$ is of course $n+1$ dimensional hyperbolic space. The projection, $\pi: G \to \mathbf{H}^{n+1}$, that takes a matrix in G to its first row, has as its fibres the right cosets of $SO(n+1)$. It is well known that G may be identified with the oriented, orthonormal frame bundle of $\mathbf{H}^{n+1}$.

3. The energy of harmonic maps between spheres. Let $C^2(S^n, S^m)$ be the class of C^2 functions between two spheres not necessarily of the same dimension. Define an action of $G \subseteq 0\,(1, n+1)$ on $C^2(S^n, S^m)$ by defining

$$g \cdot f = f \circ (\hat{g}^{-1}) \quad g \in G, f \in C^2(S^n, S^m).$$

The following lemma is well known. The proof is included for the sake of completeness.

LEMMA 1. *Fix $f \in C(S^n, S^p)$, where $n \geqq 3$. Given $\varepsilon > 0$, there exists a compact set $K \subseteq G$ such that $E(g \cdot f) < \varepsilon$ whenever $g \in G \backslash K$.*

PROOF. Using equation (6) and the definition of energy, we have

$$\begin{aligned} e(g \cdot f) &= e(f \circ \hat{g}^{-1})(x) \\ &= e(f)(\hat{g}^{-1}(x)) \frac{1}{(c \cdot x + \lambda)^2}, \end{aligned}$$

where $\pi(g^{-1}) = (\lambda, c)$. Let $M \geqq e(f)(p)\ \forall p \in S^n$. Then

$$\begin{aligned} E(g \cdot f) &= \int_{S^n} e(g \cdot f) dv \\ &\leqq C \int_{S^n} (c \cdot x + \lambda)^{-2} dv(x). \end{aligned} \tag{7}$$

Since the fibres of π are compact it is enough to show that the last integral in (7) tends to zero as $|c|^2 \to \infty$. Set $\rho = |c|$. Then $\lambda = (\rho^2 + 1)^{1/2}$. Also set $u = -a/|a| \in S^n$. Choose geodesic polar co-ordinates at u, say (r, θ). So $r \in [0, \pi)$ and $\theta \in S^{n-1} \subseteq \mathbf{R}^n$. The metric on S^n then has the form $dr^2 + \sin^2(r) d\theta^2$, where $d\theta^2$ denotes the standard metric on S^{n-1}. In terms of these co-ordinates we have

$$\int_{S^n} \frac{dv(x)}{(cx + \lambda)^2} = \mathrm{vol}(S^{n-1}) \int_0^\pi \frac{\sin^{n-1}(r)}{(\lambda - \rho \cos r)^2} dr. \tag{8}$$

It is enough to verify that the last integral tends to zero as $\rho \to \infty$. We split the integral up and use some elementary estimates.

$$\int_{\pi/2}^{\pi} \frac{\sin^{n-1} r\, dr}{(\lambda - \rho \cos r)^2} \leqq \frac{1}{\lambda^2} \int_{\pi/2}^{\pi} \sin^{n-1} r\, dr \tag{9}$$

Set $\varepsilon = \rho^{-1/2}$. Then, for C, C' independent of ε,

$$\begin{aligned}\int_0^\varepsilon \frac{\sin^{n-1} r}{(\lambda - \rho\cos r)^2}\,dr &= \int_0^\varepsilon \frac{\sin^{n-1} r(\lambda + \rho\cos r)dr}{(1+\rho^2\sin^2 r)^2} \\ &\leqq C\rho\int_0^\varepsilon \sin^{n-1} r dr \leqq C'\,\rho\cdot\rho^{-n/2} \\ &= C'_\rho{}^{(2-n)/2}.\end{aligned} \tag{10}$$

Also, for C'' and C''' independent of ρ,

$$\begin{aligned}\int_\varepsilon^{\pi/2} \frac{\sin^{n-1} r}{(\lambda - \rho\cos r)^2}\,dr &= \int_\varepsilon^{\pi/2} \frac{\sin^{n-1} r(\lambda + \rho\cos r)dr}{(1+\rho^2(\sin^2 r))^2} \\ &\leqq \frac{C''\rho}{(1+C''\rho)^2}\int_0^{\pi/2} \sin^{n-1} r\,dr.\end{aligned} \tag{11}$$

It follows that the integrals in (10), (11) and (12) all tend to zero as $\rho \to \infty$. The result is then clear.

The next lemma shows that the energy functional cannot be uniformly small on a G orbit.

