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A LIMIT ON THE LENGTH OF THE INDECOMPOSABLE MODULES OVER A HEREDITARY ALGEBRA

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Introduction. Let k be an algebraically closed field, let Λ be a hereditary k-algebra of finite type, that is, there is only a finite number of nonisomorphic indecomposable Λ-modules. Let K be an indecomposable Λ-module, let Soc K be the socle of K, $P(\text{Soc } K)$ the projective cover of Soc K, and $I(K)$ the injective envelope of K. If X is a module, let $\ell(X)$ denote the composition length of X. In this paper we show that the inequality

$$(1) \qquad \ell(P(\text{Soc } K)) + \ell(I(K)) - \ell(K) \leqq \ell(M)$$

is always true, where M is an indecomposable Λ-module of maximal length.

It is known that if $p(\Lambda)$ is the number of nonempty preprojective classes in the preprojective partition for Λ, then $p(\Lambda) = \ell(M)$, where M is as above [5]. We will apply the inequality (1) above to prove that if R is the trivial extension of Λ by $D\Lambda$, then we have

$$(2) \qquad p(R) - 1 \leqq \ell(X_0) \leqq p(R)$$

where X_0 is an indecomposable R-module of maximal length.

To show this, we first apply (1) to show that if X is an indecomposable R-module, then

$$(3) \qquad \begin{aligned} &\text{(i) } \ell(X) \leqq 1 + \ell(M), \text{ if } X \text{ is projective} \\ &\text{(ii) } \ell(X) \leqq \ell(M), \text{ if } X \text{ is not projective.} \end{aligned}$$

Using (3) and the fact that $p(R) = p(\Lambda) + 1$, we get that (2) is always satisfied.

In the last section we give some examples to illustrate what may happen if the algebra is not hereditary. First we give an example of an algebra where the inequality (1) is not satisfied by all indecomposable modules, not even by all simple modules.

As a second example, we present an algebra Λ which is not hereditary, but the inequality (1) is satisfied for all indecomposable Λ-modules, and

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if we let T be the trivial extension of Λ by $D\Lambda$, then the inequality (3) is not satisfied for all indecomposable T-modules.

The third example illustrates the fact that there are artin algebras for which the left part of inequality (2) above is not satisfied. It would be interesting to know whether the right part is generally true.

1. Statements and proofs. The main aim of this section is to prove the following

THEOREM A. *Let Λ be a hereditary algebra of finite type over an algebraically closed field. Let K be an indecomposable Λ-module and M an indecomposable Λ-module of maximal length. Then the inequality*

$$(1) \qquad \ell(P(\mathrm{Soc}\ K)) + \ell(I(K)) - \ell(K) \leqq \ell(M)$$

is always true.

We know that since Λ is hereditary of finite type over an algebraically closed field, then mod Λ, the category of finitely generated Λ-modules, is equivalent to the category of representations of a Dynkin diagram of type A_n, D_n, $n \geqq 4$, E_6, E_7 or E_8 (see [2] for description of Dynkin diagrams). We will prove the theorem by treating each of these cases separately. But before we start proving this theorem, we will first look at the special case when K is a simple Λ-module. In this case we can prove that there is an indecomposable Λ-module which has the same composition factors as the module $P(\mathrm{Soc}\ K) \amalg I(K)/K$. This is true even if Λ is hereditary of infinite type. But we point out that it is easy to find examples where Λ is hereditary of finite type, K is an indecomposable Λ-module, and such that there is no indecomposable module which has the same composition factors as $P(\mathrm{Soc}\ K) \amalg I(K)/K$.

If Γ is an artin algebra, we recall that a Γ-module N is said to have a waist if there is a nontrivial proper submodule N' of N such that every submodule of N contains or is contained in N'. N' is then a waist in N. It is easy to see that if N has a waist, then N is an indecomposable Γ-module, see [1] for details.

