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BOUNDEDNESS FOR BLOCH FUNCTIONS

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ABSTRACT. Two theorems concerning the boundedness for certain functions in the Bergman space for the unit disc are proven. Theorem 1. If f is in the Bergman space so that $|f(z)| \leqq m$ for all z in the crescent region bounded by $|z| < 1$ and $|z-x| < 1-x$, $0 < x \leqq 1/2$, then $|f(z)| \leqq m$ for all z in the unit disc. Theorem 2. If f is a Bloch function so that $\limsup_{z \to a} |f(z)| \leqq m$ for all but a finite number of a's in the boundary of the unit disc, then $f(z)$ is bounded on the unit disc.

Introduction. The Bergman p-space for the open unit disc Δ is the closure of the analytic functions in $L^p(\Delta, dA)$ where dA is area measure. In this paper the relationships between integrability and boundedness on Δ will be investigated. Let $A_p(\Delta)$ denote the Bergman p-space, $p \geqq 1$.

It is clear (maximum modulus theorem) that if $f \in A_1(\Delta)$ and f is bounded on the annular region bounded by $|z| = 1$ and $|z| = r$, $r < 1$, then f is bounded on Δ. However, for $f \in A_1(\Delta)$ and f bounded on the open crescent region bounded by $|z| = 1$ and $|z - x| = 1 - x$ for $0 < x \leqq 1/2$, it is not clear that $f(z)$ is bounded on Δ. This will be shown to be true as a corollary of a stronger result for crescent regions.

This result represents the interplay between the maximum modulus theorem and integrability. It is conjectured by the author that if $f \in A_1(\Delta)$ and $\limsup_{z \to a} |f(z)| < M$ for all but a finite number of points $a \in \partial\Delta$, then f is bounded. This conjecture will be shown to be true for the space of Bloch functions for Δ.

Notations & Definitions. Throughout this paper Δ will be used for the open unit disc and G will be the crescent region bounded by $|z| = 1$ and $|z - x| = 1 - x$ where $0 \leqq x \leqq 1/2$. Let $U = \Delta/\bar{G}$. Then ∂U is parametrized by $\Gamma(\theta) = x + (1 - x)e^{i\theta}$ where $0 \leqq \theta < 2\pi$. The closure of the analytic polynomials in the Bergman p-space for G will be denoted by $H_p(G)$. The standard Hardy p-space for the unit circle will be given by $H_p(\partial\Delta)$.

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DEFINITION 1.0. An analytic function f on Δ is a Bloch function provided

$$\sup_{z\in\Delta} |f'(z)|(1 - |z|^2) < \infty.$$

DEFINITION 2.0. An analytic function f on Δ is of Bounded Mean Oscillation with respect to area measure provided

$$\sup_{B(a,r)} \frac{1}{\pi r^2} \int_\Delta |f(z) - f(a)|\, dA < \infty,\ B(a, r) \subset \Delta.$$

DEFINITION 3.0. A meromorphic function f on Δ is normal provided

$$\sup_{z\in\Delta} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} < \infty.$$

Results.

THEOREM 1.20. *If $f(z) \in H_p(G)$, $p \geqq 1$ and $|f(z)| \leqq M$ for all $z \in G$, then the analytic extension of f to Δ is bounded on Δ.*

The proof of Theorem 1.20 relies on the following theorem for crescent regions by James Brennan.

THEOREM 1.0. [Thm 5:11, 3] *Let G be the crescent region bounded by $|z| = 1$ and $\Gamma(\theta)$. If $f \in L^p(G, dA)$, then the following are equivalent.*

(1) *$f \in H_p(G)$.*

(2) *f can be extended analytically to a function $\hat{f}$ so that $\hat{f}Q^{1/p} \in H_p(\partial U)$ where ∂U is parametrized by $\Gamma(\theta)$ and Q is an outer function in $H_p(\partial U)$.*