LEMMA 2. *Let $f \in C^2(S^n, S^m)$ be a non-null homotopic map. Then*

$$\max_{g\in G} E(g\cdot f) \geqq \frac{n}{2}\operatorname{vol}(S^n). \tag{12}$$

PROOF. Let v be a constant vector on $\mathbf{R}^{n+1}$. Then the vector field $v - (v, x)x$, $x \in S^n$, generates a one parameter group of conformal diffeomorphisms $\phi_{v,t}: S^n \to S^n$. It is easy to check that, on any compact subset $K \subseteq S^n$ such that $-a/|a| \notin K$, $\phi_{v,t}(p) \to a/|a|$ uniformly as $t \to \infty$. Define a map $F: G \to \mathbf{R}^{m+1}$ *by*

$$F(g) = \int_{S^n} f\circ g\,dv.$$

We claim that $F^{-1}\{0\} \neq \{\ \}$. To see this, consider the composition $\Phi_t: S^n \to \mathbf{R}^{m+1}$ given by $\Phi_t(v) = F(\phi_{v,t})$. Note that $\lim_{t\to\infty}\Phi_t(v) = \lim_{t\to\infty}\int_{S^n} f\circ\phi_{v,t}\,dv = \operatorname{vol}(S^n)f(v)$, by the Lebesgue dominated convergence theorm. If $F^{-1}\{0\} = \phi$, Φ_t gives a null homotopy of $\operatorname{vol}(S^n)\cdot f$ as a map from S^n to $\mathbf{R}^{m+1}\backslash\{0\}$. This implies that $f: S^n \to S^m$ is null-homotopic, contradicting our hypothesis. Hence $F^{-1}\{0\} \neq \phi$. Suppose $g_0 \in F^{-1}\{0\}$. Then

$$\int_{S^n} f\circ g_0\,dv = 0 \in \mathbf{R}^{m+1}.$$

Computing the energy of $f_0 = f\circ g_0$ gives

$$E(f_0) = \frac{1}{2}\int_{S^n}|\nabla f_0|^2\,dv \geqq \frac{\lambda_1}{2}\int_{S^n}|f_0|^2\,dv = \frac{\lambda_1}{2}\operatorname{vol}(S^n),$$

where

$$\lambda_1 = \inf_{\substack{\psi \in C'(S^n) \\ \int \psi = 0}} \left(\int |\nabla \psi|^2 dv \right) \Big/ \left(\int |\psi|^2 dv \right)$$

is the first eigenvalue of S^n. The estimate follows, since $\lambda_1 = n$ for the n sphere.

REMARK. Similar arguments can be found in the literature. See the articles of Hersch [3], Yang and Yau [6], and Li and Yau [4].

The next lemma we need computes the tension field of a conformal diffeomorphism.

LEMMA 3. *Let $g \in G$ be given by the matrix*

$$\begin{bmatrix} \lambda & c_j \\ b^i & a^i_j \end{bmatrix}.$$

The tension field of the associated conformal diffeomorphism $\hat{g}: S^n \to S^n$ is given by

$$\tau(\hat{g}) = \frac{(m-2)}{(cx+\lambda)} d\hat{g}(V), \tag{13}$$

where V is the conformal vector field on S^n given by

$$V_x = {}^t c - (cx)x, \quad x \in S^n. \tag{14}$$

PROOF. We will write the elements of $S^n \subseteq \mathbf{R}^{n+1}$ as column vectors. Moreover, the row vector $c = (c_1, \ldots, c_{n+1})$, the first row of the matrix g, and a^i denotes the i^{th} row of the matrix (a^i_j). Set ${}^t(\delta^k) = (0, \ldots, 1, 0, \ldots, 0)$, where the 1 is in the k^{th} position. Finally, let ${}^t x = (x^1, \ldots, x^{n+1})$ be the standard co-ordinate functions of $\mathbf{R}^{n+1}$ restricted to S^n. One easily computes

$$\begin{aligned} \nabla x^k &= \delta^k - x^k x, \\ dx^k(V) &= c^k - (cx)x^k. \end{aligned} \tag{15}$$

Also recall that $\Delta x^k = n x^k$, $k = 1, \ldots, n+1$, where Δ is the Laplace Beltrami operator of S^n. The well-known formula (see [1]) for the tension field of maps into S^m gives

$$\tau(\hat{g}) = \Delta \hat{g} + 2e(\hat{g})\hat{g}. \tag{16}$$

Here the Laplacian of $\hat{g}$ is computed by regarding $\hat{g}: S^n \to S^n \subseteq \mathbf{R}^{n+1}$ as an $\mathbf{R}^{n+1}$ valued function. From equation (6), it follows that $e(\hat{g}) = (n/2)(cx+\lambda)^{-2}$. Using (15) and the formula for Δx^k, we have

$$\Delta(\hat{g}^i) = -\frac{n(a^i x)}{cx + \lambda} + \frac{n(a^i x + b^i)(cx)}{(cx + \lambda)^2} - 2\frac{a^i({}^t c) - (a^i x)(cx)}{(cx + \lambda)^2} + 2\frac{a^i x + b^i}{(cx + \lambda)^3}(|c|^2 - (cx)^2). \tag{17}$$