We also recall from [1] the definition of a pasted module. Let $A \subseteq B$ and $C \subseteq D$ be Γ-modules, and suppose that there is an isomorphism $\alpha\colon B/A \to C$. If there is a module X with $B \subseteq X$ and an isomorphism $\beta\colon X/A \to D$ such that the diagram below commutes,

$$\begin{array}{ccc} X/A & \xrightarrow{\ \simeq\ } & D \\ \text{inc} & & \\ B/A & \underset{\alpha}{\xrightarrow{\ \sim\ }} & C \end{array}$$

then we say that we can paste B and D by α, and we call X the pasted module. It is shown in [**1**, Corollary 3.2] that if Γ is hereditary, we can always paste modules. We are now in a position to prove the following proposition.

PROPOSITION 1. *Let Λ be an hereditary artin algebra. Let S_i be a simple Λ-module, P_i the projective cover of S_i, and I_i the injective envelope of S_i. Then there is a Λ-module X_i which is simple or has a waist, such that $P_i \subseteq X_i$ and $X_i/rP_i \cong I_i$.* Consequently, we have $\ell(X_i) = \ell(I_i) + \ell(P_i) - \ell(S_i)$.

PROOF. We have $rP_i \subseteq P_i$ and $S_i \subseteq I_i$. Further, $P_i/rP_i \cong S_i$. That is, if we put $rP_i = A$, $P_i = B$, $S_i = C$ and $I_i = D$, we have the situation described above. Since Λ is hereditary, there is a pasted module X_i with $P_i \subseteq X_i$ and an isomorphism $\beta: X_i/rP_i \to I_i$. In [**1**, Corollary 3.4] it is shown that if we have modules M, N such that M/rM is simple, and there is an isomorphism $\alpha: M/rM \to \operatorname{Soc} N$, then if we can paste M and N by α, the pasted module has a waist or is simple. (If X_i is simple, then both P_i and I_i are simple.)

We will now prove Theorem A above.

(I) *Proof of the theorem if Λ is of type A_n.* If Λ is of type A_n, then there is up to isomorphism only one indecomposable Λ-module M of maximal length, and M has exactly one copy of each simple module as compositon factors [**2**], therefore $\ell(M) = n$. The following two lemmas are trivial consequences of the description of representations of A_n-diagrams in [**2**, Satz 2.2].

LEMMA (A-i). *If K is an indecomposable Λ-module, then K has at most one copy of each simple module as composition factors. Therefore,* Soc *K has no more than one copy of any simple module as a summand.*

LEMMA (A-ii). *If we number the vertices of the A_n-diagram in the obvious way $\underset{1}{\bullet}-\underset{2}{\bullet}\ \bullet\ \bullet\ \bullet-\underset{n}{\bullet}$, then if K is an indecomposable Λ-module and there are i, j with $i < j$ such that both of the corresponding simple modules S_i and S_j are composition factors of K, then each simple S_m, with $i \leqq m \leqq j$, is a composition factor of K.*

With the help of (A-ii) and recalling the structure of A_n-diagrams, it is not difficult to see that the following lemma is true.

LEMMA (A-iii). *Let S_i and S_j be simple summands of* Soc *K with $S_i \not\cong S_j$. Then*

(a) *$P(S_i)$ and $P(S_j)$ have no composition factor in common.*

(b) *$I(S_i)$* and *$I(S_j)$ have at most one composition factor in common. If $I(S_i)$ and $I(S_j)$ have a composition factor in common, then this common* *composition factor is a simple injective summand of* K/rK, *and if* $i < j$, *then no simple* S_m *with* $i < m < j$ *is a summand of* Soc K.

(c) $P(S_i)$ *and* $I(S_j)$ *have no composition factor in common.* $P(S_i)$ *and* $I(S_i)$ *have only one common composition factor, and this composition factor is a copy of* S_i.