The outer function $Q(z)$ has the property that for $z \in \partial U$, $|Q(z)|$ is bounded equivalent to $\delta(z) = \text{dist}(z, \partial\Delta)$. Since $\delta(z) = 1 - |z|$ for $z \in \partial U$ and

$$1/2(1 - |z|^2) \leqq 1 - |z| \leqq 1 - |z|^2,$$

an outer function whose radial limits agree in modulus with $1 - |z|^2$ will be acceptable as $Q(z)$. It is a direct calculation to show that

$$Q(z) = \frac{x}{1 - x}(1 - z)^2.$$

PROOF OF THEOREM 1.20. It suffices to prove this theorem in the case $p = 1$. Let $f \in H_1(G) \cap A_\infty(G)$ and $\hat{f}$ be the analytic extension of f to Δ. By Theorem 1.0, $(x/1 - x)(1 - z)^2\, \hat{f}(z) \in H_1(\partial U)$. By Theorem 10.1 [9],

$$F(w) = x(1 - x)^2 (1 - w)^2 \hat{f}(x + (1 - x)w) \in H_1(\partial\Delta).$$

Since $\hat{f}(z)$ is analytic on $\Gamma(\theta)$, $0 < \theta < 2\pi$, the radial limits of $F(w)$ are $x(1 - x)^2 (1 - e^{i\theta})^2 \hat{f}(x + (1 - x)e^{i\theta})$ for $0 < \theta < 2\pi$. Since $|\hat{f}(z)| \leqq K$

in G, and $\hat{f}(z)$ is continuous on $\Gamma(\theta)$, $0 < \theta < 2\pi$, $|\hat{f}(x + (1 - x)e^{i\theta})| \leqq K$ for $0 \neq \theta \neq 2\pi$.

By the factorization theorem for $H_1(\partial\Delta)$, there are functions $B(w)$, $S(w)$, and $O(w)$ where $B(w)$ is a Blaschke function, $S(w)$, is a singular inner function, and $O(w)$ is an outer function so that

$$F(w) = B(w)\, S(w)\, O(w) \text{ for } w \in \Delta.$$

Now, for $0 < \theta < 2\pi$, $x(1 - x)^2\, |1 - e^{i\theta}|^2\, |\hat{f}(x + (1 - x)e^{i\theta})| = |O(e^{i\theta})|$. Therefore, $|O(e^{i\theta})| \leqq x(1 - x)^2\, |1 - e^{i\theta}|^2\, K$ a.e.. Since $x(1 - x)^2\, K(1 - w)^2$ is outer, $|O(w)| \leqq x(1 - x)^2\, K(1 - w)^2$ for all $w \in \Delta$. It follows that

$$\begin{aligned} |1 - w|^2\, x(1 - x)^2\, |\hat{f}(x + (1 - x)w)| &= |B(w)|\, |S(w)|\, |O(w| \\ &\leqq |O(w)| \leqq x(1 - x)^2\, K|1 - w|^2. \end{aligned}$$

Hence $|\hat{f}(x + (1 - x)w| \leqq K$ for all $w \in \Delta$. Therefore $|\hat{f}(z)| \leqq K$ for all $z \in U$. Thus $\hat{f}(z)$ is bounded.

The condition that f be in $H_1(G)$ cannot be relaxed to f being analytic in the unit disc, since

$$f(z) = e^{1+z/1-z} \in A_\infty(G);$$

but this function is not bounded in Δ.

The conjecture for the Bergman space now seems plausible. As the proof of Theorem 1.20 shows, the bound for f is independent of x. Thus it appears that the Maximum Modulus Theorem could possibly be relaxed at least for a finite number of points on $\partial\Delta$.

Now we turn our attention to the proof of the conjecture for the space of Bloch functions. Cima and Graham **[6]** have shown that Bloch and $BMO_a(\Delta)$ are equivalent Banach spaces. Since $BMO_a(\Delta) \subset A_1(\Delta)$, Bloch $\subset A_1(\Delta)$.

The proof involves the Lehto-Virtanen maximum principle **[2]**, **[10]**, and some theorems of Anderson, Clunie, and Pommerenke **[2]** which are now presented.

Lehto-Virtanen maximum principle. Let $f(z)$ be meromorphic in Δ and

$$(1 - |z|^2)\frac{|f'(z)|}{1 + |f(z)|^2} < \infty.$$

Let D be an open disc so that ∂D and $\partial\Delta$ intersect in an angle β. Let G be a domain such that $\bar{G} \subset \Delta \cap D$. We suppose further that, for $z \in \partial G/B$, $|f(z)| \leqq \delta < \delta_0(\alpha, \beta)$ where

$$\delta_0(\alpha,\beta) = \frac{\sin(\beta)}{\alpha\beta}\left(1 + \left(1 + \left(\frac{\alpha\beta}{\sin(\beta)}\right)^2\right)^{1/2}\right)\exp - \left(1 + \left(\frac{\alpha\beta}{\sin(\beta)}\right)^2\right)^{1/2}.$$