Since $g \in SO(1, n + 1)$, we have the relation $a^i \cdot {}^t c = \lambda b^i$. Using this fact and adding and subtracting $(2\lambda(a^i x))/(cx + \lambda)^2$ gives

$$\Delta(\hat{g}^i) = -(n - 2)\frac{a^i x}{cx + \lambda} + \frac{n(a^i x + b^i)(cx)}{(cx + \lambda)^2} - 2\lambda\frac{a^i x + b^i}{(cx + \lambda)^2} + 2\frac{a^i x + b^i}{(cx + \lambda)^3}(|c|^2 - (cx)^2). \tag{18}$$

Adding and subtracting $2\lambda(cx)(a^i x + b^i)/(cx + \lambda)^3$ and using the identity $|c|^2 = -1 + \lambda^2$ yields the formula

$$\Delta(\hat{g}^i) = -(n - 2)\frac{a^i x}{cx + \lambda} + (n - 2)\frac{a^i x + b^i}{(cx + \lambda)^2}(cx) - 2\frac{a^i x + b^i}{(cx + \lambda)^3}. \tag{19}$$

Substituting this into (16) and using the formula for $e(\hat{g})$ yields

$$\begin{aligned} \tau(\hat{g}^i) &= (n - 2)\left(-\frac{a^i x}{cx + \lambda} + \frac{(cx)(a^i x + b^i)}{(cx + \lambda)^2} + \frac{a^i x + b^i}{(cx + \lambda)^3}\right) \\ &= \frac{(n - 2)}{(cx + \lambda)}\left(\frac{(cx)b^i - (a^i x)\lambda}{(cx + \lambda)} + \frac{a^i x + b^i}{(cx + \lambda)^2}\right). \end{aligned} \tag{20}$$

It just remains to show that the term in braces is $d\hat{g}^i(V)$.

$$\begin{aligned} d\hat{g}^i(V) &= \left(\frac{a^i_j\, dx^j}{(cx + \lambda)} - \frac{(a^i x + b^i)}{(cx + \lambda)^2}(c_k dx^k)\right)(V) \\ &= \frac{a^i_j(c_j + x^j(cx))}{(cx + \lambda)} - \frac{(a^i x + b^i)(c \cdot {}^t c - (cx)^2)}{(cx + \lambda)^2} \\ &= \frac{\lambda b^i - (a^i x)(cx)}{(cx + \lambda)} - \frac{(a^i x + b^i)(\lambda^2 - (cx)^2 - 1)}{(cx + \lambda)^2} \\ &= \frac{\lambda b^i - (a^i x)(cx)}{(cx + \lambda)} - \frac{(a^i x + b^i)(\lambda - (cx))}{(cx + \lambda)} + \frac{a^i x + b^i}{(cx + \lambda)^2} \\ &= \frac{(cx)b^i - (a^i x)}{cx + \lambda} + \frac{a^i x + b^i}{(cx + \lambda)^2}. \end{aligned} \tag{21}$$

Our formula follows.

We are now ready to prove our main theorem.

PROOF OF MAIN THEOREM. Let $f \in C^2(S^n, S^m)$ be a harmonic map from a sphere of dimension greater than two that is not null homotopic. Let

$g \in G \setminus SO(n+1)$. The composition formula for the tension field (see [1] gives

$$\begin{aligned} \tau(f \circ \hat{g}) &= df(\tau(\hat{g})) + Tr\, \Delta\, df(d\hat{g}, d\hat{g}) \\ &= \frac{(n-2)}{cx+\lambda} df(V) + \frac{1}{cx+\lambda)^2} \tau(f) \circ g \\ &= (n-2) \frac{df(V)}{(cx+\lambda)} \end{aligned} \tag{22}$$

since f is harmonic. Let $\hat{g}_t$ be a one parameter family of conformal diffeomorphisms such that

$$\hat{g}_0 = \hat{g},$$
$$\left.\frac{\partial g_t}{\partial t}\right|_{t=0} = V.$$

Then

$$\left.\frac{d(f \circ \hat{g}_t)}{dt}\right|_{t=0} = df(V) \text{ and}$$

$$\left.\frac{dE(f \circ \hat{g}_t)}{dt}\right|_{t=0} = \int_{S^n} \frac{(n-2)}{(cx+\lambda)} \|df(V)\|^2\, dv. \tag{23}$$

Since $\lambda \geqq 1$, $(c \cdot x - \lambda) \geqq \lambda - |c| = \lambda - (\lambda^2 - 1)^{1/2} > 0$. Therefore the first variation computed in equation (23) is strictly positive, for every $g \in G \setminus SO(n+1)$. By Lemma 1, we know that there is an $f_0 \in \{f \circ g : g \in G\}$ such that $E(f_0) = \max_{g \in G} E(f \circ g)$. If $f_0 = f \circ g_0$ with $g_0 \notin SO(n+1)$, we have just argued that there would exist a one parameter variation of f_0 along conformal diffeomorphisms that would increase energy. This contradicts the choice of f_0. It follows that $g_0 \in SO(n+1)$ and that $E(f_0) = E(f)$. Part (a) of the main theorem then follows. Part (b) is then a consequence of Lemma 2.

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