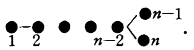
Because of Proposition 1, it is enough to look at the case where K is not simple. If K is not simple, then $\operatorname{Soc} K \subseteq rK$, and we get that $\ell(\operatorname{Soc} K) + \ell(K/rK) \leqq \ell(K)$. This means that the following inequality is true.

$$\begin{aligned}\ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(K) \\ \leqq \ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(\operatorname{Soc} K) - \ell(K/rK).\end{aligned}$$

Now look at the composition factors of the module $P(\operatorname{Soc} K) \amalg I(K)$. From (A-i) and (A-iii, b) it follows that the module $I(K)$ has at most two copies of any simple module as composition factor, and if $I(K)$ has two copies of a simple S_i, then S_i is also a composition factor of K/rK. From (A-iii, (a)) $P(\operatorname{Soc} K)$ has at most one copy of any simple module as composition factor, and from (A-iii, c) it follows that if S_i is a composition factor of $P(\operatorname{Soc} K)$ which is also a composition factor of $I(K)$, then S_i is a summand of Soc K. From (A-i) it follows that Soc K has at most one copy of S_i as a composition factor. Consequently, the module $P(\operatorname{Soc} K) \amalg I(K)$ has at most two copies of any simple module as composition factors, and if $P(\operatorname{Soc} K) \amalg I(K)$ has two copies of a simple S_i, then this simple module is a compositions factor of Soc K or of K/rK. It follows that

$$\ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(\operatorname{Soc} K) - \ell(K/rK) \leqq n = \ell(M),$$

and the inequality (1) is proved.

(II) PROOF OF THE THEOREM IF Λ IS OF TYPE D_n, $n \geqq 4$. If Λ is of type D_n, $n \geqq 4$, then we number the vertices of the D_n-diagram in the following way . From [2] we know that the module of maximal length has the dimension vector $(1, 2, \ldots, 2, 1, 1)$, and so $\ell(M) = 3 + 2(n - 3) = 2n - 3$.

By considering the indecomposable representations of a D_n-diagram, see [**2**; 3.2] we get the following lemma.

LEMMA (D-i). *No indecomposable* Λ*-module has more than two copies of any simple module as composition factors. Further, no indecomposable module* K *has more than one copy of the simple modules* S_1, S_{n-1}, S_n, *corresponding to the vertices* $1, n - 1$ *and* n *of the diagram. Since* $\operatorname{Soc} K \subseteq K$, *the same thing is true of* Soc K.

We will first consider the case when K has at most one copy of each simple module as composition factor. In this case, some of the properties of A_n-diagram still hold. We make use of the following lemma, see [**2**; 3.2 *b*)-f)].

LEMMA (D-ii). *If an indecomposable Λ-module K has no more than one copy of any simple module as composition factor, and it has one copy of each of the simple modules S_i and S_r with $1 \leqq i \leqq r \leqq n$, then it has one copy of each simple S_k with $i \leqq k \leqq r$, except possibly when $r = n$ and $k = n - 1$. Remark also that if both S_{n-1} and S_n are composition factors of K, then S_{n-2} is also a composition factor of K.*

Let now K be an indecomposable Λ-module that has no more than one copy of any simple module as composition factor. Suppose that at most one of the simple modules S_{n-1} and S_n is a summand of Soc K. Recalling the structure of the D_n-diagrams and applying (D-ii), it is not difficult to see that K satisfies lemma (A-iii) above, and one can argue as in the case of A_n-diagrams to show that