Then $f(z)$ is analytic in G and $|f(z)| \leqq \eta(\delta, \alpha, \beta)$ where $\eta = \eta(\delta, \alpha, \beta)$ is the smallest positive solution of

$$\delta = \eta \exp\left(-\frac{\alpha\beta}{2\sin(\beta)}\left(\eta + \frac{1}{\eta}\right)\right).$$

The following theorem is by Anderson, Clunie, and Pommerenke, [**2**, Theorem 4.2].

THEOREM 2.40. *Let* $f \in \mathscr{B}$ (Bloch) *and* Γ *be an arc ending at* $e^{i\theta}$. *Let* $A \subset \mathbf{C}$. *If for* $z \in \Gamma$, $\lim_{z \to e^{i\theta}} \operatorname{dist}(f(z), A) = 0$ *then, for some absolute constant* K_1,

$$\limsup_{r \to 1} \operatorname{dist}(f(re^{i\theta}), A) \leqq K_1 \|f\|_{\mathscr{B}}$$

where $\|f\|_{\mathscr{B}}$ *is the Bloch norm of* f.

The constant K_1 comes from the Lehto-Virtanen maximum principle and depends on the angle β and is independent of f. Thus the phrase absolute constant is used.

In Anderson, Clunie, and Pommerenke's proof of this theorem, the existence of certain domains were given without proof. In the next theorem, it will be necessary to understand these domains more fully. The next lemma indicates how the needed domains can be constructed.

LEMMA 2.50. *Let* $\{r_n\}$ *be a sequence of real numbers in* Δ *so that* $\lim_{n\to\infty} r_n = 1$. *Then there exists a sequence of discs,* D_n, *so that*;

(a) $r_n \in D_n$,
(b) $1 \notin \overline{D_n}$,
(c) *The circular angle formed by* $\partial\Delta$ *and* ∂D_n *makes a constant angle of* $3\pi/4$,
(d) diameter $(D_n) \to 0$ as $n \to \infty$, *and*
(e) $D_n \cap \mathbf{C}/\Delta \neq \varnothing$.

PROOF. If we allow $1 \in \overline{D_n}$ then D_n could be defined to be the disc centered at $(r_n, 1 - r_n)$ with radius $2(1 - r_n)^{1/2}$ (See Figure 4.1).

To see that D_n can be chosen as in the lemma, let $0 < \delta < \pi/4$ be an angle. Then

$$\ell(s) = s(\cos(3\pi/4 + \delta), \sin(3\pi/4 + \delta)) + (\cos(\delta), \sin(\delta))$$

is a unit speed parametrization of a line through the point $(\cos(\delta), \sin(\delta))$. This line has the property that any circle centered on this line, passing through the point $(\cos(\delta), \sin(\delta))$ makes a circular angle of $3\pi/4$ with $\partial\Delta$ (See Figure 4.2).

Let $\rho_n = \ell(\sqrt{2}(1 - r_n))$. Since ℓ is parametrized with respect to arc length, $|\rho_n - (\cos(\delta), \sin(\delta))| = \sqrt{2}(1 - r_n)$.

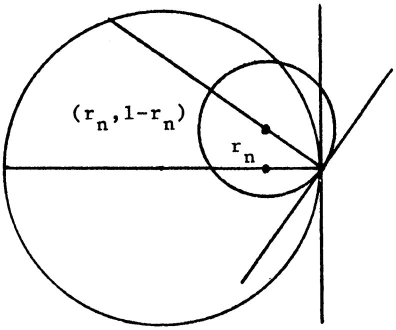

FIGURE 4.1

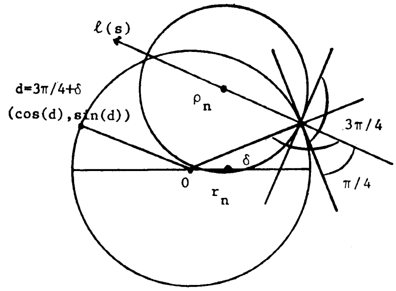

FIGURE 4.2

Let $D_n(\delta)$ be the disc centered at ρ_n whose radius is $\sqrt{2}(1 - r_n)$. It follows from elementary geometry that the circular angle formed by $\partial D_n(\delta)$ and $\partial \Delta$ is $3\pi/4$ (See Figure 4.2). Clearly,

$$|\rho_n - 1| > |\rho_n - (\cos(\delta), \sin(\delta))| = \sqrt{2}(1 - r_n).$$

Thus $1 \notin \overline{D_n}$.

It is straightforward to show that there is a $\delta_1 > 0$ so that

$$|\sqrt{2}(1-r_n)(\cos(3\pi/4 + \delta), \sin(3\pi/4 + \delta)) + (\cos\delta, \sin\delta) - (r_n, 0)|^2 \leqq 2(1 - r_n)^2$$

whenever $\delta < \delta_1$. It is also obvious that diameter $(D_n) \to 0$ and that $D_n \cap (\mathbf{C}/\bar{\Delta}) \neq \varnothing$.

The next theorem is a generalization of Theorem 2.40.

THEOREM 2.51. *Let* $R = \Delta \cap \{z \mid \operatorname{Im}(z) > 0\}$. *Let* $f \in \mathscr{B}$ *so that* $\limsup_{r\to1} |f(r)| < \infty$, *and let* U *be an open connected set so that*

$$1 \in \bar{U} \text{ and } \bar{U}/\{1\} \subset R.$$

Further, let $A = f(\bar{U}/\{1\})$. *Then, for an absolute constant* K_1,

$$\limsup_{r\to1} \operatorname{dist}(f(r), A) \leqq K_1 \|f\|_{\mathscr{B}}.$$

PROOF. Let $\beta = 3\pi/4$ and choose α so small that $\delta_0(\alpha, \beta) \geqq 1$ where δ_0 is defined in the Lehto-Virtanen maximum principle. Let $K_1 = 3/\alpha$ be the absolute constant for the theorem. We assume without loss of generality that $\|f\|_{\mathscr{B}} \leqq 1$.

Suppose the conclusion is false. Then, since $\limsup_{r\to1} |f(r)| < \infty$, there is a point w_0 and a sequence of real numbers, $\{r_n\}$, in Δ so that as $r_n \to 1$, $f(r_n) \to w_0$, and $\mathrm{dist}(w_0, A) > K_1$. By Lemma 2.50, there are discs D_n such that ∂D_n makes a circular angle of $3\pi/4$ with $\partial\Delta$, $r_n \in D_n$, $1 \notin \overline{D_n}$, and diameter $(D_n) \to 0$ as $n \to \infty$.

The subset U is connected. Thus $\bar{U}$ is connected. Let B_n denote the boundary of D_n. Further, let A_n and C_n denote the two subarcs of B_n which are contained in R (See Figure 4.3).

Since the diameter$(D_n) \to 0$ and U is connected, there exists an N such that if $n \geqq N$, then $A_n \cap U \neq \emptyset$ and $C_n \cap U \neq \emptyset$. Let G_n be the component of $(\mathbf{C}/\bar{U}) \cap D_n$ which contains r_n. Now G_n is a component of an open set and therefore is open. Clearly $\bar{G}_n \subset \bar{D}_n$. Thus $1 \notin \partial G_n \subset \bar{G}_n$. Let $z \in \partial G_n$. It is straightforward to prove that $z \in B_n \cup (\bar{U}/\{1\})$ and thus $\partial G_n \subset B_n \cup (\bar{U}/\{1\})$.