$$\ell(P(\mathrm{Soc}\ K)) + \ell(I(K)) - \ell(K) \leqq n < 2n - 3.$$

Suppose that both S_{n-1} and S_n are summands of Soc K. Then we have from (D-ii) above that K has a copy of S_{n-2} as composition factor. Since both S_{n-1} and S_n are summands of Soc K, the D_n-diagram has an oriented subquiver D_t of form $\bullet\!\to\!\bullet \cdots \bullet\!\to\!\bullet\genfrac{}{}{0pt}{}{\nearrow\bullet_{n-1}}{\searrow\bullet_{n}}$, where we have chosen $t \in N$. Such t is the least integer in the set $\{1, \ldots, n-2\}$ which has the property that all arrows of D_t point in the same direction. If $t \neq 1$, then observe that the subquiver $\underset{1}{\bullet}\!-\!\bullet \cdots\cdot \bullet\!-\!\underset{t}{\bullet}$ is an A_n-quiver. We see that $I(S_n)/S_n \cong I(S_{n-1})/S_{nn-}$, and S_n and S_{n-1} are simple projectives. Let m be least integer in the set $\{1, \ldots., n-2\}$ such that S_m is a composition factor of K, by (D-ii) there always exists such an integer m. Now if $m \geqq t$, then $P(\mathrm{Soc}\ K) \amalg I(K)$ has two copies of every simple S_i with $t \leqq i \leqq n$, and no copies of any other simple module. Since S_n, S_{n-1} and S_m are composition factors of K, we get $\ell(P(\mathrm{Soc}\,K)) + \ell(I(K) - \ell(K) \leqq 2n - 3$. If $m < t$, then $P(\mathrm{Soc}\ K) \amalg I(K)$ has three copies of the simple module S_t, but then by (D-ii) S_t is also a composition factor of K. $P(\mathrm{Soc}\ K) \amalg I(K)$ has two copies of any simple S_i with $t < i \leqq n$, but each such S_i is also a composition factor of K, and since the subquiver $\underset{1}{\bullet}\!-\!\bullet \cdot\cdot\underset{m}{\bullet}\cdot \bullet\!-\!\underset{t}{\bullet}$ is an A_n-quiver, we get $\ell(P)\mathrm{Soc}\,(K) + \ell(I(K)) - \ell(K) \leqq 2n - 3$.

We will now consider indecomposable Λ-modules which have two copies of at least one simple module. In this case we have the following lemma which follows directly from [**2**; 3.2a)].

LEMMA (D-iii). *Every indecomposable Λ-module K which has two copies of at least one simple module as composition factors, is of the following form: K has one copy of each of the simple modules S_{n-1} and S_n. There* *exists a unique $u \in N$, with $1 < u \leqq n-2$, such that K has two copies of any simple S_i with $u \leqq i \leqq n-2$. Further, there exists a unique $v \in N$, with $1 \leqq v < u$, such that K has one copy of any simple S_i with $v \leqq i \leqq u-1$, and if $v > 1$, K has no copy of any simple S_i with $1 \leqq i < v$.*

Since $\ell(M) = 2n - 3$, to prove the theorem it is enough to show that if K is as in (D-iii), then $P(\operatorname{Soc} K) \amalg I(K)/K$ has at most one copy of two of the simple modules S_{n-2}, S_{n-1}, S_n, at most one copy of either S_u or S_v, and at most two copies of any other simple module as composition factors.

If both or none of the simple modules S_{n-1} and S_n are summands of $\operatorname{Soc} K$, then it is easy to see that $P(\operatorname{Soc} K) \amalg I(K)/K$ has exactly one copy of each of these simple modules. Suppose that one, but not both, of the simple modules S_{n-1} and S_n is in $\operatorname{Soc} K$, say S_n is in $\operatorname{Soc} K$. If S_{n-2} is also in $\operatorname{Soc} K$, then there are three copies of S_n in $P(\operatorname{Soc} K) \amalg I(K)$ (note that S_n is a composition factor of $P(S_{n-2})$, and three copies of S_{n-2} (note that S_{n-2} is a composition factor of $I(S_n)$), but there are only two copies of S_{n-1}. If S_{n-2} is not in $\operatorname{Soc} K$, then there can be at most three copies of each of the simple modules S_{n-1} and S_{n-2} in $P(\operatorname{Soc} K) \amalg I(K)$, and two copies of S_n. Therefore $P(\operatorname{Soc} K) \amalg I(K)/K$ has in any case at most one copy of two of the three simple modules S_{n-2}, S_{n-1} or S_n, and no more than two copies of the third one.