Now suppose $\bar{G}_n \not\subset \Delta$. So there is a $q \in \partial\Delta \cap \partial G_n$. Thus $q \neq 1$. Since $\bar{U}/\{1\} \subset R$, there is a ball, $B(q, r)$, such that $B(q, r) \cap \bar{U} = \emptyset$. Since $q \in \partial G_n$ there is a $q_1 \in B(q, r) \cap G_n$. The subset G_n is open and connected, and therefore is arcwise connected. Thus there is an arc, Γ, in G_n from r_n to q_1. By the construction of D_n, $R \cap (\mathbf{C}/\bar{D}_n)$ has two components in R. Let E_1 be the component whose closure contains A_n, and E_2 be the component whose boundary contains C_n. Since $A_n \cap U \neq \emptyset$ and $C_n \cap U \neq \emptyset$, there exist points t_1 and s_1 such that $t_1 \neq 1$, $s_1 \neq 1$, $t_1 \in U \cap E_1$, and $s_1 \in U \cap E_2$. The subset U is also arcwise connected. Thus there exists an arc, $\Gamma_1(x)$, defined on $[0, 1]$ such that $\Gamma_1(0) = t_1$, $\Gamma_1(1) = s_1$, and $\Gamma_1 \subset U$. Let $x_1 = \sup\{x \in [0, 1] \mid \Gamma_1(x) \in A_n\}$ and $x_2 = \inf\{x \in [x_1, 1] \mid \Gamma_1(x) \in C_n\}$. Clearly $\Gamma_1(1) \notin A_n$. Thus $x_1 \neq 1$. Since $x_1 = \sup\{x \in [0, 1] \mid \Gamma_1(x) \in A_n\}$, $\Gamma_1(x_1) \in A_n$. Similarly $\Gamma_1(x_2) \in C_n$. Thus $x_1 \neq x_2$. Let J be the jordan curve formed by Γ_1 on $[x_1, x_2]$ and B_n so that r_n is in the interior of J. Since r_n is in the interior of J and q_1 is in the exterior of J, $\Gamma \cap \Gamma_1 \neq \emptyset$. But this is impossible since $\Gamma \subset G_n \subset D_n \cap (\mathbf{C}/\bar{U})$ and $\Gamma_1 \subset U$. Hence $\partial G_n \cap \partial\Delta = \emptyset$. Therefore $\bar{G}_n \subset \Delta$.

Define $g(z) = (\alpha(f(z) - w_0))^{-1}$ for $z \in \Delta$, $f(z) \neq w_0$. Thus $g(z)$ is meromorphic in Δ and

$$\frac{(1 - |z|^2)\,|g'(z)|}{1 + |g(z)|^2} = \frac{(1 - |z|^2)\,|f'(z)|\,\alpha^3\,|f(z) - w_0|^2}{\alpha^2|f(z) - w_0|^2}$$
$$= (1 - |z|^2)\,|(f'(z)|\,\alpha \leqq \alpha\|f\|_{\mathscr{B}} \leqq \alpha.$$

Let $z \in \partial G_n/B_n$. Then by the previous arguments $z \in \bar{U}/\{1\}$. Thus $f(z) \in A$. Since $\mathrm{dist}(w_0, A) > K_1$,

$$|g(z)| = \frac{1}{\alpha|f(z) - w_0|} \leqq \frac{1}{\alpha K_1} = \frac{1}{\alpha(3/\alpha)} = 1/3 < 1/2.$$

Since $\delta_0(\alpha, \beta) \geqq 1$, the Lehto-Virtanen maximum principle yields that g

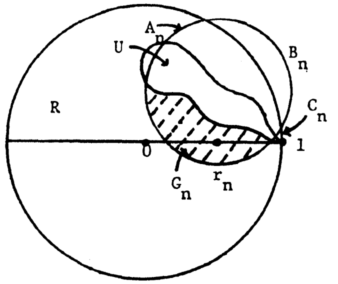

FIGURE 4.3

is uniformly bounded in G_n by $\eta(\delta, \alpha, \beta)$ which is independent of n. But this is impossible since $|g(r_n)| \to \infty$ as $f(r_n) \to w_0$. Hence,

$$\limsup_{r\to 1} \operatorname{dist}(f(r), A) \leqq K_1 \|f\|_{\mathscr{B}}.$$

THEOREM 2.60. *If* $f \in \mathscr{B}$ *and* $K > 0$ *so that* $\limsup_{z\to a} |f(z)| \leqq K$ *for* $a \in \partial\Delta/\{1\}$, *then* $\limsup_{r\to 1} |f(r)|$ *is bounded.*

PROOF. We begin the proof with the construction of an arc in Δ ending at 1. Let $\{p_n\}$ be a sequence of points in $\partial\Delta \cap \{z \mid \operatorname{Im}(z) > 0\}$ so that $p_n \to 1$ as $n \to \infty$. Further, let p_n be chosen so that $\pi/2 > \arg(p_n) > \arg(p_{n+1})$. Let $p_0 = i$.

Since $\limsup_{z\to a} |f(z)| \leqq K$ for $a \in \partial\Delta/\{1\}$, for each $a \in \partial\Delta$ such that $\arg(p_n) \geqq \arg(a) \geqq \arg(p_{n+1})$, there is a disc, $D_n(a)$, centered at a so that $|f(z)| < K + 1$ for $z \in D_n(a)$. Since for $n \geqq 0$, $C_n = \{a | a \in \partial\Delta, \arg(p_n) \geqq \arg(a) \geqq \arg(p_{n+1})\}$ is compact, there is a finite subset $a_1, \ldots, a_{R_n}$ so that

$$\bigcup_{i=1}^{R_n} D_n(a_i) \subset C_n.$$

Clearly there is an $r_n < 1/2^n$ so that

$$A_n = \{z \mid \arg(p_n) \geqq \arg(z) \geqq \arg(p_{n+1}),\ 1 - |z| \leqq r_n\}$$

is a subset of the union of the $D_n(a_i)$, $1 \leqq i \leqq R_n$. Let $\Gamma_n = \{z \in A_n | 1 - |z| = r_n\}$. Let $q_0 = (1 - r_0)i$ and let $q_n = (1 - r_{n-1})e^{i\arg(p_n)}$ for $n \geqq 1$. Finally, for $n \geqq 1$, let β_n be the line segment from q_n to s_n (See Figure 4.5). Then, for $n \geqq 0$, $\Gamma_n \cup \beta_{n+1}$ is the image of an arc from q_n to s_{n+1} which is parametrized by arc length. Since $\operatorname{length}(\beta_n) \leqq 1/2^n$ and

$$\sum_{n=0}^{\infty} \operatorname{length}(\Gamma_n) \leqq \pi/2, \text{ then } \sum_{n=0}^{\infty} \operatorname{Length}(\Gamma_n \cup \beta_{n+1}) < \infty.$$

Let $\alpha = \sum_{n=0}^{\infty} \operatorname{length}(\Gamma_n \cup \beta_{n+1})$ and $\delta_n = \operatorname{length}(\Gamma_n \cup \beta_{n+1})$. Let, for $n \geqq 0$,

$$\gamma_n: \left[\sum_{i=0}^{n-1} \delta_i, \sum_{i=0}^{n} \delta_i\right] \to \Gamma_n \cup \beta_{n+1}$$

be a unit speed parametrization of $\Gamma_n \cup \beta_{n+1}$, (When $n = 0$ define the sum to be zero). Define

$$\gamma: [0, \alpha] \to \bigcup_{n=0}^{\infty} (\Gamma_n \cup \beta_{n+1}) \cup \{1\} \text{ by}$$

$$\gamma(t) = \begin{cases} \gamma_n(t) & \text{if } t \in \left[\sum_{i=0}^{n-1} \delta_i, \sum_{i=0}^{n} \delta_i\right]. \\ 1 & \text{if } t = \alpha \end{cases}$$

The arc $\gamma(t)$ is a homeomorphism.

Let $A = f(\Gamma/\{1\})$. Since $|f(z)| < K + 1$ for $z \in \Gamma/\{1\}$, A is bounded. Clearly, $\lim_{z\to 1} \text{dist}(f(z), A) = 0$ for $z \in \Gamma$. Thus by Theorem 2.40, with $\beta = 3\pi/4$, there is an absolute constant K_1 depending only on $3\pi/4$ so that

$$\limsup_{r\to 1} \text{dist}(f(r), A) \leqq K_1 \|f\|_{\mathscr{B}}.$$

It follows that there is a $\delta > 0$ so that if $1 - \delta < r < 1$, then $\text{dist}(f(r), A) < K_1 \|f\|_{\mathscr{B}} + 1$. So for any r, $1 - \delta < r < 1$, there is an $a(r) \in A$ such that $|f(r) - a(r)| < K_1 \|f\|_{\mathscr{B}} + 1$. Therefore, $|f(r)| < K_1\|f\|_{\mathscr{B}} + 1 + |a(r)| \leqq K_1 \|f\|_{\mathscr{B}} + K + 2$. Hence $\limsup_{r\to 1} |f(r)|$ is bounded by $K_1 \|f\|_{\mathscr{B}} + K + 2$.

COROLLARY 2.62. *If $f \in \mathscr{B}$ and $\limsup_{z\to a} |f(z)| \leqq K$ for $a \in \partial\Delta/\{1\}$, then there is a constant $M > 0$ so that $|f(r)| \leqq M$ for $r \in (-1, 1)$.*

PROOF. By Theorem 2.60, $|f(r)|$ is bounded near 1. Since $\limsup_{r\to -1} |f(z)| < K + 1$ and f is analytic on $(-1, 1)$, the bound, M, exists.