Now, if S_i is a simple projective summand of $\operatorname{Soc} K$ with $u \leqq i \leqq n-2$, then there are two copies of S_i in $\operatorname{Soc} K$, four copies of S_i in $P(\operatorname{Soc} K) \amalg I(K)$, and two copies in $P(\operatorname{Soc} K) \amalg I(K)/K$. If S_i is a nonprojective summand of $\operatorname{Soc} K$ with $i \geqq u$, then from (D-iii) and the structure of D_n-diagrams one can infer that $i = u$, and there are in this case three copies of S_u in $P(\operatorname{Soc} K) \amalg I(K)$, and one copy in $P(\operatorname{Soc} K) \amalg I(K)$. In this situation there can be two copies of S_v in $P(\operatorname{Soc} K) \amalg I(K)/K$, in all other cases there are at most one. If S_i is a summand of $\operatorname{Soc} K$ with $i < u$, then there are at most three copies of S_i in $P(\operatorname{Soc} K) \amalg I(K)$, and at most one in $P(\operatorname{Soc} K) \amalg I(K)/K$. If S_t is a simple injective Λ-module with $u < t \leqq n-2$, then there are four copies of S_t in $P(\operatorname{Soc} K) \amalg I(K)$, two copies in $P(\operatorname{Soc} K) \amalg I(K)/K$. If S_t is a simple injective with $v < t \leqq u$, then there are at most three copies of S_t in $P(\operatorname{Soc} K) \amalg I(K)$, and at most two copies in $P(\operatorname{Soc} K) \amalg I(K)/K$. If S_i is a noninjective simple modula which is not a composition factor of $\operatorname{Soc} K$, then there are at most two copies of S_i in $P(\operatorname{Soc} K) \amalg I(K)$. The conclusion follows.

(III) Λ IS OF TYPE E_6, E_7, E_8. The inequalities

$$\ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(K) \leqq 11 \qquad E_6$$

$$\ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(K) \leqq 17 \qquad E_7$$

$$\ell(P(\operatorname{Soc} K)) + \ell(I(K)) - \ell(K) \leqq 29 \qquad E_8$$

have been established by checking them for all possible orientations with help of a computer. This computer work has been done by Ø. Bakke.

We will now show how this theorem can be applied to find a limit on the length of the indecomposable modules over the trivial extension R of Λ by $D\Lambda$, and hence to the relation between the number of nonempty preprojective classes, $p(R)$, and the maximal length of indecomposable R-modules. The aim is to show that $p(R) - 1 \leqq \ell(X_0) \leqq p(R)$, where X_0 is an indecomposable R-module of maximal length.

Let Λ be as above, and R the trivial extension of by $D\Lambda$. We will consider right Λ-modules and right R-modules. Let $Q = D\Lambda$. We recall that there is an embedding of categories $\operatorname{mod}\Lambda \to \operatorname{mod}R$ [4]. We first get the following

PROPOSITION 2. *Let X be an indecomposable R-module, and let M be an indecomposable Λ-module of maximal length.*

(i) *If X is projective injective, then $\ell(X) \leqq 1 + \ell(M)$. We have an equality sign above if and only if Λ is a Nakayama algebra, or either the projective cover or the injective envelope of X/rX in* $\operatorname{mod}\Lambda$ *is isomorphic to M.*