THEOREM 2.70. *If $f \in \mathscr{B}$ and $\limsup_{z\to a} |f(z)| \leqq K$ for $a \in \partial\Delta/\{1\}$, then $f(z)$ is bounded on the unit disc.*

PROOF. Suppose the conclusion is false. Then there exists a sequence

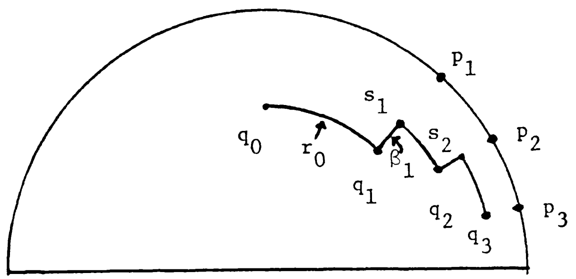

FIGURE 4.4

of points, $\{z_n\}$, contained in Δ so that $|f(z_n)| \to \infty$ as $n \to \infty$. By Theorem 2.40 and Corollary 2.62, there is a constant M_1 so that $|f(r)| \leqq M_1$ for $r \in (-1, 1)$.

Let $M > \max\{M_1 + 2 + K_1 \|f\|_{\mathscr{B}}, K + 1\}$ where K_1 is the absolute constant of Theorem 2.51. Let $B(O, M)$ be the ball centered at zero with radius M.

Since $|f(r)| \leqq M_1$ for $r \in (-1, 1)$,

$$\operatorname{dist}(f(r), \mathbf{C}/\overline{B(O, M)}) > K_1 \|f\|_{\mathscr{B}}.$$

Let $U = f^{-1}(\mathbf{C}/\overline{B(O, M)})$. Since $|f(z_n)| \to \infty$, $U \neq \emptyset$. Without loss of generality assume that $U \subset R$ where $R = \{z \mid \operatorname{Im}(z) > 0\} \cap \Delta$. This can be done since either $\Delta \cap \{z \mid \operatorname{Im}(z) > 0\}$ or $\Delta \cap \{z \mid \operatorname{Im}(z) < 0\}$ contains an infinite number of the z_n's.

Let V be the component of U so that there is a $z_0 \in V$ such that $|f(z_0)| > M$. Since $\limsup_{z\to a} |f(z)| \leqq K$ for $a \in \partial\Delta/\{1\}$, $\partial V \cap (\partial\Delta/\{1\}) = \emptyset$. Similarly since $M > M_1$, $\partial V \cap [-1, 1) = \emptyset$.

Now I claim $1 \in \bar{V}$. Suppose 1 is not in $\bar{V}$. Then $\bar{V} \subset R$. Since f is analytic on Δ, f assumes its maximum on V at a point $\rho \in \partial V$. Since $1 \notin \bar{V}$, $\rho \neq 1$. Thus f is analytic at ρ. Therefore there is a ball, $B(\rho, \varepsilon)$, so that $B(\rho, \varepsilon) \subset R$ and $|f(w)| > M$ for $w \in B(\rho, \varepsilon)$. Since $\rho \in \partial V$, $B(\rho, \varepsilon) \cap V \neq \emptyset$. Thus $B(\rho, \varepsilon) \cup V$ is connected and open. But this is impossible since $B(\rho, \varepsilon) \cup V$ properly contains V and V was a component of $f^{-1}(\mathbf{C}/\overline{B(O,M)})$. Thus $1 \in \bar{V}$.

It now follows from Theorem 2.51 that

$$\limsup_{r\to 1} \operatorname{dist}(f(r), f(\bar{V}/\{1\})) \leqq \|f\|_{\mathscr{B}} K_1.$$

But this is impossible since $\operatorname{dist}(f(r), \mathbf{C}/\overline{B(O, M)}) > K_1\|f\|_{\mathscr{B}}$. Hence f is bounded on Δ.

THEOREM 2.80. *Let E be a finite subset of $\partial\Delta$. If $f \in \mathscr{B}$ and* $\limsup_{z\to a} |f(z)| \leqq K$ *for $a \in \partial\Delta/E$, then f is bounded on Δ.*

The proof is similar to that of Theorem 2.70.

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