(ii) *If X is an indecomposable nonprojective R-module, then*

$$\ell(X) \leqq \ell(M).$$

PROOF. We recall from [4] that if X is an R-module, then X can be written as $X = (U \otimes Q \xrightarrow{\phi} V)$, where U, V are Λ-modules, and ϕ is a Λ-morphism. To prove (i), we apply Proposition 1. Let X be an indecomposable projective injective R-module. Then we have from [4] that X is of form $X = (P \otimes Q \xrightarrow{\phi} I)$ where P is an indecomposable projective Λ-module, ϕ is an isomorphism, and I is an indecomposable injective Λ-module such that $P/rP \cong \operatorname{Soc} I$. We get $\ell(X) = \ell(P) \perp\!\!\!\perp \ell(I)$. But from Proposition 1 we know that there is an indecomposable Λ-module Y such that $\ell(Y) = \ell(P) + \ell(I) - 1$. We get $\ell(X) = \ell(P) + \ell(I) = \ell(Y) + 1 \leqq \ell(M) + 1$, since M is an indecomposable Λ-module of maximal length.

If P or I is isomorphic to M, then it is easy to see that we have an equality sign. Suppose now that for an indecomposable projective R-module X we have $\ell(X) = \ell(M) + 1$, that is, if $X = (P \otimes Q \xrightarrow{\sim} I)$, then $\ell(P) + \ell(I) = \ell(M) + 1$, or $\ell(Y) = \ell(P) + \ell(I) - 1 = \ell(M)$, where Y is the pasted module corresponding to P and I. An hereditary artin algebra of finite type has, up to isomorphism, only one indecomposable module of maximal length, see [2], therefore $Y \cong M$. But the pasted module Y has waist (see Proposition 1), so M has a waist. Since Λ is of finite type, this means that M has a simple top or a simple socle, see [1, theorem 2.2]. Therefore, Λ has to be of type A_n and, furthermore, it has to be of one of the following types:

1) $\underset{1}{\bullet}\rightarrow\bullet\cdots\bullet\rightarrow\bullet$; 2) $\underset{a_n}{\bullet}\rightarrow\bullet\cdots\cdots\underset{a_1}{\bullet}\rightarrow\underset{a_0}{\bullet}\leftarrow\underset{b_1}{\bullet}\cdots\cdots\bullet\leftarrow\underset{b_m}{\bullet}$, where $m \geqq 1$, $n \geqq 1$; or 3) $\underset{a_n}{\bullet}\leftarrow\bullet\cdots\cdots\underset{a_1}{\bullet}\leftarrow\underset{a_0}{\bullet}\rightarrow\underset{b_1}{\bullet}\cdots\bullet\rightarrow\underset{b_m}{\bullet}$, $n \geqq 1$, $m \geqq 1$.

In the first case, Λ is Nakayama. Then if X is a projective R-module, $\ell(X) = \ell(M) + 1$. In the second case, M is injective, in the third case M is projective. This completes the proof of (1) in the theorem.

If X is an indecomposable nonprojective R-module, we know from **[4]** that X is either of type $X = (N \otimes Q \rightarrow 0)$, where N is an indecomposable Λ-module, or $X = (U \otimes Q \xrightarrow{\lambda} V)$, where U is a projective Λ-module, V is injective, and $\ker \lambda = K$ is an indecomposable noninjective Λ-module with $K \rightarrow U \otimes Q$ an injective envelope in mod Λ**[4]**. If X is of the first type, it follows trivially that $\ell(X) \leqq \ell(M)$, where M is the indecomposable Λ-module of maximal length. Therefore, it is enough to look at X of second type to prove part (ii) of Proposition 2. We will make use of the following lemma.

LEMMA. *If* $X = (U \otimes Q \xrightarrow{\lambda} V)$, *with* $\ker \lambda = K$, *then* $\ell(X) = \ell(P(\mathrm{Soc} K)) + \ell(I(K)) - \ell(K)$, *where as above* $I(K)$ *is the injective envelope of* K *in* modΛ, *and* $P(\mathrm{Soc}\ K)$ *is the projective cover of* Soc K *in* modΛ.

PROOF. If $X = (U \otimes Q \xrightarrow{\lambda} V)$ with $\ker \lambda = K$, then $U \otimes Q \cong I(K)$ and $U/rU \cong \mathrm{Soc}\ I(\mathrm{Soc}\ K) = \mathrm{Soc}\ K$. That means $U \cong P(\mathrm{Soc}\ K)$. The sequence $0 \rightarrow K \rightarrow I(K) \xrightarrow{\lambda} V \rightarrow 0$ is exact, therefore $V \cong I(K)/K$. We get $\ell(X) = \ell(U) + \ell(V) = \ell(P(\mathrm{Soc}\ K)) + \ell(I(K)) - \ell(K)$.

Using this lemma and the theorem above, part (ii) of the proposition follows immediately.

Now, let $p(R)$ be the number of nonempty preprojective classes in the preprojective partition for R. We will now easily get

THEOREM B. *Let* Λ *and* R *as above. If* X_0 *is an indecomposable* R*-module of maximal length, then*

$$p(R) - 1 \leqq \ell(X_0) \leqq p(R),$$

and $\ell(X_0) = p(R)$ *if and only if the indecomposable* Λ*-module of maximal length is projective or injective.*

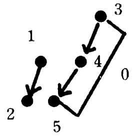
PROOF. In [3] it is shown that $p(R) = \ell(M) + 1$, where M is an indecomposable Λ-module of maximal length. Since M is also an indecomposable R-module we have $\ell(X_0) \geqq \ell(M)$. From Proposition 2 we get $\ell(X_0) \leqq p(R)$, and if $\ell(X_0) = p(R) = \ell(M) + 1$, we have the situation described in Proposition 2(i), and the conclusion follows.

3. Examples. In this section we want to look at some examples of algebras which are not hereditary, and see what happens to the inequalities discussed in §2. First we will present an algebra which has the property that the inequality (1) above is not satisfied by all indecomposable Λ-modules, not even by all the simple modules.

EXAMPLE 1. Let Λ be a self-injective Nakayama algebra, and suppose that the common length of the indecomposable projectives is n, where $n > 1$. Then we know that this is the maximal length of the indecomposable Λ-modules. If S_i is any simple module, then

$$\ell(P(S_i)) + \ell(I(S_i)) - \ell(S_i) = 2_n - 1 > n.$$

EXAMPLE 2. Let Λ be the algebra given by the tree with one zero relation.



There are 12 nonisomorphic indecomposable Λ-modules, and the maximal length is 4. One can check that

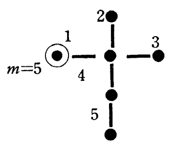
$$\ell(P(\mathrm{Soc}\ K)) + \ell(I(K)) - \ell(K) \leqq 4$$

for all indecomposable Λ-modules K.

But let T be the trivial extension of Λ by $D\Lambda$. Then one can show that T corresponds to a Brauer tree of form •—•—•—•—•—•. There are two nonprojective Λ-modules of length 5, so the statement of Proposition 2(ii) is not true in this case.

As a last example, we want to present a self-injective algebra for which $\ell(X_0) < p(R) - 1$, where X_0 is an indecomposable module of maximal length.

EXAMPLE 3. Let Λ be an algebra corresponding to the Brauer tree



There is one exceptional vertex of multiplicity $m = 5$. One can show that the maximal length of indecomposable modules is 12, while $p(\Lambda) = 17$.

REFERENCES

1. M. Auslander, E. L. Green, I. Reiten, *Modules with waists*, Illinois J. Math. (1975), 467–477.

2. P. Gabriel, *Unzerlegbare Darstellungen* I, Man. math. 6 (1972) 71–103, Springer Verlag.

3. B. Rohnes, *Preprojective partitions for trivial extensions of hereditary algebras*, To appear in Communications in algebra.

4. H. Tachikawa, *Representations of trivial extensions of hereditary algebras*, Proc. of the Ottawa Conference, Springer Lecture Notes **832** (1980), 579–599.

5. D. Zacharia, *The preprojective partition for an hereditary artin algebra*, to appear.

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